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MATHEMATICIANS AND THE NEW SELECTIVE SERVICE REGULATIONS

On January 10, 1944, Selective Service Headquarters released amended Local Board Memoranda 115 and 115B, which become effective on February 1, and also an amended Activity and Occupational Bulletin No. 33-6, which becomes effective on February 15. These vitally concern all chairmen of departments of mathematics for they deal with the occupational classification of both teachers and undergraduate students. This communication contains a summary of the provisions of these directives.

Mathematicians are still included in the List of Critical Occupations. It is specifically stated that the titles included in Part II (Professional and Scientific Occupations) of the List of Critical Occupations "shall be considered as also including persons engaged in full-time teaching of these professions. A person may be considered as engaged in full-time teaching if he devotes not less than 15 hours per week in contact with students in actual classroom or laboratory instruction." The problem of securing the proper classification of a *teacher* of mathematics differs in accordance with the age group into which he falls.

I. *Deferment of Mathematicians in 18–21 Age Group*

All requests for new or additional occupational deferments for registrants in this age group who are teachers of mathematics must be made on Director of Selective Service Form 42-A Special. An original and two copies of this form must be presented by the employer to the State Director in whose state is located the registrant's principal place of employment.

Effective February 1, no registrant in this group, at the time he is classified, may be considered as a "necessary man" entitled to be placed in Class II-A or Class II-B unless:

(a) the State Director of Selective Service has endorsed his Form 42-A Special with a statement that, based on the information furnished therein, he recommends that the local board except the registrant from the general restrictions against occupational deferment of registrants aged 18 through 21.

(b) he is classified as (1) belonging to the Personnel of the Merchant Marine or Army Transportation Corps or (2) as a student who qualifies under AOB 33-6.

Registrants in this group who are classified II-A or II-B on February 1, 1944, will, in general, not have deferments terminated before the expiration date. (This, however, does not apply to students covered in the new AOB 33-6.)

The President and Secretary of the Society have addressed a memorandum to State Directors of Selective Service, calling attention to the vital need for these mathematicians to handle the mathematics in the Army Specialized Training Program and the Navy College Training Program.

II. *Deferment of Mathematicians in 22–37 Age Group*

In making requests for occupational deferment, one copy of Form 42-A must be filed with the local board, as heretofore. The local board may, without going further, classify the registrant in II-A or II-B, as requested.

If the registrant is not classified in II-A or II-B and if reference to the United States Employment Service was not made by the local board prior to classification in I-A, I-A-O or IV-E, it must be made immediately after such classification. Upon suitable certification from the USES, the local board must reopen the case. The cases of registrants who are qualified for professional and scientific occupations will be forwarded by the USES to the National Roster. In all cases in which reference to the USES is made, the local board is directed not to issue an order to report for induction until it has received a report from the USES or until the expiration of 30 days after referral, whichever occurs first. If, during the 30-day period, the USES certifies to the local board that the registrant is qualified and that his removal would adversely affect the maintenance of his employer's required production, the local board must reopen the case and consider the new evidence as a basis for further deferment.

Even though a registrant is not employed in the area in which his local board is located, the registrant's local board must refer the case to the local USES office in the local board area and any further reference of the case to the USES office in which the registrant is employed will be undertaken and accomplished by the USES.

In view of these provisions, *department chairmen are urged to register their needs for personnel with the USES office where the college or university is located and in the USES office where the registrant's local board is located.* It is probable that the USES will be unable to supply any mathematicians but this is the best way to establish the fact that there is a shortage. If the USES does not have calls for mathematicians, physicists, engineers, chemists, et cetera, it will have no reason to believe that there is a shortage.

The USES offices are required to refer cases involving scientific personnel to the National Roster. (Information has come to the Secretary to the effect that the USES offices are not now referring cases to the Roster.) If department chairmen believe that the advice given by the local USES office to the local board is inadequate or incorrect, the employer is advised to call the attention of the USES office to the instructions in Sections 5300–5309 of the USES Manual (which contain instructions involving the National Roster). If Roster advice is not sought, the chairman will do well to report all facts in the case directly to the Roster.

Department chairmen are urged to make the original requests for occupational deferment as strong as possible and to make all possible appeals. Despite the curtailment of certain college training programs, chairmen are advised to continue requests for deferment, for the shortage of qualified mathematicians is still critical.

III. *Department of Undergraduates*

In Activity and Occupation Bulletin 33-6 the position is taken that the Army and Navy Training Programs will provide adequately for the needs of the armed forces. Student deferment is for civilian needs in war production and in support of the war effort.

Students who will graduate on or before July 1, 1944, in a considerable number of fields (including engineering branches, bacteriology, chemistry, forestry, geophysics, mathematics, meteorology, optometry, pharmacy, and physics including astronomy) are eligible for occupational deferment on certification by the institution and the National Roster.

An undergraduate who is a full-time student of chemistry, engineering, geology, geophysics, or physics who will graduate after July 1, 1944, will be eligible for occupational deferment, if he is properly certified by his institution and the National Roster and will graduate within 24 months from the date of certification and provided the national quota of 10,000 is not exceeded. Mathematics was not included in the list mentioned above, in spite of the strong protests of our Washington representatives. For undergraduate students in the 18–21 age group, requests for deferment are made in duplicate on DSS Form 42 Special; for students over 22 years of age, Form 42 is used.

A national quota of 10,000 has been established for students who should be occupationally deferred at any one time by reason of pursuing courses of study in chemistry, engineering, geology, geophysics, and physics. Students deferred to graduate on or before July 1, 1944, and students deferred for reasons other than pursuing a course of study will not be counted against this quota.

Requests for information needed by the National Roster to set up institutional quotas, et cetera, have already gone out to college presidents. Information regarding the procedures to be followed by institutions and departments concerned will be issued by the National Roster to college presidents in the near future. The departments concerned should be on the lookout for such information.

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Department chairmen are requested to inform the Secretary of cases in which mathematicians are improperly classified. In this manner the War Policy Committee, through its Washington representatives, will be better able to present the needs of the mathematicians.

M. H. STONE,
President
J. R. KLINE,
Secretary

January 17, 1944.

MAPPINGS BY MEANS OF SYSTEMS OF ANALYTIC FUNCTIONS OF SEVERAL COMPLEX VARIABLES

W. T. MARTIN

1. Introduction. An analytic mapping of a domain D in the space E_{2k} of k complex variables $z_1, \cdots, z_k$ is a mapping defined by

$$(1.1) \qquad T: \quad z'_j = f_j(z_1, \cdots, z_k), \qquad j = 1, \cdots, k,$$

where the $f_j(z)$ are analytic in D. We shall consider only univalent (schlicht) domains D contained in the finite portion of the space E_{2k}. An important result in the theory of analytic mappings states that the mapping is topological (that is, 1-1 and bi-continuous) if, and only if, the Jacobian

$$(1.2) \qquad J(z) \equiv \partial(f_1, \cdots, f_k)/\partial(z_1, \cdots, z_k)$$

different from zero at each point of D (Carathéodory [12]).[1]

An analytic mapping T is called an *inner mapping* of a domain D if $TD \subset D$. It is called an *automorphism* of D if T is 1-1 and if $TD = D$.

In 1907 Poincaré [17] showed that, given two domains D and D', it is not always possible to map D onto D' analytically. Since that time, several general problems have been considered. One of these problems is to indicate some general rules which tell whether or not two given domains can be mapped analytically upon each other. A ond problem is to determine a family of special domains, in terms some simple properties, the family to be such that every domain l be mapped analytically onto one of these special domains.

In this talk I shall deal with a special case of the first of these two problems together with certain results on the second problem. The work on the first problem which will be presented is based upon Henri Cartan's theory of mappings of domains onto domains of circular type. The work given on the second problem is based upon Bergman's theory of representative domains. In preparing this talk, I have used freely the excellent résumé on analytic mapping contained in the book on several complex variables by Behnke and Thullen [4]. I have also used freely material from the manuscript by Bochner and the speaker of a book on several complex variables, now in preparation [10]. I am indebted to Professor Bochner for permission to use

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An address delivered before the New York meeting of the Society on October 30, 1943, by invitation of the Program Committee; received by the editors November 9, 1943.

[1] Numbers in brackets refer to the Bibliography at the end of the paper.

material from this manuscript. The approach used in much of thi talk follows the approach of that manuscript, and in addition certain so far unpublished results from it are given here.

Let D be a domain containing the origin, and let (1.1) be an analytic mapping of D onto a domain D', with a fixed origin. We shall frequently write (1.1) in the form

$$(1.3)\qquad T:\ z_j' = a_{1j}z_1 + \cdots + a_{kj}z_k + \text{(higher powers)}, \qquad j = 1, \cdots, k,$$

to mean that the analytic functions $f_j(z)$ defining T have the expansions

$$(1.4)\qquad f_j(z) = a_{1j}z_1 + \cdots + a_{kj}z_k + \text{(higher powers)}$$

in a neighborhood of the origin.

Again, if D is a domain containing the origin and if $f(z_1, \cdots, z_k)$ is analytic in D we shall sometimes have occasion to develop f in the neighborhood of the origin in terms of homogeneous polynomials (a diagonal series)

$$(1.5)\qquad f(z_1, \cdots, z_k) = \sum_{n=0}^{\infty} P_n(z_1, \cdots, z_k).$$

The expression $P_n(z)$ is a homogeneous polynomial of degree n and is given by

$$(1.6)\qquad P_n(z) = \frac{1}{2\pi}\int_0^{2\pi} f(z_1e^{i\theta}, \cdots, z_ke^{i\theta})e^{-in\theta}d\theta$$

for z in a sufficiently small neighborhood N of the origin. From (1.6) we see that if $|f(z)| \leqq M$ in N, then

$$(1.7)\qquad |P_n(z)| \leqq M \quad \text{for} \quad z \in N, \qquad n = 0, 1, 2, \cdots$$

2. **Some uniqueness properties of analytic mappings.** H. Cartan [13] and C. Carathéodory [12] have proved two very elegant results on the uniqueness of analytic mappings. We shall give these two results here:

THEOREM 1 (CARTAN). *Let D be a domain containing the origin an let T be an analytic mapping of D into a bounded domain $D' \subset D$. I the linear part of T is the identity, that is, if T has the form*

$$(2.1)\quad T:\ z_j' = f_j(z_1, \cdots, z_k) = z_j + \textit{(higher powers)}, j = 1, \cdots, k,$$

then T is actually the identity

$$f_j(z) \equiv z_j, \qquad j = 1, \cdots, k. \tag{2.2}$$

PROOF. Expand the $f_j(z)$ in terms of homogeneous polynomials:

$$f_j(z) = z_j + P^j_{r_j}(z) + P^j_{r_j+1}(z) + \cdots \tag{2.3}$$

with r_j an integer not less than 2. In writing $f_j(z)$ in this form, we are assuming that the polynomials P^j_n of degree n for $1 < n < r_j$ all vanish identically. Now iterate the mapping. We find that T^2 has the expansion

$$T^2\colon\ z_j'' = z_j' + P^j_{rj}(z') + \cdots = z_j + 2P^j_{r_j}(z) + \cdots, \tag{2.4}$$

and in general

$$T^m\colon\ z_j^{(m)} = z_j + mP^j_{r_j}(z) + \cdots. \tag{2.5}$$

Now T maps D into a bounded domain $D' \subset D$, hence the same is true of all the iterates T^2, T^3, $\cdots$. Thus the mapping functions

$$f_j^{(m)}(z) = z_j + mP^j_{r_j}(z) + \cdots \tag{2.6}$$

are uniformly bounded in D. By the result given at the end of §1, this implies that

$$\left| mP^j_{r_j}(z) \right| \leqq M, \qquad \text{for } j = 1, \cdots, k;\ m = 1, 2, \cdots,$$

for z in a sufficiently small neighborhood of the origin. This implies that $P^j_{r_j}(z) \equiv 0$ and hence (2.2) holds.

Carathéodory's uniqueness theorem follows almost immediately from Cartan's theorem.

THEOREM 2 (CARATHÉODORY). *There exists at most one* 1-1 *analytic mapping of a domain D onto a bounded domain D' which maps a point O of D onto a point O' of D' and whose first partial derivatives in O have prescribed values.*

PROOF. If S and T are two such mappings, then the mapping ST^{-1} is a mapping of D' onto itself and the hypotheses of Theorem 1 are satisfied. Thus

$$ST^{-1} = I \quad \text{or} \quad S = T.$$

An important corollary has been derived from Theorem 2. First we make a definition. A *circular* domain is a domain which is mapped onto itself by all the mappings

$$T(\theta)\colon\ z_j' = e^{i\theta} z_j, \qquad j = 1, \cdots, k;\ 0 \leqq \theta < 2\pi.$$

A circular domain is called *proper* if it contains the origin. Circular domains were introduced by Behnke [1] and Carathéodory [11] and have been investigated by various writers. We shall return to them in §4. For the present we give only the following result.

COROLLARY. *Every automorphism of a proper bounded circular domain which leaves the origin fixed is a homogeneous linear transformation.*

PROOF. Let

$$A: \quad z_j' = \sum_{n=1}^{\infty} P_n^j(z_1, \cdots, z_k), \qquad j = 1, \cdots, k,$$

be any such automorphism. Then $T(\theta)A$ has the form

$$T(\theta)A: \quad z_j' = \sum_{n=1}^{\infty} e^{i\theta} P_n^j(z_1, \cdots, z_k),$$

and $AT(\theta)$ has the form

$$AT(\theta): \quad z_j' = \sum_{n=1}^{\infty} P_n^j(z_1 e^{i\theta}, \cdots, z_k e^{i\theta}) = \sum_{n=1}^{\infty} e^{in\theta} P_n^j(z).$$

Since D is circular and A is an automorphism, both transformations $T(\theta)A$ and $AT(\theta)$ are automorphisms of D, with the origin fixed. Also, both have the same linear parts. Hence, by Theorem 2, they are the same. This can happen, however, only if $P_n^j(z) \equiv 0$ for $n > 1$, that is only if A is linear.

The corollary has been proved by Behnke [2], Bergman [5], H. Cartan [13], and Welke [18]. Bergman's proof, which is by a different method, will be given in §5.

3. Inner mappings and automorphisms with a fixed point. Cartan's theorem tells us that if we have an inner analytic mapping T of a domain D containing the origin into a bounded domain D', and if the linear part of T is the identity, then T is actually the identity. Now suppose we have a general inner analytic mapping of a bounded domain D with the origin as a fixed point:

$$(3.1) \qquad T: \quad z_j' = a_{1j} z_1 + \cdots + a_{kj} z_k + \cdots, \qquad j = 1, \cdots, k.$$

Denoting by Δ_T the value of the functional determinant at the origin:

$$(3.2) \qquad \Delta_T = \left| a_{ij} \right|_{i,j=1,\cdots,k},$$

Cartan [14] and Carathéodory [12] have shown that

$$|\Delta_T| \leqq 1 \tag{3.3}$$

holds for every inner mapping, with the equality

$$|\Delta_T| = 1 \tag{3.4}$$

holding if and only if T is an automorphism. This result is very important in the theory of analytic mapping and will be useful to us in a later section.

The result has recently been generalized a little by Bochner and the speaker [10]. Denoting by $\lambda_1, \cdots, \lambda_k$ the characteristic roots of the matrix (a_{ij}), they have shown that

$$|\lambda_1| \leqq 1, \cdots, |\lambda_k| \leqq 1 \tag{3.5}$$

holds for every inner transformation (3.1) and

$$|\lambda_1| = \cdots = |\lambda_k| = 1 \tag{3.6}$$

holds if and only if T is an automorphism.

We shall not give the proofs of these results.

4. Groups of automorphisms of a bounded domain with a fixed point.

Near the end of §2 we introduced the notion of a circular domain, a domain which with a point (z^0) contains also all the points

$$(z_1^0 e^{i\theta}, \cdots, z_k^0 e^{i\theta}), \qquad 0 \leqq \theta < 2\pi. \tag{4.1}$$

Cartan [13] has generalized this notion as follows: Let $(m_1, \cdots, m_k)$ be k integers (positive, negative, or zero). Then an $(m_1, \cdots, m_k)$-circular domain is one which, with a point (z^0), contains all the points

$$(z_1^0 e^{im_1\theta}, \cdots, z_k^0 e^{im_k\theta}), \qquad 0 \leqq \theta < 2\pi. \tag{4.2}$$

The domain is called *proper* if it contains the origin.

The domains of circular type play in several respects the role of the circle in one complex variable. It is, therefore, natural to question whether every bounded domain D can be mapped analytically onto some domain of circular type, or if not, which ones can. In the first place, an $(m_1, \cdots, m_k)$-circular domain admits an infinite number of automorphisms with the origin as a fixed point, namely those of (4.2). Therefore, if it is possible to map analytically a given domain D onto an $(m_1, \cdots, m_k)$-circular domain, the domain D must admit an infinite number of automorphisms leaving an interior point fixed. Cartan [13] (for $k=2$) has shown that this condition is also sufficient. Following the résumé given by Behnke and Thullen [4], we shall give a brief outline of Cartan's argument.

THEOREM 3 (CARTAN). *The group G of all automorphisms of a bounded domain D which leave the origin fixed* ($O \in D$) *is isomorphic with a compact group Γ of homogeneous linear transformations; the determinant of each transformation of Γ is of absolute value unity.*

By Carathéodory's theorem, Theorem 2, the group Γ consists merely of the linear parts of the members of G, and by (3.4) the determinant of each transformation of G is of absolute value unity.

An infinite compact group Γ of homogeneous linear transformations (in the space of two complex variables) always leaves invariant an Hermitian form

$$Aw\bar{w} + Bz\bar{z} + Cw\bar{z} + \bar{C}\bar{w}z \tag{4.3}$$

(Weyl [19]). By a suitable linear transformation L this Hermitian form will be transformed into a normalized form

$$w\bar{w} + z\bar{z} \tag{4.4}$$

which is invariant under the corresponding group $\Gamma^* = L\Gamma L^{-1}$.

THEOREM 4 (Cartan). *Every infinite compact group of homogeneous linear transformations, which leave invariant the form* (4.4), *contains a subgroup of the form*

$$T_{m,p}(\theta):\quad w' = we^{im\theta}, \qquad z' = ze^{ip\theta}, \qquad 0 \leqq \theta < 2\pi. \tag{4.5}$$

Let G_0 be the subgroup of G which is isomorphic to the group $\{T_{m,p}(\theta)\}$. Denoting by $A(\theta)$ the isomorph in G of $T_{m,p}(\theta)$, we know that $A(\theta)$ must have the form

$$A(\theta):\quad \begin{aligned} w' &= f(w, z; \theta) = e^{im\theta}w + \cdots, \\ z' &= g(w, z; \theta) = e^{ip\theta}z + \cdots. \end{aligned} \tag{4.6}$$

Now define

$$\begin{aligned} F(w, z) &= \frac{1}{2\pi}\int_0^{2\pi} f(w, z; \theta)e^{-im\theta}d\theta, \\ G(w, z) &= \frac{1}{2\pi}\int_0^{2\pi} g(w, z; \theta)e^{-ip\theta}d\theta. \end{aligned} \tag{4.7}$$

Then Cartan showed that except possibly for certain exceptional cases the mapping

$$S:\quad w' = F(w, z), \qquad z' = G(w, z) \tag{4.8}$$

is a 1-1 analytic mapping of D onto a proper (m, p)-circular domain with $SO=O$. (It is easily seen that the linear part of S is the identity.) In the exceptional cases, Cartan showed that S can be replaced by a mapping $\tilde{S}$ which has the requisite properties. Hence, we have:

THEOREM 5 (CARTAN). *If a bounded domain D admits an infinite group G of automorphisms with the origin as a fixed point ($O\in D$), then D can be mapped in a* 1-1 *analytic manner on a bounded proper (m, p)-circular domain D' with the origin going into the origin.*

The structure of the group tells one what type of circular domain it will go into.

It is known that there exist bounded domains D which possess the following property: given any point P in D, the automorphisms of D which leave P fixed are finite in number [13]. Hence, *not every bounded domain can be mapped onto an (m, p)-circular domain.*

This concludes our résumé of Cartan's work. Certain aspects of it have been generalized. We shall indicate this generalization.

Let D be a bounded domain in E_{2k} containing the origin, and let G be a group of automorphisms of D with the origin as a fixed point. If $T(\alpha)$, $T(\beta)$, and so on, are elements of G, and if we write the expansion of $T(\alpha)$ in the form

$$\begin{aligned} T(\alpha):\quad z_j' &= a_1^j(\alpha)z_1 + \cdots + a_k^j(\alpha)z_k \\ &\quad + \sum_{n_1+\cdots+n_k\geqq 2} a^j_{n_1\cdots n_k}(\alpha)z_1^{n_1}\cdots z_k^{n_k}, \end{aligned} \tag{4.9}$$

then Carathéodory's uniqueness theorem, Theorem 2, states that if

$$a_i^j(\alpha) = a_i^j(\beta), \qquad i, j = 1, \cdots, k, \tag{4.10}$$

then

$$a^j_{n_1\cdots n_k}(\alpha) = a^j_{n_1\cdots n_k}(\beta), \quad j = 1, \cdots, k;\ n_1+\cdots+n_k \geqq 2. \tag{4.11}$$

Actually a little more than this has recently been proved [10], namely:

THEOREM 6 (BOCHNER AND MARTIN). *If $T(\alpha)$, $T(\beta)$, and so on, are elements of a group G of automorphisms of a bounded domain D with the origin as a fixed point, and if $(j, n_1, \cdots, n_k)$ is any fixed multi-index, then corresponding to any $\epsilon>0$, there exists a $\delta>0$ such that*

$$\sum_{\rho,\sigma=1}^{k} \left| a_\sigma^\rho(\alpha) - a_\sigma^j(\beta) \right| \leqq \delta \tag{4.12}$$

implies

$$\left| a^j_{n_1\cdots n_k}(\alpha) - a^j_{n_1\cdots n_k}(\beta) \right| \leqq \epsilon. \tag{4.13}$$

(The a's are the coefficients in the expansion of the T's as in (4.9).)

The proof of this theorem uses strongly the uniqueness theorem of Carathéodory. We shall omit the proof. By use of Theorem 6, Bochner and the speaker [10] have obtained the following theorem which is related to Cartan's theorem, Theorem 5.

THEOREM 7. *If* $\{T(\alpha)\}$ *is a group G of automorphisms of a bounded domain D with the origin as a fixed point, then there exists a change of coordinates S in the neighborhood of the origin*

$$S\colon\quad z_j' = z_j + (\textit{higher powers}), \qquad j = 1, \cdots, k,$$

such that the given automorphisms are linear transformations in the new coordinates,

$$L(T(\alpha)) = ST(\alpha)S^{-1}.$$

We shall outline the proof of this theorem as given in [10].

By Theorem 2 each $T(\alpha)$ is uniquely determined by $L(T(\alpha))$ and hence by the k^2 complex numbers $a_\mu^j(\alpha)$, $\mu, j = 1, \cdots, k$. Thus, every element of G can be identified with a point in Euclidean space E of $2k^2$ dimensions, and G itself with a bounded point set in E. Also since

$$L(T(\alpha)\cdot T(\beta)) = L(T(\alpha))\cdot L(T(\beta)),$$

the group product $\beta\alpha$ of any two elements α, β of G is given by

$$a_m^j(\beta\alpha) = \sum_{\mu=1}^{k} a_\mu^j(\beta)a_m^\mu(\alpha), \qquad j, m = 1, \cdots, k.$$

Thus in the ordinary Euclidean topology of E, the group product $\beta\alpha$ is a continuous function of β and α.

Now, since G is bounded, there exists an additive measure μ on G with the following properties among others:

(i) the measure of the total set G is 1,

(ii) the measure is group invariant, and

(iii) every function $f(\alpha)$ on G which is uniformly continuous on G (in the topology of E) is integrable.

Now Theorem 6 states that each coefficient of $T(\alpha)$ is uniformly continuous on G. Hence, by property (iii) each coefficient of $T(\alpha)$ is integrable relative to our measure. It follows that the transformation $L(T(\alpha^{-1}))\cdot T(\alpha)$ is measurable and it can be shown that the desired change of coordinates S is given by

$$S = \int_G L(T(\alpha^{-1})\cdot T(\alpha)d\mu(\alpha).$$

5. Bergman's representative domains. In connection with the second problem of the theory of analytic mapping mentioned in the introduction, Bergman [5] has introduced the notion of a *representative domain* of a class of *equivalent domains*, and in a series of papers he has developed several interesting aspects of the theory. A portion of his work has been simplified by Welke [18] and parts of it have been summarized by Behnke and Thullen [4] and by Bochner and the speaker [10]. In this section we shall follow the latter summary for the most part.

First we give two definitions due to Bergman.

Normalized mapping—An analytic mapping of a domain D containing the origin is called normalized (with respect to the origin) if its linear part is the identity.

Equivalent domains—Two domains D and D' are called equivalent if they can be mapped onto each other by a 1-1 analytic normalized mapping.

We shall use these two notions later in this section.

Let D be a domain in E_{2k} and denote by $\mathcal{L}_2$ the set of all functions $f(z_1, \cdots, z_k)$ analytic in D and such that the norm

$$\|f\| = \left[\int_D |f(z)|^2 d\omega_z \right]^{1/2} \tag{5.1}$$

is finite $(d\omega_z = dx_1dy_1 \cdots dx_kdy_k)$.

If D has finite volume then every bounded analytic function in D belongs to $\mathcal{L}_2$. If D is also bounded this includes all polynomials. For the sake of simplicity, we restrict ourselves to bounded domains D. The space $\mathcal{L}_2$ is then a Hilbert space.

Now take a point in D, call it the origin and expand all functions $f(z)$ of $\mathcal{L}_2$ in power series about the origin. It can be shown that there exists a unique minimal function

$$f_0(z) = 1 + (\text{higher powers}) \tag{5.2}$$

such that

$$\|f_0\| < \|f\| \tag{5.3}$$

for every f of $\mathcal{L}_2$ with the development

$$f(z) = 1 + (\text{higher powers}). \tag{5.4}$$

The same is true for functions of the form

$$f_j(z) = z_j + (\text{higher powers}), \tag{5.5}$$

that is for each $j=1, \cdots, k$ there exists a unique function $f_j(z)$ of $\mathcal{L}_2$ for which

$$\|f_j\| < \|f\| \tag{5.6}$$

holds for every $f \in \mathcal{L}_2$ in D with the development

$$f(z) = z_j + \text{(higher powers)}. \tag{5.7}$$

Bergman has shown that these $(k+1)$-minimal functions possess some beautiful forms of invariance. Consider a 1-1 normalized analytic transformation

$$z_j = z_j(z_1', \cdots, z_k') = z_j' + \text{(higher powers)} \tag{5.8}$$

from D onto a domain D' with nonvanishing Jacobian

$$J(z') = \partial(z_1, \cdots, z_k)/\partial(z_1', \cdots, z_k'). \tag{5.9}$$

Now, by the ordinary transformation of variables

$$\int_D |f(z)|^2 d\omega_z = \int_{D'} |f(z(z'))|^2 |J(z')|^2 d\omega_{z'}, \tag{5.10}$$

and it is not hard to see that

$$f(z) \leftrightarrow f(z(z'))J(z') \tag{5.11}$$

is a 1-1 transformation of $\mathcal{L}_2(z)$ onto $\mathcal{L}_2(z')$. Also since

$$J(z') = 1 + \text{(higher powers)} \tag{5.12}$$

we have

$$f_j^D(z(z')) \equiv f_j^{D'}(z')J(z'), \qquad j = 0, 1, \cdots, k, \tag{5.13}$$

where $f_j^D(z)$ and $f_j^{D'}(z')$ are the minimal functions for D and D' respectively. Hence at least in a neighborhood of the origin

$$f_j^D(z(z'))/f_0^D(z(z')) \equiv f_j^{D'}(z')/f_0^{D'}(z'), \qquad j = 1, \cdots, k, \tag{5.14}$$

and thus these quotients are absolute invariants. Omitting the superscript D, we denote these functions by $w_1(z), \cdots, w_k(z)$ and we observe that they have the expansion

$$w_j(z) = z_j + \text{(higher powers)}, \qquad j = 1, \cdots, k. \tag{5.15}$$

Now assume that our domain D is such that for D these k functions have the special form

$$w_j(z) \equiv z_j, \qquad j = 1, \cdots, k. \tag{5.16}$$

Transcribe them by (5.8) onto D'. Being invariants they go over into

$$w_j'(z') = z_j(z_1', \cdots, z_k') = z_j' + \text{(higher powers)}. \tag{5.17}$$

Hence they cannot have the specialized form (5.16) in the domain D' unless the transformation (5.8) is the identity $z_j \equiv z_j'$. In other words the specialized form (5.16) cannot occur for more than one domain D in a class of equivalent domains and thus the specialized form (5.16) singles out the domain D from all other domains D' which are equivalent to D under a 1-1 analytic mapping which preserves the origin. This specialized domain is called the *representative* of its class by Bergman.

Now let K be a class of equivalent domains and assume that the class has a representative domain R. Then each domain D of K can be mapped onto R in a 1-1 manner by a normalized mapping. Denote this mapping by

$$T:\ z_j' = g_j(z), \qquad z \in D, \tag{5.18}$$

and consider the mapping

$$S:\ z_j' = \overset{D}{w}_j(z) \tag{5.19}$$

where the $w_j^D(z)$ are the invariant quotients for D. We do not assume that S is 1-1. But by the invariance we know that

$$\overset{D}{w}_j[\overset{-1}{g_1}(z'), \cdots, \overset{-1}{g_k}(z')] \equiv \overset{R}{w}_j(z'), \tag{5.20}$$

and since R is representative

$$\overset{R}{w}_j(z') \equiv z_j', \qquad j = 1, \cdots, k.$$

Hence

$$\overset{D}{w}_j[\overset{-1}{g_1}(z'), \cdots, \overset{-1}{g_k}(z')] \equiv z_j', \tag{5.21}$$

that is

$$\overset{D}{w}_j(z) \equiv g_j(z). \tag{5.22}$$

Thus S gives a 1-1 normalized mapping of D onto R.

This yields a very beautiful result due to Bergman.

THEOREM 8 (BERGMAN). *Let K be a class of equivalent domains, equivalent with respect to* 1-1 *normalized analytic mappings. For $D \in K$ let*

$$\overset{D}{f_0}(z) = 1 + (\textit{higher powers}), \tag{5.23}$$

$$(5.24) \qquad \overset{D}{f}_j(z) = z_j + (\textit{higher powers}), \qquad j = 1, \cdots, k,$$

be the $k+1$ (*unique*) *minimal functions for* D *and define*

$$(5.25) \qquad \overset{D}{w}_j(z) = \overset{D}{f}_j(z)/\overset{D}{f}_0(z), \qquad j = 1, \cdots, k.$$

There exists at most one domain R *of* K *for which the functions* $w_j^R(z)$ *have the specialized form*

$$\overset{R}{w}_j(z) \equiv z_j, \qquad j = 1, \cdots, k.$$

If there is actually such a domain R *then every domain* D *of* K *can be mapped onto* R *by a* 1-1 *normalized analytic mapping, and the mapping which does this is given by*

$$(5.26) \qquad z_j' = \overset{D}{w}_j(z), \qquad j = 1, \cdots, k; z \in D.$$

It can easily be seen that there is a representative domain if for at least one domain D of K the mapping

$$z_j' = \overset{D}{w}_j(z)$$

is 1-1.

Bergman has observed that every bounded proper circular domain Δ is a representative domain. To see this write the minimal function $f_0(z)$ for a bounded proper circular domain Δ in the form

$$(5.27) \qquad f_0(z) = 1 + P_1(z) + P_2(z) + \cdots,$$

where $P_n(z)$ is a homogeneous polynominal of degree n. Then

$$(5.28) \qquad f_0(z_1e^{i\theta}, \cdots, z_ke^{i\theta}) = 1 + e^{i\theta}P_1(z) + e^{2i\theta}P_2(z) + \cdots.$$

Now the volume element $d\omega_z$ is invariant under the circular transformation

$$z_j \leftrightarrow z_je^{i\theta}.$$

Hence the function (5.28) has the same norm as (5.27). Hence both are minimal. But since the development of each starts with 1 and since the minimal functions are unique, (5.27) and (5.28) must be identical. Thus

$$P_n(z) \equiv 0, \qquad n = 1, 2, 3, \cdots,$$

and

$$f_0(z) \equiv 1.$$

In a similar manner we see that

$$f_j(z) \equiv z_j, \qquad j = 1, \cdots, k,$$

and thus

$$w_j(z) \equiv z_j.$$

This yields the desired result.

Now suppose we have a bounded domain D which is equivalent to a bounded proper circular domain Δ. On applying Bergman's theorem, Theorem 8, we have a procedure for mapping D onto Δ; we merely calculate the $k+1$ minimal functions

$$f_\mu(z), \qquad \mu = 0, 1, \cdots, k,$$

for D and map by

$$z_j' = f_j(z)/f_0(z), \qquad j = 1, \cdots, k.$$

By use of complex orthonormal functions Bergman [6] has simplified the calculation of the minimal functions.

A second important consequence of this result has been derived by Bergman and Welke, namely the corollary of §2.

Every origin preserving automorphism of a bounded circular domain Γ *containing the origin is a homogeneous linear transformation.*

Proof. Let

$$T: \quad z_j = a_{1j}z_1' + \cdots + a_{kj}z_k' + \cdots$$

be an automorphism of Δ. It can be normalized by a homogeneous linear transformation L, and this homogeneous linear transformation carries Δ into a circular domain $\Delta' = L\Delta$. Now

$$TL: \quad z_j = z_j' + (\text{higher powers}), \qquad j = 1, \cdots, k,$$

is an automorphism of Δ'. Since Δ' is circular it is representative and hence the only 1-1 normalized automorphism of Δ' is the identity, that is $TL = I$ or $T = L^{-1}$. Thus T is linear.

6. Mappings of unbounded domains. In this section I want to give briefly and without proof a few results on mappings of unbounded domains. First we shall recall Cartan's theorem, Theorem 1, which states that if we have an inner (analytic) mapping T of a domain D containing the origin onto a bounded domain D', and if the linear part of T is the identity, then T is actually the identity.

This theorem does not hold if D' is unbounded. In fact, Bieberbach [9], using a result of Fatou [15, 16], has shown that there exist two entire functions $z' = z + (\text{higher powers})$, $w' = w + (\text{higher powers})$, which map the entire space E_4 onto a proper subset of E_4 in a topological manner. Thus they define an inner mapping whose linear part is the identity and yet the mapping is not the identity.

In spite of this Behnke and Peschl [3] have been able to prove a very beautiful generalization of Cartan's theorem for certain unbounded domains.

THEOREM 9 (BEHNKE AND PESCHL). *Let D be a domain containing the origin and let D have the following property: There exist k bounded analytic functions*

$$g_j(z) = \sum_{n=n_j}^{\infty} P_n^j(z), \qquad j = 1, \cdots, k, \tag{6.1}$$

in D such that the determinant

$$J(z) = \partial(P^1_{n_1}, \cdots, P^k_{n_k})/\partial(z_1, \cdots, z_k) \tag{6.2}$$

is not identically zero. Then every inner analytic mapping of D whose linear part is the identity is actually the identity.

The development (6.1) is the development of $g_j(z)$ in a series of homogeneous polynomials; there is no restriction on the value of the lowest degree n_j.

Using the result of Behnke and Peschl, Bochner and the speaker [10] have carried over part of the results of §3 to a class of unbounded domains. Their result is: *If D is not bounded but possesses the alternative property indicated in Theorem* 9, *if T is an inner mapping of D with the origin as a fixed point, if* (3.4) *holds, and if the linear parts* $\{L_T, L_{T^2}, L_{T^3}, \cdots\}$ *are uniformly bounded in D, then T is an automorphism.*

7. **Concluding remarks.** The field of analytic mappings is wide and contains numerous results. It has been impossible to give or even to summarize all of these results; as a consequence many interesting and important results have been omitted. Many of these will be found in the résumé of mapping contained in chapter 7 of the tract by Behnke and Thullen [4]. Several are given in the book by Bergman [6]. I would like to refer the listener to these two books for excellent treatments of various aspects of the theory.

Among the more recent papers on mapping I would like to mention two by Bergman and Spencer [7, 8] which have appeared since the publication of the books just mentioned. These papers point in a somewhat different direction from those considered in this talk, and they contain references to other recent treatments.

BIBLIOGRAPHY

(The references listed below include only those directly referred to in the talk; much more complete bibliographies are contained in the books by Behnke and Thullen [**4**] and Bergman [**6**].)

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15. P. Fatou, *Sur les fonctiones méromorphes de deux variables*, C. R. Acad. Sci. Paris vol. 175 (1922).

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18. H. Welke, *Über die analytischen Abbildungen von Kreiskörpern und Hartogsschen Bereichen*, Math. Ann. vol. 103 (1930) pp. 437–449.

19. H. Weyl, *Theorie der Darstellung kontinuierlicher halb-einfacher Gruppen durch lineare Transformationen*, Math. Zeit. vol. 23 (1925) pp. 271–309.

SYRACUSE UNIVERSITY

THE OCTOBER MEETING IN NEW YORK

The three hundred ninety-ninth meeting of the American Mathematical Society was held at Columbia University on Saturday, October 30, 1943. The attendance included the following one hundred thirty-one members of the Society:

C. F. Adler, A. A. Albert, R. L. Anderson, T. W. Anderson, R. G. Archibald, L. A. Aroian, E. G. Begle, Stefan Bergman, Felix Bernstein, Lipman Bers, J. H. Bigelow, Gertrude Blanch, A. H. Bowker, C. B. Boyer, A. B. Brown, R. C. Buck, Hobart Bushey, J. H. Bushey, S. S. Cairns, I. S. Carroll, L. M. Comer, T. F. Cope, Richard Courant, A. P. Cowgill, A. T. Craig, M. D. Darkow, R. L. Dietzold, Jesse Douglas, Jacques Dutka, Samuel Eilenberg, E. E. Fedder, J. M. Feld, A. D. Fialkow, Tomlinson Fort, R. M. Foster, W. C. G. Fraser, K. O. Friedrichs, R. E. Fullerton, Godofredo García, H. P. Geiringer, Abe Gelbart, David Gilbarg, B. P. Gill, Lewis Greenwald, Laura Guggenbühl, E. J. Gumbel, Jacques Hadamard, P. R. Halmos, Olaf Helmer, M. R. Hestenes, L. S. Hill, Einar Hille, Abraham Hillman, T. R. Hollcroft, Harold Hotelling, J. L. Howell, E. M. Hull, L. C. Hutchinson, W. H. Ingram, S. A. Joffe, S. A. Kiss, H. S. Kieval, J. R. Kline, B. O. Koopman, M. E. Ladue, Cornelius Lanczos, Solomon Lefschetz, B. A. Lengyel, Norman Levinson, D. C. Lewis, E. R. Lorch, A. N. Lowan, E. J. McShane, L. A. MacColl, H. F. MacNeish, A. J. Maria, D. H. Maria, W. T. Martin, L. W. Miller, E. C. Molina, Deane Montgomery, E. J. Moulton, G. W. Mullins, F. J. Murray, C. A. Nelson, G. E. Noether, P. B. Norman, C. O. Oakley, F. W. Perkins, George Piranian, E. L. Post, G. B. Price, L. L. Rauch, H. W. Reddick, Eric Reissner, Moses Richardson, J. F. Ritt, E. K. Ritter, S. L. Robinson, P. C. Rosenbloom, J. E. Rosenthal, M. G. Scherberg, O. H. Schmidt, Seymour Sherman, Max Shiffman, J. A. Shohat, James Singer, D. M. Smiley, M. F. Smiley, P. A. Smith, Wolfgang Sternberg, J. J. Stoker, R. R. Stoll, E. G. Straus, W. C. Strodt, Fred Supnick, Feodor Theilheimer, W. J. Thron, Peter Treuenfels, H. F. Tuan, J. W. Tukey, D. F. Votaw, W. R. Wasow, Louis Weisner, A. P. Wheeler, W. F. Whitmore, S. S. Wilks, John Williamson, Jacob Wolfowitz, Alexander Wundheiler, Oscar Zariski.

There was a session for contributed papers in the morning at which Professor Deane Montgomery presided. Professor J. F. Ritt presided at the afternoon session.

A business meeting was held at the opening of the afternoon session at which the Society voted to amend Article III, Section 1, and Article IX, Sections 1 and 2 of the By-Laws to read:

ARTICLE III. EDITORIAL COMMITTEES

Section 1. There shall be six Editorial Committees—one of four members for the Bulletin, one of three members for the Transactions, one of three members for the Mathematical Reviews, one of three members for the Colloquium Publications, one of three members for the Mathematical Surveys, and one consisting of three representatives of the Society on the Board of Editors of the American Journal of Mathematics.

ARTICLE IX. PUBLICATIONS

Section 1. The Society shall publish an official organ called the Bulletin of the American Mathematical Society. It shall publish a journal called the Transactions of the American Mathematical Society, the object of which shall be to make known important researches presented at meetings of the Society. It shall publish a periodical called Mathematical Reviews containing abstracts or reviews of current mathematical literature. It shall publish a series of volumes called Colloquium Publications which shall embody in book form new mathematical developments. It shall publish a series of monographs called Mathematical Surveys which shall furnish expositions of the principal methods and results of particular fields of mathematical research. It shall also cooperate in the conduct of the American Journal of Mathematics.

Section 2. The editorial management of the Bulletin, Transactions, Mathematical Reviews, Colloquium Publications, and Mathematical Surveys, and the participation of the Society in the management of the American Journal of Mathematics shall be in charge of the respective Editorial Committees as provided in Article III, Section 1.

Following the business meeting Professor W. T. Martin of Syracuse University gave an address entitled *Mappings by means of systems of analytic functions of several complex variables.*

Titles and cross references to the abstracts of papers read follow below. Papers whose abstract numbers are followed by the letter *t* were read by title. The papers numbered 1–8 were read in the morning session and papers 9–22 were presented by title. Paper 2 was read by Mr. Buck, paper 3 by Dr. Bers; paper 4 was read for the authors by Professor Jesse Douglas. Mr. Haimo was introduced by Professor F. L. Griffin, and Mr. Salzer by Professor A. N. Lowan.

1. Max Shiffman: *Isoperimetric inequalities and continuity of area for classes of surfaces.* (Abstract 49-11-272.)

2. R. C. Buck and Harry Pollard: *Convergence and summability properties of subsequences.* (Abstract 49-11-256.)

3. Lipman Bers and Abe Gelbart: *On functions satisfying certain systems of partial differential equations.* (Abstract 50-1-20.)

4. Edward Kasner and John DeCicco: *A generalized theory of contact transformations.* (Abstract 49-9-235.)

5. J. A. Shohat: *Parseval formula in its application to Van der Pol's and generalized equations.* (Abstract 49-11-293.)

6. Norman Levinson: *On a nonlinear differential equation of the second order.* (Abstract 49-11-264.)

7. R. R. Stoll: *Representations of completely simple semigroups.* (Abstract 49-11-248.)

8. L. A. MacColl: *Geometrical characterizations of some families of dynamical trajectories.* (Abstract 49-11-299.)

9. Stefan Bergman: *The determination of singularities of functions*

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satisfying a partial differential equation from the coefficients of their series development. (Abstract 50-1-18-*t*.)

10. Lipman Bers and Abe Gelbart: *On a class of functions defined by certain partial differential equations.* (Abstract 50-1-19-*t*.)

11. R. C. Buck: *A note on subsequences.* (Abstract 49-11-254-*t*.)

12. R. C. Buck: *Multiple sequences.* (Abstract 49-11-255-*t*.)

13. R. C. Buck: *Partition of space.* (Abstract 49-11-296-*t*.)

14. Herbert Federer: *Surface area.* I. (Abstract 49-11-259-*t*.)

15. Herbert Federer: *Surface area.* II. (Abstract 49-11-260-*t*.)

16. Franklin Haimo: *Periodic functions on algebraic systems.* (Abstract 49-1-2-*t*.)

17. A. N. Lowan and H. E. Salzer: *Table of coefficients for inverse interpolation.* (Abstract 49-9-224-*t*.)

18. A. P. Morse: *A theory of covering and differentiation.* (Abstract 49-11-266-*t*.)

19. A. P. Morse and J. F. Randolph: *The ϕ rectifiable subsets of the plane.* (Abstract 49-11-267-*t*.)

20. F. J. Murray: *On the existence of quasi-complements in Banach spaces.* (Abstract 49-11-268-*t*.)

21. H. E. Salzer: *Coefficients for numerical integration with central differences.* (Abstract 49-9-237-*t*.)

22. Antoni Zygmund: *On certain integrals.* (Abstract 49-11-276-*t*.)

T. R. HOLLCROFT,
Associate Secretary

THE ANNUAL MEETING OF THE SOCIETY

The fiftieth Annual Meeting of the American Mathematical Society was held at the Museum of Science and Industry, Chicago, Illinois, Friday and Saturday, November 26 and 27, 1943, in conjunction with meetings of the Mathematical Association of America. Over two hundred persons registered including the following one hundred sixty-five members of the Society:

R. P. Agnew, A. A. Albert, C. B. Allendoerfer, B. H. Arnold, Max Astrachan, W. L. Ayres, R. W. Babcock, Reinhold Baer, R. M. Ballard, Harry Bateman, J. C. Bell, S. F. Bibb, B. H. Bissinger, L. M. Blumenthal, J. L. Brenner, R. W. Brink, J. O. Brown, R. H. Bruck, E. L. Buell, H. E. Burns, Herbert Busemann, Albert Cahn, W. D. Cairns, W. B. Carver, W. B. Caton, W. F. Cheney, E. W. Chittenden, R. V. Churchill, A. B. Coble, L. W. Cohen, N. B. Conkwright, A. H. Copeland, J. J. Corliss, D. R. Curtiss, M. M. Day, John DeCicco, J. J. Dennis, Jesse Douglas, Arnold Dresden, D. M. Dribin, Samuel Eilenberg, Wade Ellis, Paul Erdös, H. J. Ettlinger, H. P. Evans, H. S. Everett, G. M. Ewing, J. V. Finch, N. J. Fine, Edward Fleisher, L. R. Ford, R. H. Fox, J. S. Frame, Morris Friedman, T. C. Fry, J. H. Giese, J. W. Givens, Michael Golomb, Cornelius Gouwens, L. M. Graves, V. G. Grove, D. W. Hall, P. R. Halmos, R. W. Hamming, W. L. Hart, C. T. Hazard, M. H. Heins, E. D. Hellinger, Edward Helly, Fritz Herzog, E. H. C. Hildebrandt, T. H. Hildebrandt, D. L. Holl, A. S. Householder, H. K. Hughes, Ralph Hull, M. G. Humphreys, Mildred Hunt, H. D. Huskey, M. H. Ingraham, B. W. Jones, William Karush, Robert Kates, D. E. Kearney, I. F. Keeler, Fred Kiokemeister, J. R. Kline, L. A. Knowler, W. C. Krathwohl, E. P. Lane, R. E. Langer, E. H. Larguier, C. G. Latimer, D. H. Leavens, R. H. Lence, A. L. Lewis, M. I. Logsdon, A. T. Lonseth, Eugene Lukacs, Janet MacDonald, C. C. MacDuffee, Morris Marden, A. V. Martin, W. T. Martin, Karl Menger, E. J. Mickle, H. J. Miser, W. L. Mitchell, C. W. Moran, Marston Morse, E. J. Moulton, S. B. Myers, A. L. Nelson, C. V. Newsom, K. L. Nielsen, Ivan Niven, E. A. Nordhaus, E. P. Northrop, C. D. Olds, Isaac Opatowski, W. V. Parker, B. C. Patterson, Tibor Radó, G. Y. Rainich, R. B. Rasmusen, W. T. Reid, Haim Reingold, W. H. Roever, R. G. Sanger, A. T. Schafer, R. D. Schafer, O. F. G. Schilling, G. E. Schweigert, M. E. Shanks, H. A. Simmons, G. W. Smith, I. S. Sokolnikoff, C. F. Stephens, N. E. Steenrod, A. H. Stone, D. M. Stone, M. H. Stone, E. B. Stouffer, R. L. Swain, J. L. Synge, R. M. Thrall, W. J. Trjitzinsky, S. M. Ulam, H. S. Wall, S. E. Warschawski, Warren Weaver, K. W. Wegner, J. V. Wehausen, C. P. Wells, E. T. Welmers, M. E. Wescott, M. D. Wetzel, G. W. Whitehead, G. T. Whyburn, L. R. Wilcox, R. L. Wilder, K. P. Williams, R. S. Wolfe, J. W. T. Youngs, Daniel Zelinsky.

Sessions for the reading of contributed papers were held Friday at 9:30 A.M. and 2:00 P.M. and Saturday at 10:45 A.M. and 3:15 P.M. The presiding officers for these sessions were Professors L. R. Ford, R. E. Langer, R. L. Wilder, and Dean M. H. Ingraham.

Friday morning following the session for contributed papers, Professor Reinhold Baer of the University of Illinois delivered an address entitled *The higher commutator subgroups*. Professor A. A. Albert presided for this address.

Friday evening Professor Harry Bateman delivered the seventeenth Josiah Willard Gibbs Lecture on *The control of elastic fluids*. President M. H. Stone presided.

The annual business meeting and election of officers was held on Saturday morning, President Stone presiding. At the close of the business session President Stone announced the sixth award of the Bôcher Memorial Prize to Professor Jesse Douglas of Brooklyn College for his memoirs:

(a) *Green's function and the problem of Plateau*, American Journal of Mathematics, volume 61 (1939) pp. 545–589.

The most general form of the problem of Plateau, American Journal of Mathematics, volume 61 (1939) pp. 590–609.

(b) *Solution of the inverse problem of the calculus of variations*, Transactions of the American Mathematical Society, volume 50 (1941) pp. 71–128.

Professor Douglas then gave a résumé of his prize-winning papers.

Saturday afternoon with President Stone presiding, Professor Marston Morse of the Institute for Advanced Study delivered the Retiring Presidential Address entitled *New settings for topology in analysis*.

Sessions of the Mathematical Association of America were held on Saturday evening and Sunday morning. At the business meeting of the Association on Sunday morning Professors C. V. Newsom and G. M. Ewing presented a resolution of appreciation to the officials of the Museum of Science and Industry and the local committee on arrangements on behalf of the two mathematical organizations.

At the meeting of the Board of Trustees at 6:00 P.M., November 26, 1943, in the Hotel Windermere East, there was no quorum present and the Board adjourned to January 8, 1944. The Council met at 9:30 P.M. on November 26, 1943, in the Hotel Windermere East.

The Secretary announced the election of the following twelve persons to membership in the Society:

Mr. R. H. Bing, University of Texas;
Mr. Patrik Butler, University of Texas;
Dr. Shiing-shen Chern, Institute for Advanced Study;
Professor Russell Dunholter, College of Engineering and Commerce, University of Cincinnati;
Professor William Henry Fagerstrom, College of the City of New York;
Professor Alois Joseph Lorenz, St. Michael's College, Winooski Park, Vt.;
Dr. Earl John Mickle, Ohio State University;

Mr. Calvin Northrup Mooers, Naval Ordinance Laboratory, Washington, D. C.;
Dr. Chaim Leib Pekeris, Massachusetts Institute of Technology;
Professor Louis Wright Roberts, Howard University;
Dr. John Daniel Rommel, Jr., U.S.N.R., San Diego, Calif.;
Dr. Gail Sellers Young, Jr., Purdue University.

It was reported that the following had been elected as nominees on the institutional memberships of the institutions indicated:

Columbia University: Messrs. Robert Nathan Ascher, Edmund Churchill, and Huseyin Demir, Miss Aida Kalish, Messrs. Howard Levene, Robert King Ritt, and Cengiz Uluçay.
Michigan State College: Mr. Lawrence Duncan Childs.
University of Minnesota: Mr. Henry Damon Colson.
Ohio State University: Dr. Leslie Haynes Miller, Mr. Herbert Charles Parrish.
University of Pennsylvania: Mr. Richard Crowley Campbell.

The following appointments by Acting President C. C. MacDuffee were reported: as an auditor of the Society's accounts for 1943, Professor T. F. Cope; as tellers for the election at the 1943 Annual Meeting, Professors W. C. Krathwohl and E. R. Lorch, Dr. W. C. Strodt; as an editorial committee for Mathematical Surveys, to serve until December 31, 1944, Professors F. D. Murnaghan (Chairman), A. A. Albert, and T. H. Hildebrandt; as a Committee on the Award of the Cole Prize in Algebra, to be awarded at the 1944 Annual Meeting for papers published during the period 1939–1943, Professors E. T. Bell (Chairman), A. A. Albert, and N. H. McCoy; as a member of the Committee on Places of Meetings, for a period of three years beginning 1944, Professor C. R. Adams (committee now consists of Professors W. B. Carver, Chairman, C. R. Adams, and E. W. Chittenden); as new members of the Committees to Select Hour Speakers: for Annual and Summer Meetings, Professor J. D. Tamarkin (committee now consists of Professors J. R. Kline, Chairman, J. D. Tamarkin, and R. L. Wilder); for Eastern Sectional Meetings, Professor E. J. McShane (committee now consists of Professors T. R. Hollcroft, Chairman, E. J. McShane, and Saunders MacLane); for Western Sectional Meetings, Professor A. A. Albert (committee now consists of Professors W. L. Ayres, Chairman, A. A. Albert, and R. E. Langer); for Far Western Sectional Meetings, Professor Harry Bateman (committee now consists of Professors A. D. Michal, Chairman, Harry Bateman, and W. M. Whyburn).

Times and places of meetings during 1944 were set as follows: April 29 in Berkeley, California; April 28–29 in Chicago, Illinois; April 28–29 in New York City.

Certain invitations to give hour addresses were announced: Professor E. L. Post for the February, 1944, meeting in New York City; Professors André Weil and E. R. Lorch for the April, 1944, meeting in New York City; Professor C. C. MacDuffee for the 1944 Summer Meeting.

The Secretary reported the resignations of Professors Saunders MacLane and G. B. Price from the Committee on Publicity (committee now consists of Professors R. M. Foster, Chairman, J. B. Rosser, and Dr. J. M. Thompson).

The Secretary reported that Professor Tibor Radó had accepted the invitation of the Council to deliver a set of Colloquium Lectures in 1945 and that the title of the Lectures will be *Length and area.*

The Council voted to invite Professor John von Neumann to deliver the eighteenth Josiah Willard Gibbs Lecture at the 1944 Annual Meeting.

The Secretary reported that the ordinary membership in the Society is now 2594, including 208 nominees of institutional members and 67 life members. There are also 87 institutional members. The total attendance at all meetings in 1943 was 1,028; the number of papers read was 278; there were 12 hour addresses, 6 symposium addresses, 3 Colloquium Lectures, 1 Gibbs Lecture, and a Retiring Presidential Address; the number of members attending at least one meeting was 619. The Secretary also reported the completion of the biennial membership campaign held during November when approximately 3,600 college teachers of mathematics, including 560 members of the Mathematical Association of America, were invited to membership in the Society.

At the annual election which closed on November 27, and at which 486 votes were cast, the following officers were elected:

Vice Presidents, Professors Einar Hille and G. T. Whyburn.

Associate Secretaries, Professors W. L. Ayres, A. D. Michal, and G. B. Price.

Member of the Editorial Committee of the Bulletin, Professor R. E. Langer.

Member of the Editorial Committee of the Transactions, Professor E. J. McShane.

Member of the Editorial Committee of the Colloquium Publications, Professor G. T. Whyburn.

Member of the Editorial Committee of Mathematical Reviews, Professor J. D. Tamarkin.

Member of the Editorial Committee of the American Journal of Mathematics, Professor Hassler Whitney.

Members-at-large of the Council, Professors Harry Bateman, Garrett

Birkhoff, M. R. Hestenes, Harold Hotelling, Nathan Jacobson, R. D. James.

Professor J. W. T. Youngs of Purdue University was appointed Acting Associate Secretary to serve during the time that Associate Secretary G. B. Price is engaged in work for the government.

The Council adopted the following resolution concerning the work of Associate Secretary M. H. Ingraham in connection with the financial affairs of the Society:

> With the retirement of Dean M. H. Ingraham from an Associate Secretaryship in the American Mathematical Society, the Council wishes to place on record an expression of its profound appreciation of his services rendered to the Society and to the cause of mathematics.
>
> In the early nineteen thirties with at best a static income and with a rapidly expanding program of research publications, the responsible officers of the Society faced a problem of unusual difficulty. Various temporary expedients had been tried but the fact remained that either new funds of considerable magnitude must be tapped or the size of the journals drastically cut. Secretary Ingraham was at that time in large part responsible for putting into successful practice the scheme of individual contributing memberships by which members made voluntary increases in dues. But it soon became apparent that these extra funds were not sufficient to meet the situation. With the argument that the mathematicians had themselves made a real contribution to the balancing of the budget, it seemed possible to seek the financial backing of the universities. Professor Ingraham, one of the prime movers in this fruitful idea, was selected to interview university authorities. The Rockefeller Foundation, which over a period of years had been generous to the Society, was willing to pay the expenses of an extended trip among the mathematical centers.
>
> Secretary Ingraham was remarkably successful not only in enlisting the support of the universities but also in leaving behind a friendlier and more understanding attitude toward research in mathematics. In each of the intervening years he kept in touch with the institutions which subscribed and, as a result, a sum of seventy-five hundred dollars has flowed annually into the treasury, thus enabling the publication program to be kept at a high level. And in a variety of other ways the Society has benefited from his leadership.
>
> Perhaps no member of the Society is more widely known among his colleagues or has a larger circle of friends. The Council desires to extend hearty thanks on behalf of the Society and to wish him every success in his new duties at the University of Wisconsin.

Since the 1943 Annual Meeting was held before the close of the Society's fiscal year, it was impossible to present a full report of the Treasurer. (Excerpts from the final report of the Treasurer however appear on pages 29–32 of this Bulletin.) For the information of the Council a general statement on the current finances of the Society was presented. It was reported: (1) that total receipts for the fiscal year would exceed estimates and that the net excess of receipts over disbursements would probably be about $5,000; (2) that income from individual and from institutional memberships would be at least as great as in 1942; (3) that the income from the sale of the Transactions would exceed estimates by nearly $1,000, principally because of an unusual demand for back volumes; (4) that during 1943 the Society had received a full year's income, amounting to almost $4,000, from the estate of Robert Henderson, a grant of $2,500 from the Rockefeller Foundation for the use of the War Policy Committee, and the final payment on the principal of the Reilly Estate (the total receipts from the Reilly Estate amount to $23,651.47 and have been added to the pooled investments of the Society).

The Librarian reported the following additions to the library: 41 books, 81 bound volumes of periodicals, 7 pamphlets, 306 dissertations in printed form, 149 dissertations on microfilm and 1 volume on microfilm. He reported that during the year considerable progress had been made on the project of securing copies of all American doctoral dissertations for the library. From the gift to the library by Professor R. C. Archibald, former Librarian of the Society, a microfilm reader has been purchased which is installed in the reading room and is available for use by members of the Society.

The Bulletin Editorial Committee reported that it would publish approximately 1,000 pages in the 1943 volume and that, because the present supply of unpublished manuscripts is very small, prompt publication is now possible. The editors also pointed out that, while shorter papers are desired, there is no longer an adherence to any strict rule of length.

The Transactions Editorial Committee reported that it expected to publish about 1,065 pages in 1943 and that, in spite of the fact that there is no present backlog of papers, at least Volume 55 would be of normal size. It was reported that Professor A. A. Albert would be Managing Editor during 1944–1945 and that Professors G. A. Hedlund, Nathan Jacobson, W. T. Reid, N. E. Steenrod, and Antoni Zygmund had been appointed Associate Editors, to replace Professors T. H. Hildebrandt, R. E. Langer, H. P. Robertson, and Gabor Szegö. The retirement of Professor Einar Hille from the Editorial Committee of the Transactions marks the end of an unusual record of twenty-one years of continuous connection with the editorial work of this journal. The Society has been fortunate to have had his wise counsel and devoted service for this long period.

The Mathematical Reviews Editorial Committee reported that there had been a marked decrease in the amount of mathematical material published throughout the world, that the editors had been able to obtain material from Russia regularly, and that an almost complete supply of material published in Germany up to the beginning of 1943 had been secured.

The Council voted to place Mathematical Surveys on the same basis as the Colloquium Series in the regulations concerning special privileges for enlisted men. The regulations as amended at the Summer and Annual Meetings were distributed to all members of the Society on January 1.

President Stone reported to the Council that the War Policy Committee is actively studying problems concerning the proper use of mathematicians in the war. *Department chairman are urgently requested to inform the Secretary concerning cases in which the new deferment procedures lead to the improper classification of a mathematician.* (The new procedures were described in a memorandum published in the November issue of this Bulletin.) It was further reported that an active subcommittee of the War Policy Committee (Professor W. L. Hart, Chairman) is studying the mathematical aspects of the various war training programs. A report was presented by the Committee on Available Teachers of Collegiate Mathematics. The Committee reported that the present supply of available teachers would not be adequate to meet the anticipated demands and urged all members of the Society to inform the Committee of persons who are available for appointment as teachers in the war training programs.

Titles and cross references to papers read at the meeting follow below. Papers 1 to 9 were read Friday morning, papers 10 to 18 Friday afternoon, papers 19 to 26 Saturday morning, papers 27 to 36 Saturday afternoon, and papers 37 to 56, whose abstract numbers are followed by the letter *t*, were read by title. Paper 6 was read by Professor Ulam, 11 by Dr. Caton, 16 by Dr. Wetzel, 25 by Professor Eilenberg, 26 by Dr. Erdős, and 27 by Mr. Fine. Mr. Kenneth Levenberg was introduced by Professor H. B. Curry, Dr. Angeline J. Brandt by Professor R. M. Thrall, and Mr. H. E. Salzer by Professor A. N. Lowan.

1. John DeCicco: *Dynamical and curvature trajectories in space.* (Abstract 49-11-297.)

2. Janet MacDonald: *Conjugate nets in asymptotic parameters.* (Abstract 50-1-46.)

3. Alice T. Schafer: *Two singularities of space curves.* (Abstract 49-11-300.)

4. Isaac Opatowski: *Isoperimetric problems in bending of cantilevers.* (Abstract 50-1-40.)

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5. Kenneth Levenberg: *A method for the solution of certain nonlinear problems in least squares.* (Abstract 49-11-286.)

6. L. W. Cohen and S. M. Ulam: *On the algebra of systems of vectors and some problems in kinematics.* (Abstract 50-1-38.)

7. G. Y. Rainich: *Noncommutative relations.* (Abstract 50-1-15.)

8. P. R. Halmos: *In general a measure preserving transformation is mixing.* (Abstract 50-1-44.)

9. J. L. Brenner: *The linear homogeneous group.* II. (Abstract 50-1-3.)

10. K. L. Nielsen: *On operators for linear partial differential equations.* (Abstract 50-1-30.)

11. W. B. Caton and Einar Hille: *On the class of functions* $H_p(1/2)$. (Abstract 50-1-23.)

12. S. E. Warschawski: *On Theodorsen's method of conformal mapping of nearly circular regions.* (Abstract 50-1-37.)

13. A. T. Lonseth: *The propagation of error in linear problems.* (Abstract 50-1-28.)

14. W. T. Reid: *Expansion theorems for boundary problems of the calculus of variations.* (Abstract 50-1-31.)

15. Anne L. Lewis: *Sufficiency proofs for the problem of Bolza in the calculus of variations.* (Abstract 50-1-27.)

16. H. S. Wall and Marion Wetzel: *Positive definite forms and convergence theorems for continued fractions.* (Abstract 49-9-216.)

17. W. J. Trjitzinsky: *Problems of representation and uniqueness for functions of a complex variable.* (Abstract 50-1-35.)

18. A. H. Copeland: *The nature of turbulence.* (Abstract 50-1-39.)

19. Dorothy M. Stone: *An algebraic characterization of measure algebras.* Preliminary report. (Abstract 49-11-249.)

20. J. V. Wehausen: *A remark concerning a set of completely continuous transformations.* (Abstract 49-11-274.)

21. J. W. T. Youngs: *On surfaces of class* K_1. Preliminary report. (Abstract 49-11-309.)

22. G. E. Schweigert: *Periodic analysis for onto-homeomorphisms of semi-locally-connected continua.* Preliminary report. (Abstract 50-1-51.)

23. M. E. Shanks: *Monotone decompositions of continua.* (Abstract 50-1-50.)

24. R. H. Fox: *The complete homotopy group.* (Abstract 49-11-306.)

25. Samuel Eilenberg and Saunders MacLane: *Cohomology theory in groups.* (Abstract 50-1-9.)

26. Paul Erdös and S. M. Ulam: *Some combinatorial problems in set theory.* Preliminary report. (Abstract 50-1-25.)

27. N. J. Fine and Ivan Niven: *The probability that a determinant be congruent to a* (mod *m*). (Abstract 50-1-11.)

28. M. M. Day: *Arithmetic of ordered systems.* (Abstract 50-1-6.)

29. R. M. Thrall: *On the decomposition of modular tensors.* II. (Abstract 49-11-250.)

30. Ralph Hull: *A theorem on the unit groups of simple algebras.* (Abstract 50-1-14.)

31. A. A. Albert: *Quasiquaternion algebras.* (Abstract 50-1-2.)

32. L. R. Wilcox: *Modularity in Birkhoff lattices.* (Abstract 50-1-17.)

33. R. H. Bruck: *Some results in the theory of quasigroups.* (Abstract 49-11-244.)

34. Angeline J. Brandt: *The free Lie ring and Lie representations of the full linear group.* (Abstract 49-11-243.)

35. Wade Ellis: *Relations satisfied by linear operators on a vector space.* (Abstract 50-1-10.)

36. V. G. Grove: *The transformation of Čech.* (Abstract 50-1-45.)

37. Bernard Friedman: *A method of approximating the complex roots of polynomial equations.* (Abstract 49-11-280-*t*.)

38. D. G. Bourgin and C. W. Mendel: *Orthonormal sequences.* (Abstract 50-1-21-*t*.)

39. S. M. Ulam: *Theory of the operation of product of sets.* III. Preliminary report. (Abstract 50-1-36-*t*.)

40. Edward Kasner and John DeCicco: *Union-preserving transformations of space.* (Abstract 49-11-298-*t*.)

41. M. H. Heins: *On the problem of Milloux for functions analytic throughout the interior of the unit circle.* (Abstract 49-11-262-*t*.)

42. R. E. Johnson: *On the equation* $\chi\alpha=\gamma\chi+\beta$ *over an algebraic division ring.* (Abstract 49-11-246-*t*.)

43. M. E. Shanks: *The space of metrics on a compact metrizable space.* II. (Abstract 49-11-308-*t*.)

44. H. B. Curry: *The method of steepest descent for nonlinear minimization problems.* (Abstract 49-11-278-*t*.)

45. Dorothy M. Stone: *The representation of abstract measure functions.* (Abstract 49-11-273-*t*.)

46. R. H. Bruck: *Some results in the theory of linear non-associative algebras.* (Abstract 50-1-4-*t*.)

47. Morris Marden: *A recurrence formula for the solutions of certain partial differential equations.* (Abstract 50-1-29-*t*.)

48. Claude Chevalley: *Some properties of ideals in rings of power series.* (Abstract 50-1-5-*t*.)

49. H. E. Salzer: *New tables and facts involving sums of four tetrahedral numbers.* (Abstract 50-1-16-*t*.)

50. A. A. Albert: *Quasigroups.* II. (Abstract 50-1-1-*t*.)

51. L. R. Wilcox: *Theory of traffic light networks.* I. (Abstract 50-1-42-*t*.)

52. A. R. Schweitzer: *On functional equations with solutions containing arbitrary functions.* III. (Abstract 50-1-32-*t*.)

53. G. T. Whyburn: *Mapping classes for locally connected continua.* (Abstract 50-1-52-*t*.)

54. B. H. Colvin: *The expansion problem associated with a third order ordinary differential system of highly irregular type.* (Abstract 50-1-24-*t*.)

55. Alfred Brauer: *On certain limits.* (Abstract 50-1-22-*t*.)

56. R. L. Moore: *Concerning webs in the plane.* (Abstract 50-1-49-*t*.)

W. L. AYRES,
Associate Secretary

APPENDIX

Excerpts from Report of Treasurer

December 10, 1943

To the Board of Trustees of the
American Mathematical Society

Gentlemen:

I have the honor to submit herewith the report of the Treasurer for the fiscal year ended November 30, 1943. The following remarks may be of interest.

Investment Portfolio

On November 30, 1943 the market value of the securities held for Invested Funds was $6,208 less than book value. This is to be compared with the total amount, $17,416, held in accounts "Reserve for Investment Losses" and "Profit on Sales of Securities." This total amount of reserves has now reached nearly ten per cent of the total value of Invested Funds. In view of the conservative character of many of the securities, these reserves may be considered sufficient protection against contingent depreciation in market value. The considerable increase, $8,843, in this profit account during the year reflects two policies embodied in the recommendations of the Custodian and followed by the Board of Trustees—sales of railroad bonds which had been held on the books at depreciated values, and sales of certain common stocks when rising market values increased the percentage of common stocks in the portfolio beyond the point deemed desirable.

The market value of securities held for Current Funds was $654 less than the book value. Profit on sales of securities for Current Funds amounting to $612 is being carried as a reserve against losses to these funds.

The investment portfolio now includes government bonds 29%, other bonds 24%, preferred stocks 11%, common stocks 28%, cash in savings banks 8%.

Income from Investments

Income received during the year from investment of Current Funds amounted to $1,254, exclusive of income earmarked for International Congress, $96, and Reilly Fund, $22. This represents a return of slightly under 2¾%, computed on average book value of investments.

Income on Invested Funds amounted to $6,352, representing a return of 3¾% computed similarly. Total investment income from all sources was thus $7,724, representing a return of nearly 3½%. In comparing this income and yield with $8,794 and approximately 4% for the fiscal year 1942, there is reflected to some extent the changes in the character of the securities resulting from the sales mentioned above and the purchase of substantial amounts of government bonds. Also the figures for 1942 were somewhat too high to be truly representative, because of the receipt of a considerable sum of accrued back interest on certain railroad bonds then held. In view of the decreased rate of return, it is particularly fortunate that the Henderson Estate provided additional income of $4,438.

Increase in Surplus

The increase in Surplus of $9,553, as compared with $5,153 in the preceding fiscal year requires some explanation. If it be noted that 1942 income from the Henderson Estate amounting to $2,275 had not been added to Surplus on November 30, 1942, but was so added during the current fiscal year, more proper figures for comparison are seen to be $7,278 for 1943 and $7,428 for 1942. That the net result of the year's operations proved about as favorable as that for the preceding year is particularly satisfactory in view of the founding of Mathematical Surveys, made possible partly by the use of income from special funds (in the case of the Reilly Fund to such an extent that there is a small deficit in income account!), of somewhat increased expenses for other purposes, of the support of the War Policy Committee, and of the decrease in income from the Society's investments. However, a grant of $2,500 from the Rockefeller Foundation, half of which has been received, prevented the support of the War Policy Committee from seriously depleting Surplus. And an increase in membership that resulted in an increase in dues, together with a slight increase in subscriptions to the Bulletin and to the Transactions, were other favorable factors that enabled the Society to come successfully through a year which at its beginning threatened to be rather difficult financially.

Respectfully submitted,

BENNINGTON P. GILL,
Treasurer

BALANCE SHEET

Assets

	November 30, 1943	November 30, 1942
CURRENT FUNDS:		
Cash	$ 19,846.11	$ 19,177.78
Investments	36,731.06	33,470.47
	$ 56,577.17	$ 52,648.25
INVESTED FUNDS:		
Cash	$ 1,273.67	$ 1,409.62
Investments	176,721.29	163,195.68
	$177,994.96	$164,605.30
TOTAL ASSETS	$234,572.13	$217,253.55

Liabilities

	November 30, 1943	November 30, 1942
CURRENT FUNDS:		
Mathematical Reviews	$ 14,252.62	$ 16,184.69
Colloquium	5,761.69	2,805.04
Mathematical Surveys	840.03	
Reprinting Funds	2,239.10	1,329.15
Prize Funds and Other Special Funds Accumulated Income	2,296.18	4,848.52
Reilly Fund Principal		3,968.13
International Congress	6,146.06	6,050.33
Miscellaneous	80.99	173.99
Income from Henderson Estate		2,275.00
Sinking Fund	592.34	515.99
Profit on Sales of Securities	612.86	295.73
Surplus	23,755.30	14,201.68
	$ 56,577.17	$ 52,648.25
INVESTED FUNDS:		
Endowment Fund Principal	$ 60,110.68	$ 60,110.68
Prize Funds and Other Special Funds	31,033.22	26,935.07
Life Membership and Subscription Reserve	4,434.81	4,835.09
Colloquium	5,000.00	5,000.00
Mathematical Reviews	60,000.00	60,000.00
Reserve for Investment Losses	4,385.89	3,537.46
Profit on Sales of Securities	13,030.36	4,187.00
	$177,994.96	$164,605.30
TOTAL LIABILITIES	$234,572.13	$217,253.55

SUMMARY STATEMENT OF INCOME AND EXPENDITURES

1942–1943

	1943		1942	
	Receipts	*Disbursements*	*Receipts*	*Disbursements*
GENERAL RECEIPTS:				
Dues—Ordinary Memberships	$ 16,514.71		$ 15,681.19	
Dues—Contributing Memberships	859.63		1,054.00	
Dues—Institutional Memberships	6,616.52		7,347.35	
Initiation Fees	378.84		665.00	
Investment Income	8,612.83		4,643.14	
Miscellaneous	39.96		34.73	
Grant from Rockefeller Foundation for War Policy Committee	1,250.00			
GENERAL DISBURSEMENTS:				
Secretaries		$ 5,590.02		$ 5,136.86
Treasurer		1,093.58		794.36
Officers' Traveling		581.23		643.61
Library		893.15		569.95
Committee Expenses		1,584.17		67.92
Emergency		4.90		5.71
Total	$ 34,272.49	$ 9,747.05	$ 29,425.41	$ 7,218.41
Excess of General Receipts	$ 24,525.44		$ 22,207.00	
PUBLICATION:				
Bulletin	$ 1,710.80	$11,536.88	$ 1,673.13	$12,101.37
Bulletin Reprinting	626.71		467.30	
Transactions	5,886.83	8,309.42	5,787.17	9,373.59
Transactions Reprinting	1,172.40	889.16	274.58	201.59
Colloquium	3,951.38	994.73	3,814.97	6,197.07
Mathematical Reviews	10,235.88	12,167.95	10,809.17	11,548.93
Mathematical Surveys	3,600.89	2,760.86		
Semicentennial Publications	76.35		95.90	
American Journal		2,500.00		2,500.00
Total	$ 27,261.24	$39,159.00	$ 22,922.22	$41,922.55
Excess of Cost of Publication		$11,897.76		$19,000.33
MISCELLANEOUS:				
Special Funds (including Congress)	$ 1,273.41	$ 3,600.00	$ 1,442.39	$ 2,100.00
Profit on Sales of Securities	9,160.49		2,713.28	
Income from Henderson Estate			2,275.00	
Gift from R. C. Archibald		80.97	100.00	
Credits not Allocated		12.03	73.99	103.86
From Surplus:				
Transactions Reprinting		750.00		100.00
Mathematical Reviews		300.00		
Mathematical Surveys		1,000.00		
Total	$ 10,433.90	$ 5,743.00	$ 6,604.66	$ 2,303.86
Difference	$ 4,690.90		$ 4,300.80	
Net Change in Assets	$ 17,318.58		$ 7,507.47	
ASSETS BEGINNING OF YEAR	$217,253.55		$209,746.08	
ASSETS END OF YEAR	$234,572.13		$217,253.55	

EDWARD BURR VAN VLECK—IN MEMORIAM

The death of Edward Burr Van Vleck on June 2, 1943, at almost eighty years of age, in Madison, Wisconsin, will bring regret to a wide circle of friends and to American mathematicians generally. His creative mathematical work, his important role in the development of American mathematics since 1890, and his qualities of personality were such as to make all feel that something of inestimable and characteristic value has departed and yet remains in treasured remembrance.

Van Vleck was born in Middletown, Connecticut, on June 7, 1863, of Knickerbocker stock, for the first Van Vleeck came from Amsterdam, Holland, to New Amsterdam (New York) in 1658. His father was a distinguished American mathematical astronomer, Professor John Monroe Van Vleck, in whose memory has been erected the Van Vleck Observatory at Wesleyan University, given by a brother. The scientific tradition, thus doubly established in the Van Vleck family, has been notably continued to a third generation by his son, Professor John Hasbrouck Van Vleck, now Professor of Mathematical Physics in Harvard University.

The general account of Edward Burr Van Vleck's life is simple. As a boy, college student, and instructor, he developed in the typical New England environment of Middletown, living at home with his parents and three sisters (one of whom survives him) until about twenty-five years of age, obtaining an A.B. degree from Wesleyan University at twenty-one years of age and an A.M. degree three years later. Then he spent two years as a graduate student at Johns Hopkins University, where he veered from the field of physics to that of mathematics. After this came three further years at the University of Göttingen, Germany, where he was inspired by the great mathematician Felix Klein. He obtained his Ph.D. degree in 1893 at Göttingen at about thirty years of age. Thus he entered upon his mathematical research work somewhat later than usual, a fact which he always felt to be regrettable.

His first college post was that of instructor for two years at the University of Wisconsin, after which he taught for ten years at his alma mater, Wesleyan University, becoming associate and full professor there. In 1906 he went to the University of Wisconsin as Chairman of the Department of Mathematics, and, except for a half-year (1919–20) when he was visiting lecturer in the Department of Mathematics at Harvard University, and for various periods of travel in this country and abroad, remained in Madison thereafter. He became professor emeritus in 1929.

Behind this academic outline there was always a full and gracious life. Van Vleck married Miss Hester Laurence Raymond of North Lyme, Connecticut, shortly after he returned from his studies in Germany. In both Middletown and Madison, the Van Vlecks added much to the college and community life. Their lovely home in Madison, overlooking Lake Mendota, will be remembered by many as a center from which radiated kindness, friendship, and delightful hospitality to colleagues and their families, to students, and indeed to all those about them.

One may conjecture that Van Vleck was a very thoughtful and rather shy boy. Throughout his life he disliked all disputes, so that, even to his own disadvantage, he would avoid controversies with others. He liked urbane living and achieved it to a notable extent. However, when questions of principle arose, he stood his ground firmly and said precisely what he thought, although in polite form.

To those who knew him during his maturer years, he appeared as a clearsighted and liberal-minded idealist never deviating from the highest standards for himself, and yet fully appreciating the qualities of others who had a different outlook. His life was built around the motives of friendship and genuine scientific interest to a greater degree than is usually the case in the American academic scene.

Van Vleck's principal outside interests were reading and travel, both of which he shared with Mrs. Van Vleck. During the last half of his life he collected Japanese prints, and grew to be a connoisseur in this field.

On the mathematical side, Van Vleck owed much, for stimulus and inspiration, to Felix Klein. His thesis treated the development in continued fractions of integrals, forming solutions of certain linear differential equations of the second order, such as Lamé's integral. These integrals fall into natural families, any three "contiguous" members being connected by homogeneous linear difference equations. These integrals had appeared earlier in classical developments in continued fractions due to Gauss and others. This was a field which lay at the very heart of that part of analysis in which Klein was most interested. In his thesis Van Vleck extended considerably the types of such developments in a manner somewhat analogous to that devised by Pade in the similar development of arbitrary analytic functions. His thesis showed that not only was the young Van Vleck a master of the field involved but that he possessed definite creative power.

The question of the convergence of these continued fractions was left untouched in the thesis. For continued fractions of special form, this was taken up by him in three papers published in the second and fourth volumes of the Transactions of the American Mathematical Society (1901, 1904). A well known and general theorem there proved is that if in the continued fraction

$$\frac{1}{\alpha_1 + i\beta_1 +} \frac{1}{\alpha_2 + i\beta_2 +} \cdots \qquad i = (-1)^{1/2}$$

the real quantities α_i have the same sign while the real quantities β_i are alternately positive and negative, then the continued fraction will converge if the series $\sum(\alpha_u + i\beta_u)$ converges.

In his Colloquium Lectures of 1903, *Divergent series and continued fractions*, he first reviews the recent work of Poincaré, Stieltjes and others on divergent series (Part I) and then passes on to consider continued fractions (Part II). The two subjects are intertwined because "Both historically and prospectively one of the most suggestive and important methods of investigating power series is by the instrumentality of algebraic continued fractions." Van Vleck shows clearly how, from the broad point of view, the theory of continued fractions and their generalizations is closely related to the theory of linear homogeneous difference equations whose coefficients are polynomials in a complex variable. He wrote only one paper directly concerned with linear difference equations (Trans. Amer. Math. Soc., 1912) in which he extended a classic theorem of Poincaré. Doubtless the reason for this limited publication was the rapidity with which appeared new developments in the field by Nörlund and Galbrun abroad and by Carmichael, myself, and others in this country. However, while instructor at Wisconsin I had attended a graduate course given by Van Vleck in which he discussed various open problems concerning linear difference equations in a suggestive and stimulating way; while Carmichael shortly thereafter was a graduate student with me at Princeton. One must therefore look upon Van Vleck as an essential factor in the American contributions to linear homogeneous difference equations.

I have no doubt that when Van Vleck saw this promising field in which he had begun to work taken up so quickly by others, there was no tinge of regret on his part, for he would know that there were many other beautiful fields to be explored, and would feel that it was relatively unimportant from the larger point of view what the formal assignment of credit might be, so long as the subject itself was developed. It was enough for him that all should work together earnestly and sincerely for the increase of mathematical knowledge.

For a while Van Vleck seriously contemplated writing a book on functional equations, but the apparent lack of mathematical unity in the field prevented him from doing so. He wrote interesting papers dealing with special functional equations, in particular those satisfied by the elliptic ϑ-functions (Trans. Amer. Math. Soc., 1916); this last was the joint work of himself and one of his most gifted students, F. T. H'Doubler.

The range covered by Van Vleck in his more than thirty published articles was wide. He had a very keen analytic mind well adapted to the study of the niceties of point set theory, as manifested for instance in his article on non-measurable sets (Trans. Amer. Math. Soc., 1916). He also occupied himself with questions concerning the roots of algebraic equations and of the hypergeometric series, with special functions defined by differential equations, with the combination of substitutions, linear substitutions, and so on. In my opinion he resembled Bôcher in the respect that as the years went on, the self-imposed demand for elegance and simplicity exercised an inhibitive influence upon his production.

His general articles, *The influence of Fourier's series upon the development of mathematics* (Science, 1912), *The rôle of the point-set theory in geometry and dynamics* (Presidential Address, Bull. Amer. Math. Soc., 1915), *Current tendencies of mathematical research* (ibid., 1916), and *On the location of roots of polynomials and entire functions* (ibid., 1928) were thought-provoking essays and showed not only a wide knowledge and balanced perspective but an unusual ability to find the important open problems in the fields concerned. For example, in the second of these articles he states that Poincaré's definition of (dynamical) probability is equivalent to one in terms of the Lebesgue integral, although he does not give his proof. Carathéodory's well known proof of this equivalence appeared only in 1919.

In the teaching of undergraduates and of graduate students Van Vleck was clear and stimulating. At Wisconsin he gave himself unsparingly to the direction of the thesis work of undergraduates specializing in mathematics. As Chairman of the Department of Mathematics at Wisconsin for many years, he was deeply considerate of the members of the staff, completely democratic in spirit, and absolutely impartial. The friendship between him and his colleague Professor Ernest Skinner was one of great depth on both sides.

The mathematical reputation of Van Vleck was an international one. His relations with French mathematics were particularly close and sympathetic. Moreover his affection and admiration for France were profound. He was decorated "Officier de l'instruction publique" in 1920 in recognition of his services as teacher and investigator and for his work during the First World War. The University of Groningen on the three hundredth anniversary of its founding made him an Honorary Doctor in Mathematics and Physics in 1914. He, together with E. H. Moore and W. F. Osgood, received an honorary LL.D. at Clark University in 1909. At the twenty-fifth anniversary celebration of the University of Chicago in 1916, he received the honorary degree of Doctor of Science. His election to membership to the National Academy of Sciences came in 1911.

Van Vleck was President of the American Mathematical Society for the period 1913–1915. In earlier days he had been an effective and conscientious Editor of the Transactions. E. H. Moore, Maxime Bôcher, and H. S. White were others among his near contemporaries who were similarly honored by election to both of these important offices. All of these men served American mathematics faithfully and well. It will be remembered by many that, at one moment when the unity of the American Mathematical Society was somewhat in jeopardy, Van Vleck exerted an especially constructive influence.

It would have been interesting to have had Van Vleck himself formulate his outlook upon the world. Of his own volition he would of course never have done so. But, had he been somehow persuaded to give his philosophy of living, it would have expressed a large measure of tolerance towards others. As a young colleague of his at Wisconsin, I recall asking him about a great writer whose personal life had been far from admirable. Van Vleck replied very quickly: "Don't you think that what he has done for the world of literature more than made up for any personal defects?" Above all, he would have emphasized the virtues of kindness and friendship, of disinterested devotion to intellectual and other ideals, and of a due regard for the amenities of our civilization.

GEORGE D. BIRKHOFF

BOOK REVIEWS

Differentialgeometrien in den Kugelräumen. Vol. II. *Laguerresche Differentialkugelgeometrie.* By T. Takasu. Tokyo, Maruzen, 1939. 20+444 pp.

This volume is the second of a series of three, in which Professor Takasu attempts to give a unified and systematical representation of his work in the differential geometry of sphere space. Concentrating on the geometries of Möbius, Laguerre, and Lie, he presents in this volume his results in Laguerre geometry. In his exposition he follows as closely as possible the usual pattern of euclidean differential geometry, to the systematic study of which the author has contributed in several previous papers.

Following this scheme of development of his subject, Professor Takasu has divided his field into five parts. The first chapter explains the fundamental elements and the coordinates of Laguerre geometry, and shows how we can obtain a correspondence between euclidean three space and a Laguerre plane by means of minimal (isotropic) projection. The group of congruent transformations in space corresponds, in this transformation, to the Laguerre group of the plane. Then follows, in Chapter II, the Laguerre generalization of plane euclidean theory of curves, which is the theory of Laguerre invariants of oriented circles. In Chapter III we find the analogous theory of oriented spheres, corresponding to space curves in ordinary geometry. The Laguerre analogues of surfaces are congruences of spheres, which are discussed in Chapter IV. The final chapter deals with systems of cones, which correspond to line congruences in ordinary space.

This gives us quite a good picture of the structure and of the content of Laguerre differential geometry. The author has placed all previous theories, as those of Darboux, of Ribaucour, of Study, in their proper place and has enriched his work with the investigations of Blaschke and his school. He has shown how much more there is to Laguerre geometry than we could have expected from the study of even the richest other author, and how his methods allow the systematic penetration of all fields into which ordinary differential geometry has thrown its light. We mention, as examples out of a great many other results, the author's treatment of "*L*-developables" ("*L*-Torsen"), of *L*-minimal surfaces, and of Bonnet's plane coordinates. He also ventures into differential geometry in the large, as in a Laguerre version of the four vertex theorem.

We believe that Professor Takasu would have increased the usefulness of his work, if he had dwelled longer on the elementary material of his first chapter. The reader who tackles his book without previous knowledge of Laguerre geometry will find the going hard at the beginning. Once he has overcome the initial difficulties, however, he will be richly rewarded by the great number of beautiful results which fall into his lap, and by the mastery of a method which will allow him to find many more results by his own effort.

The book has a complete bibliography and an excellent index.

D. J. STRUIK

The theory and applications of harmonic integrals. By W. V. D. Hodge. Cambridge, University Press; New York, Macmillan, 1941. 9+281 pp. $4.50.

This is one of those books which everyone who specializes in a particular branch of group theory, of the theory of algebraic surfaces, of the theory of Riemann surfaces, of topology or of the tensor analysis should consult. It shows how all these different fields are connected, and not connected in some superficial way or in the form of an analogy, but in an essential manner, so that interesting and profound theorems in one field cannot be understood without a thorough knowledge of other fields. In reading this book one is reminded of books like Klein's "Ikosaeder," which is also a blend of several important fields. The task of the reviewer of such a book is hard, because he has seldom the enviable mastery of the different branches of mathematics which the author possesses. At the same time he must praise the author for the beautiful exposition of so many and different fields.

There are chapters on Riemannian manifolds, on integrals and their periods, on harmonic integrals, on their applications to algebraic varieties and on their applications to the theory of continuous groups. The first chapter, on Riemannian manifolds, is divided into a part on tensor calculus and into a part on the topology of such manifolds. All these topics are prepared in a very careful way, with precautions which will satisfy strict demands of rigor. There are references at the end of each chapter.

To indicate the contents of this book by simply copying the chapter headings is to do a great injustice to the author and his work. There is a leading thought in the choice of subject matter, and that is the study of harmonic integrals. Harmonic integrals, however, are not the capricious invention of an imaginative scholar. They appear quite naturally in the generalization of the problems set by

Riemann's theory of integration on an algebraic surface, and in particular if we ask for the existence of m-fold integrals on an algebraic manifold of m dimensions which are everywhere finite and of zero period. The investigation is facilitated by the Cartan notation in which differentials alternate and in which the exterior derivative of a "p-form" $A_{i_1 \cdots i_p} dx^{i_1} \cdots dx^{i_p} = p!$. A is defined in such a way that Stokes' theorem takes the form $\int A = \int A_x$, the second integral taken over a chain C_p and the first one over its boundary. This again requires not only some understanding of tensor algebra, but also of the topology of manifolds, which the author provides in his early chapters. There are two theorems of DeRham (J. Math. Pures Appl. (9) vol. 10 (1931) p. 115) which lead up to the heart of the problem and which are demonstrated in this book. The first of these theorems states (p. 88) that if Γ_p^i $(i=1, \cdots, R_p)$ is any base for the p-cycles of a manifold M, then there exists a regular closed ϕ-form on M for which the periods $\nu^i = \int \phi$ (on Γ_p^i) are R_p arbitrary real numbers. The second theorem of DeRham states (p. 100) that if ϕ is a closed form on M whose integral has all its periods equal to zero, then ϕ is a null form. Closed forms are forms with exterior derivatives $A_x = 0$, and null forms are the exterior derivatives of other forms, hence forms with zero periods.

The author derives DeRham's second theorem from his main existence theorem for harmonic integrals. This requires the introduction of a Riemannian metric. A function which satisfies the Laplacian equation in this metric is called harmonic. A p-form P is harmonic when it is regular and closed, as well as its dual, the $(n-p)$-form P^*; $p!P$ and $p!P^*$ correspond to the harmonic tensors $P_{i_1 \cdots i_p}$ and $g^{1/2} P^{k_1 \cdots k_p}{}_{k_1 \cdots k_p k_{p+1} \cdots k_n}$. A harmonic integral is the integral of a harmonic form. The gradient of a harmonic function ϕ_i is a simple example of a harmonic tensor. The real and imaginary paths of an integral of a complex variable are harmonic.

One of the main objects of the third chapter is the proof that on a Riemannian manifold M such harmonic integrals actually exist. The existence theorem states (p. 119) that there exists a harmonic integral having arbitrarily assigned periods on R_p independent p-cycles of M. The proof requires considerable labor and demands the knowledge of the theory of integral equations. Once established, the main existence theorem is used for the study of algebraic varieties. It is interesting that an ancient theorem of G. Mannoury (Nieuw Archief voor Wiskunde vol. 4 (1898) p. 112)—the deeper meaning of which had always puzzled the reviewer—is used to show how Riemannian manifolds can be constructed corresponding to an algebraic variety V_m

without singularities in a projective space S_r. In this representation the S_r is given by a real analytical locus of $2r$ dimensions. A large number of applications are presented together with a classification of harmonic integrals. This again connects with Lefschetz' classification of p-cycles. Period matrices of so-called "effective" integrals are studied in detail, "effective" integrals being integrals of p-forms for which a certain related $(p-2)$-form is a null form. Among the many results are several due to Severi and others which are new. In the application to continuous groups the representation of closed semi-simple groups by group manifolds is used. The main purpose of this chapter is to show how harmonic integrals provide a convenient method of discussing the invariant integrals of Cartan's theory.

A flaw in the proof of the theorem establishing the existence of harmonic integrals with preassigned periods has been corrected by H. Weyl in the Annals of Mathematics vol. 44 (1943) pp. 1–6.

The nineteen pages devoted to the exposition of the tensor calculus can be recommended to those who like a short and precise introduction into this theory.

D. J. STRUIK

NOTES

Beginning with this issue the BULLETIN OF THE AMERICAN MATHEMATICAL SOCIETY will be copyrighted. It is not the desire of the editors to restrict in any way the use of mathematical material from the BULLETIN nor to deny the right to quote from its text. However, it is desired to regulate the republication or sale of papers that have been published in the BULLETIN.

The editors of the Bulletin wish to make grateful public acknowledgment of the services rendered by the following persons who have refereed papers: R. P. Agnew, Leon Alaoglu, A. A. Albert, G. E. Albert, Reinhold Baer, M. A. Basoco, E. F. Beckenbach, E. T. Bell, Harry Blumberg, Herbert Busemann, R. V. Churchill, George Comenetz, H. S. M. Coxeter, M. M. Day, R. H. Fox, K. O. Friedrichs, Guido Fubini, D. W. Hall, Selig Hecht, M. H. Heins, M. R. Hestenes, T. H. Hildebrandt, Ralph Hull, Nathan Jacobson, R. L. Jeffery, Irving Kaplansky, A. J. Kempner, R. B. Kershner, E. P. Lane, R. E. Langer, Walter Leighton, D. C. Lewis, E. R. Lorch, Szolem Mandelbrojt, G. M. Merriman, W. E. Milne, F. J. Murray, Oystein Ore, G. H. Peebles, F. W. Perkins, E. L. Post, G. B. Price, Hans Rademacher, Tibor Radó, H. W. Raudenbush, W. T. Reid, Arthur Rosenthal, A. C. Schaeffer, Peter Scherk, I. J. Schoenberg, Wladimir Seidel, F. C. Smith, N. E. Steenrod, A. H. Stone, Otto Szász, Alfred Tarski, E. W. Titt, J. W. Tukey, S. M. Ulam, R. J. Walker, H. S. Wall, Morgan Ward, Anna P. Wheeler, W. M. Whyburn, L. R. Wilcox, Aurel Wintner, Leo Zippin, and Antoni Zygmund.

The editors of the Transactions wish to acknowledge the services of the following persons, not members of the Editorial Board, who have been consulted regarding papers offered for publication in volumes 53 and 54: Emil Artin, Reinhold Baer, E. G. Begle, Garrett Birkhoff, R. P. Boas, R. H. Bruck, E. H. Buchanan, Herbert Busemann, Alonzo Church, R. P. Dilworth, G. A. Hedlund, R. L. Jeffery, N. H. McCoy, E. J. McShane, F. J. Murray, George Pólya, C. E. Rickart, R. M. Robinson, O. F. G. Schilling, D. C. Spencer, E. R. Stabler, Otto Szász, J. D. Tamarkin, D. V. Widder, and Antoni Zygmund.

Professor Philip Franklin of Massachusetts Institute of Technology has received an alumni medal from the College of the City of New York.

Mr. J. Lifshitz of the National University of Mexico has been awarded a Guggenheim fellowship to study at Harvard University under the direction of Professor G. D. Birkhoff.

Professor F. G. Reynolds of the College of the City of New York has been given the title of professor emeritus.

Professor Shiing-shen Chern, Dr. Kurt Gödel, Professor Wolfgang Pauli, and Professor C. L. Siegel are in residence at the Institute for Advanced Study for this academic year.

Dr. Claire F. Adler has been appointed to an assistant professorship at New York University.

Dr. G. E. Albert of Ohio State University has been promoted to an assistant professorship

Dr. Warren Ambrose of Princeton University has been appointed to an assistant professorship at the University of Michigan.

Dr. Valentin Bargmann has been appointed an assistant at the Institute for Advanced Study.

Assistant Professor P. O. Bell of the University of Kansas has been promoted to an associate professorship.

Assistant Professor D. G. Bourgin of the University of Illinois has been promoted to an associate professorship.

Assistant Professor J. D. Burk of the University of Toronto has been promoted to an associate professorship.

Assistant Professor C. L. Buxton of Clarkson College of Technology, Potsdam, New York, has been promoted to an associate professorship of physics.

Associate Professor Ethel B. Callahan of Shepherd State Teachers College, Shepherdstown, West Virginia, has been appointed to an associate professorship at Hartwick College, Oneonta, New York.

Assistant Professor R. H. Cameron of Massachusetts Institute of Technology has been promoted to an associate professorship.

Associate Professor E. J. Camp of Macalester College, St. Paul, Minnesota, has been promoted to a professorship.

Dr. Josephine H. Chanler of the University of Illinois has been promoted to an assistant professorship.

Dr. R. F. Clippinger of Carnegie Institute of Technology has been promoted to an assistant professorship.

Mr. A. J. Coleman has been appointed lecturer in mathematics at Queen's University in Kingston, Ontario, Canada.

Assistant Professor N. B. Conkwright of the University of Iowa has been promoted to an associate professorship.

Mr. W. S. H. Crawford of the University of Minnesota has been appointed to an assistant professorship at Mount Allison University.

Sister James S. Creane has been appointed to an assistant professorship at Fontbonne College, St. Louis, Missouri.

Associate Professor Wayne Dancer of the University of Toledo has been promoted to a professorship.

Associate Professor P. H. Daus of the University of California at Los Angeles has been promoted to a professorship.

Mr. U. P. Davis of the University of Florida has been promoted to an assistant professorship.

Professor R. C. Dragoo of Southwestern Institute of Technology has been appointed to an assistant professorship at the University of Oklahoma.

Assistant Professor J. L. Doob of the University of Illinois has been promoted to an associate professorship.

Associate Professor W. L. Duren of Tulane University has been promoted to a professorship.

Mr. R. M. Foster of Bell Telephone Laboratories has been appointed professor and head of the department of mathematics of the Polytechnic Institute of Brooklyn.

Assistant Professor A. H. Fox of Union College has been promoted to an associate professorship.

Dr. L. M. Garrison of Alabama Polytechnic Institute has been appointed to an assistant professorship at Louisiana Polytechnic Institute.

Assistant Professor W. O. Gordon of Pennsylvania State College has been promoted to an associate professorship.

Mr. J. B. Greeley of the Polytechnic Institute of Brooklyn has been appointed to an assistant professorship at Lafayette College.

Mr. H. L. Harter of the University of Illinois has been appointed to a professorship of physics at Missouri Valley College, Marshall, Missouri.

Dr. A. D. Hestenes of Carnegie Institute of Technology has been promoted to an assistant professorship.

Assistant Professor E. H. C. Hildebrandt of the State Teachers College, Upper Montclair, New Jersey, has been appointed to an assistant professorship at Northwestern University.

Assistant Professor Evan Johnson of Pennsylvania State College has been promoted to an associate professorship.

Assistant Professor P. W. Ketchum of the University of Illinois has been promoted to an associate professorship.

Assistant Professor L. A. Knowler of the University of Iowa has been promoted to an associate professorship.

Mr. Max LeLeiko has been appointed lecturer in mathematics at the University of North Carolina.

Dr. A. T. Lonseth of Iowa State College has been promoted to an assistant professorship.

Assistant Professor C. T. Male of Union College has been promoted to an associate professorship.

Dr. H. J. Miles of the University of Illinois has been promoted to an assistant professorship.

Associate Professor F. H. Miller of Cooper Union School of Engineering has been promoted to a professorship and has been appointed head of the department of mathematics.

Mr. H. G. Means has been appointed to an associate professorship at Lehigh University.

Assistant Professor G. E. Moore of the University of Illinois has been promoted to an associate professorship.

Assistant Professor S. B. Myers of the University of Michigan has been promoted to an associate professorship.

Mr. J. D. Novak of the University of Minnesota has been appointed head of the department of mathematics and physics at MacMurray College, Jacksonville, Illinois.

Dr. W. H. Pell and Dr. Vladimir Morkovin of Brown University have accepted positions as research engineers in aerodynamic design at the Bell Aircraft Corporation, Buffalo, New York.

Dr. H. V. Price of the State University of Iowa has been promoted to an assistant professorship.

Associate Professor W. C. Randels of the University of Oklahoma has resigned to serve with the North American Aviation Company in Los Angeles.

Professor H. W. Reddick of Cooper Union has been appointed adjunct professor at New York University.

Assistant Professor P. K. Rees of Southern Methodist University has been appointed to a professorship at Southwestern Louisiana Institute.

Assistant Professor G. de B. Robinson of the University of Toronto has been promoted to an associate professorship.

Assistant Professor W. J. Robinson of Marshall College, Huntington, West Virginia, has been appointed to an assistant professorship at Allegheny College.

Dr. D. H. Rock of Brown University has been appointed to an assistant professorship at Rhode Island State College.

Assistant Professor W. O. Rogers of Pennsylvania State College has been promoted to an associate professorship.

Assistant Professor George Sauté of Cleveland College has been appointed to an associate professorship at Rollins College.

Assistant Professor A. C. Schaeffer of Stanford University has been promoted to an associate professorship.

Dr. H. M. Schwartz of the University of Illinois has been appointed to an assistant professorship at the University of Idaho.

Dr. G. E. Schweigert of the University of Missouri has been appointed to an assistant professorship at Purdue University.

Dr. R. E. Street has been appointed to a visiting lectureship in physics at Dartmouth College.

Assistant Professor H. P. Thielman of Iowa State College has been promoted to an associate professorship.

Professor C. C. Wagner of Pennsylvania State College has been named assistant dean of the School of Liberal Arts.

Dr. George Whaples of Indiana University has been appointed to an assistant professorship at the University of Pennsylvania.

Associate Professor J. K. Whittemore of Yale University has retired.

Associate Professor James Edgar Davis of Central Y.M.C.A. College, Chicago, Illinois, has been appointed an associate at the University of Illinois.

Drs. O. K. Bower, Leonard Bristow, M. M. Day, and J. W. Peters of the University of Illinois have been promoted from the rank of instructor to that of associate.

The following appointments to instructorships are announced: Albright College, Reading, Pennsylvania: Mr. C. C. Miesse; University of Arizona: Dr. O. B. Ader; Barnard College, Columbia University: Miss Louise M. Comer; Brown University: Mr. W. E. Barnes, Mr. W. M. Kincaid, Mr. P. T. Mielke; University of Chicago: Dr. Anne L. Lewis, Dr. Janet MacDonald; Cooper Union: Mr. S. G. Roth; Duke University: Dr. L. I. Wade; Hunter College: Miss Louise W. Miller; University of Illinois: Mr. W. F. Atchison, Dr. Theodore Bedrick, Miss Janie C. Lapsley; Massachusetts Institute of Technology: Mr. W. S. Loud; University of Michigan: Mr. B. J. Lockhart, Dr. C. A. Truesdell; University of Minnesota: Mr. H. D. Colson, Dr. C. M. Jensen; University of Nebraska: Mr. H. W. Linscheid; Pennsylvania State College: Dr. A. B. Cunningham; Princeton University: Mr. Richard Bellman; Purdue University: Dr. Fred Kiokemeister; University of Rochester: Dr. Dorothy L. Bernstein; State College of Washington: Dr. P. F. Neményi; Yale University: Mr. A. S. Day, Mr. R. E. Fullerton, Mr. J. S. Leech.

Dr. G. T. Bennett of Emmanuel College, Cambridge, England, died October 11, 1943, at the age of seventy-five years.

ABSTRACTS OF PAPERS

SUBMITTED FOR PRESENTATION TO THE SOCIETY

The following papers have been submitted to the Secretary and the Associate Secretaries of the Society for presentation at meetings of the Society. They are numbered serially throughout this volume. Cross references to them in the reports of the meetings will give the number of this volume, the number of this issue, and the serial number of the abstract.

ALGEBRA AND THEORY OF NUMBERS

1. A. A. Albert: *Quasigroups*. II.

A loop is a quasigroup with an identity element. In this part it is shown that H is a normal divisor of a loop G if and only if $(xH)(yH) \subset (xy)H$, $xH \subset (xH)h$, $(xy)H \subset x(yH)$ for every x and y of G and h of H. The intersection and union of two normal divisors of G are normal divisors of G, and the standard theorems used to prove the Jordan-Hölder theorem and the Schreier refinement theorem are valid for loops (although the proofs are very different). The paper shows how to extend the notion of solvable group to loops and also proves that various results of the theory of groups are also valid for loops. A construction is given of all loops with a given normal divisor and a given quotient loop, and the theory is applied to give an explicit determination of all loops of order six with a subloop of order three. Finally it is shown that all quasigroups of order five not isotopic to the group are isotopic to each other. (Received October 15, 1943.)

2. A. A. Albert: *Quasiquaternion algebras*

A quasiquaternion algebra has a basis 1, i, j, ji over a field F such that $i^2 = a + bi$, $ij = j(1-i)$, $j^2 = c$ for a, b, c in F, $c \neq 0$, $b \neq 1$. Also $b = 0$ when F has characteristic different from two and one writes $A = (a, c)$ in this case. All quadratic subalgebras are determined and it is shown that (a, c) is isomorphic to (a_0, c_0) if and only if $a_0 = a$, $c_0 = d^2c$ for $d \neq 0$ in F. The results in the characteristic two case are slightly different. A quasiquaternion algebra is a division algebra if and only if the algebras $F[i]$ and $F[j]$ are fields. If F is a finite field of q elements there are division algebras of this kind over F if and only if q is odd. Then there are $1/2(q-1)$ such algebras not isomorphic in pairs. The question as to when two algebras over F of characteristic not two are isotopic is completely solved for quasiquaternion division algebras. (Received October 15, 1943.)

3. J. L. Brenner: *The linear homogeneous group*. II.

Let (x_i) represent an n-tuple (vector) whose elements are residue classes mod p^r (p, prime; r, positive integer). The p^{nr} vectors (x_i) form a group $\mathfrak{A}$ under the operation vector addition: $(x_i)+(y_i)=(z_i)$, $z_i \equiv x_i + y_i$ (mod p^r). An automorphism of $\mathfrak{A}$ may be defined by specifying the n images (a_{ji}) of the generators $e_j = (0, \cdots, 1_j, \cdots, 0)$, where det $(a_{ji}) \neq 1$ (mod p). $\mathfrak{G}_{p,n,r}$ is the group of these automorphisms. In this article the lattices of normal and of characteristic subgroups of $\mathfrak{G}$ are described; the lattices are distributive except when $n = p = 2$, in which case they are not distributive. $\mathfrak{N}_n$ consists of all matrices in $\mathfrak{S}$ which fix every element of order p^e in $\mathfrak{A}$. $\mathfrak{N}_e$ is characteristic in $\mathfrak{S}$. This article will appear in Ann. of Math. vol. 45 (January, 1944). (Received November 1, 1943.)

4. R. H. Bruck: *Some results in the theory of linear non-associative algebras.*

An algebra is defined to be isotopically simple if every isotope is simple. The theory of linear algebras is shown to be reducible to that of isotopically simple algebras. The following exhaustive classification of algebras is convenient: (I) all algebras with unit elements, and their isotopes; (II) all algebras with right units in which every element is a left-hand divisor of zero, and their isotopes; (III) all algebras anti-isomorphic to those of class II; (IV) all algebras in which every element is both a left-hand and right-hand divisor of zero. Isotopically simple algebras of order n over an arbitrary field are constructed as follows: of class I, for all $n \neq 2$; of classes II and III, for all $n > 3$. Isotopically simple algebras of order 2 are (isotopic to) quadratic fields. Certain types of Lie algebras (class IV) are shown to be isotopically simple. New simple algebras of all orders and new division algebras of orders 4, 8 and countable infinity are defined in terms of quasigroup algebras and generalized quasigroup algebras. The notion of generalized quasigroup algebra stems from the extension problem for quasigroups, of which a complete solution is given here for quasigroups of arbitrary order. (Received October 6, 1943.)

5. Claude Chevalley: *Some properties of ideals in rings of power series.*

The paper is concerned mainly with the behavior of prime ideals in power series rings under extension of the basic field. The theorem of A. Weil on the existence of the field of definition of a prime ideal in the ring of polynomials is generalized for prime ideals in rings of power series. (Received October 8, 1943.)

6. M. M. Day: *Arithmetic of ordered systems.*

Operations of ordered addition and ordered multiplication of (partially) ordered systems are defined to include as special cases the ordinal and cardinal sum and product and the ordinal exponentiation of G. Birkhoff (Duke Math. J. vol. 9 (1942) pp. 283–302). If $(T, \geqq)$ and $(S_t, \geqq)$ are ordered systems, the ordered product over $(T, \geqq)$ of the $(S_t, \geqq)$ has as elements the functions f defined on T with $f(t)$ in S_t for every t: $f \geqq' f_1$ means that if $f(t) \neq f_1(t)$ there exists $t_1 \geqq t$ such that $f(t_1) > f_1(t_1)$ (essentially lexicographic ordering); the relation $\geqq$ in the ordered product is the least transitive relation including $\geqq'$. The product over the system of integers of a family of two-element well-ordered systems is a simple example in which $\geqq'$ is not transitive. The principal results depend on the fundamental theorem: If $(T, \geqq)$ is a number and each $(S_t, \geqq)$ is transitive, the relation $\geqq'$ is almost transitive; that is if $f_1 \geqq' f_2 \geqq' f_3 \geqq' \cdots \geqq' f_n$, there exists f_0 such that $f_1 \geqq' f_0 \geqq' f_n$. (Received October 22, 1943.)

7. R. P. Dilworth: *A decomposition theorem for partially ordered sets.*

Let P be a partially ordered set. P has finite *width* k if k is the least integer such that every set of $k+1$ elements contains a comparable pair. It is shown that *if P has finite width k, then P is the set sum of k disjoint chains.* An almost trivial application yields the P. Hall theorem on representatives of sets. Indeed, the above theorem seems to be the natural infinite extension of the P. Hall theorem. Application to distributive lattices gives the following result: *Let $m = \max h(a)$ where $h(a)$ is the number of elements covering an element a of a distributive lattice L. Then L is a sublattice of a direct product of m chains and m is the least number of chains for which this holds.* (Received October 22, 1943.)

8. Roy Dubisch and Sam Perlis: *On the radical of a non-associative algebra.* (Preliminary report.)

The radical of an associative algebra may be defined in many equivalent ways. In the case of a non-associative algebra most of these definitions provide subsets which are not always ideals or which lack many of the common properties of the radicals of associative algebras. One of these common properties is preserved by the definition made recently by Albert for non-associative algebras obeying a simple hypothesis, but most of the other properties are lost. It is the purpose of this paper to investigate several definitions yielding ideals with at least one of the desired, well known attributes of a radical, to see if any two of these ideals coincide or satisfy an inclusion relation and to see if any one of them has some properties of particular interest. The paper begins with the special case of alternative algebras by giving a new and simple proof of Zorn's theorem that the totality R of properly nilpotent elements forms an ideal. The method consists of showing that the definition given in 1941 by one of the authors provides an ideal H and that $H = R$. Albert's radical N is shown to coincide with H and R for this case. But there are examples of non-alternative algebras for which the property $H = R = N$ fails. (Received October 29, 1943.)

9. Samuel Eilenberg and Saunders MacLane: *Cohomology theory in groups.*

Let π be a discrete group which acts as a group of left operators on an abelian group G. A function f of n variables in π and with values in G is called an n-cochain. The coboundary δf of f is an $(n+1)$-cochain defined by setting $\delta f(x_1, \cdots, x_{n+1}) = x_1[f(x_2, \cdots, x_{n+1})] - f(x_1x_2, \cdots, x_{n+1}) + f(x_1, x_2x_3, \cdots, x_{n+1}) \cdots \pm f(x_1, \cdots, x_{n-1}, x_nx_{n+1}) \mp f(x_1, \cdots, x_n)$. Since $\delta\delta = 0$, the customary construction yields a cohomology group $H_n(\pi, G)$ of π with coefficients in G. By a suitable definition of a cup product, the groups $H_n(\pi, G)$ give rise to a cohomology ring. The group $H_1(\pi, G)$ is the group of crossed homomorphisms of π in G (functions such that $f(x_1x_2) = f(x_1) + x_1f(x_2)$) modulo the principal homomorphisms (functions $f(x) = x(g) - g$, for a constant $g \in G$). The group $H_2(\pi, G)$ is the group of group extensions of G by π, with the indicated operators. If π is represented as a factor group F/R of a free group F, then $H_n(\pi, G) = H_{n-2}(\pi, \mathrm{Hom}\,(R, G))$, with the group π suitably operating on the group $\mathrm{Hom}\,(R, G)$ of homomorphisms of R into G. (Received November 5, 1943.)

10. Wade Ellis: *Relations satisfied by linear operators on a vector space.*

Let α, β be two linear operators on a linear vector space V of dimension n. If $n = 2$, all general relations on α, β are consequences of the three known relations, of which two involve only one operator each (Cayley), and the third expresses $\alpha\beta + \beta\alpha$ in terms of α, β, 1 (Laguerre). If $n > 2$, there are, in addition to those of a similar type, other relations (with invariants of certain kinds as coefficients) which it is the purpose of this paper to find and express. When $n = 3$, an important distinction is made between the case (I) (of simultaneous symmetrizability) where there is a coordinate system in which α, β are both represented by symmetric matrices, and the case (II) where this is not true. In I the relations are expressed as determinants of order seven, with all their elements rational invariants. In II the coefficients must involve a quadratic irrationality δ; it is convenient to use determinants of order nine whose elements involve δ and rational invariants. For $\delta=0$, these relations reduce to those of case I. Special cases of the relations occur in connection with the study of certain differential equations of physics. (Received October 23, 1943.)

11. N. J. Fine and Ivan Niven: *The probability that a determinant be congruent to a* (mod *m*).

A complete solution is given to the problem of evaluating $P_n(a, m)$, the probability that a determinant of order n having integral elements be congruent to a modulo m. (Received October 15, 1943.)

12. A. L. Foster and B. A. Bernstein: *Symmetric definition and duality theorem for rings.*

It is shown that commutative rings with unit are capable of symmetric definition and possess an elementary principle of duality of which the Boolean algebra duality is an instance. In this paper only non-ideal theoretic consequences of this duality are treated, and applications are made to the special case of fields. (Received November 29, 1943.)

13. Ralph Hull: *A theorem on the unit groups of simple algebras.*

Let $\mathfrak{A}$ be a normal simple algebra of degree n over an algebraic number field $\mathfrak{F}$. If $\mathfrak{A}$ satisfies condition R, that is, $n>2$ or $\mathfrak{A}$ is unramified at at least one infinite prime place of $\mathfrak{F}$ when $n=2$, then any two distinct maximal orders of $\mathfrak{A}$ have distinct unit groups. The proof is based on the splitting-field theory of algebras $\mathfrak{A}$ which satisfy R (Eichler, Math. Zeit. vol. 43 (1938) pp. 481–494) and local arithmetical considerations. Condition R is indispensable in general. The theorem can be used to describe the Brandt groupoid of normal ideals of $\mathfrak{A}$. (Received October 18, 1943.)

14. Jakob Levitzki: *A characteristic minimal condition for semi-primary rings.*

A ring S is called semi-primary if the sum R of all two-sided nilpotent ideals of S is nilpotent and S/R is semi-simple. It has been proved by Hopkins (C. Hopkins, *Nil-rings with minimal condition for admissible left ideals*, Duke Math. J. vol. 4 (1938) pp. 664–667) that a ring S with minimal condition for left-ideals is semi-primary. But neither this condition, nor the weaker assumptions found later by other authors are necessarily satisfied by each semi-primary ring. In the present note it is shown that the following minimal condition is necessary as well as sufficient for semi-primary rings: Each descending chain of the form $L_1 \supset L_2 \supset L_3 \supset \cdots$, where the L_i are left-ideals containing R, and each descending chain of the form $A_1 \supset A_1 \cdot A_2 \supset A_1 \cdot A_2 \cdot A_3 \supset \cdots$, where the A_i are two-sided ideals contained in R, is finite. (Received October 27, 1943.)

15. G. Y. Rainich: *Noncommutative relations.*

The problem of factorization of polynomials in a ring discussed previously (abstract 48-1-51) leads to the consideration of systems of relations (in terms of multiplication and addition) on indeterminates (which first appear as indeterminate coefficients). The situation is studied from several points of view. I (Abstract point of view): One set of relations may imply another. A set may be contradictory. Examples are given of complete sets; a set is complete when any relation compatible with it is implied by it. II (Realization): Here a linear vector space is considered and linear operators on it which satisfy the same relations as those that are given. The connection with I is given by the fact that relations satisfied in an invariant subspace imply relations in the whole space. In III a ring is considered generated (with the aid of a field) by operators satisfying given relations. The special case when the relations involve multiplication only correspond to a group algebra. IV deals with relations satisfied by operators as a result of their being operators on a vector space of a given dimensionality. (Received October 23, 1943.)

16. H. E. Salzer: *New tables and facts involving sums of four tetrahedral numbers.*

The author has a second empirical theorem about tetrahedral numbers, that is, $(n^3-n)/6$ for integral n. Every tetrahedral number greater than 1 is the sum of four other non-negative tetrahedrals. This theorem has been verified for the first 200 cases in a table expressing every tetrahedral from 4 through 1373701 as a sum of four non-negative tetrahedrals. With the exception of 153, the first 200 triangular numbers $n(n+1)/2$ can each be expressed as the sum of four non-negative tetrahedrals. There are only 45 integers less than or equal to 1000 which cannot be expressed as the sum of four non-negative tetrahedrals. All numbers ending in 0, 5, or 6 which are less than or equal to 2006 are expressible as a sum of four non-negative tetrahedrals. This includes the first 201 cases of each type. It is interesting to note that the smallest example of a number ending in 4 which is not expressible as a sum of four non-negative tetrahedrals occurs at 1314. Thus here is an instance where a statement is true in the first 131 cases, but fails in the 132nd. (Received October 13, 1943.)

17. L. R. Wilcox: *Modularity in Birkhoff lattices.*

The following theorem connecting G. Birkhoff's upper semi-modular lattices with the author's M-symmetric lattices is proved. Let a lattice be called upper semimodular if $a+b$ covers a, b when a and b cover ab; let a lattice be called M-symmetric if $(a+b)c=a+bc$ for every $a\leqq c$ implies $(d+c)b=d+cb$ for every $d\leqq b$; finally, let a lattice be called of finite dimensional type if every a, b with $a<b$ have a finite principal chain connecting them. Then a lattice of finite dimensional type is upper semimodular if and only if it is M-symmetric. The purpose of this theorem is to replace the condition of Birkhoff, forceful only when some chain condition is assumed, by a strictly algebraic condition which is suitable for use in the infinite dimensional case. (Received October 19, 1943.)

ANALYSIS

18. Stefan Bergman: *The determination of singularities of functions satisfying a partial differential equation from the coefficients of their series development.*

Let $U(z, \bar{z})=A_{00}+\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}A_{mn}z^m\bar{z}^n$ be a (complex) solution of the equation $L(U)\equiv U_{z\bar{z}}+a_1U_z+a_2U_{\bar{z}}+a_3U=0$ where a_k, $k=1, 2, 3$, are entire functions of two variables $z=x+iy$, $\bar{z}=x-iy$, x, y real. Using the results of the papers Rec. Math.

(Mat. Sbornik) N.S. vol. 2 p. 1169, and Trans. Amer. Math. Soc. vol. 53 (1943) p. 130, §7, the author shows that the positions (that is, the coordinates x, y) of singularities of $U(z, \bar{z})$ are determined by the sequence $\{A_{m0}\}$, $m=0, 1, 2, \cdots$, and are independent of the a_k, $k=1, 2, 3$. He investigates further the connections between other subsequences of $\{A_{mn}\}$ and the positions of singularities. For instance the A_{m1} and the $\alpha_{m0}^{(\nu)}$, $\nu=1, 2, 3$, $m=0, 1, 2, \cdots$, where $a_\nu=\sum\alpha_{mn}^{(\nu)}z^m\bar{z}^n$, determine the positions of the singularities. If $a_2=\bar{a}_1$ and a_3 is real and $u=\sum D_{mn}z^m\bar{z}^n$ is a real solution of $L(u)=0$, then the D_{m0} and the $\alpha_{0m}^{(1)}$, $m=0, 1, \cdots$, determine the positions of singularities. Finally the author shows that many properties of the singularities are determined by the A_{m0} and are independent of the a_k, $k=1, 2, 3$. (Received October 26, 1943.)

19. Lipman Bers and Abe Gelbart: *On a class of functions defined by certain partial differential equations.*

This paper contains a detailed mathematical treatment of the results presented in the Quarterly of Applied Mathematics vol. 1 (1943) pp. 168–188, and several new results on the properties of the functions satisfying the system of equations $\sigma_1(x)u_x=\tau_1(y)v_y$, $\sigma_2(x)v_y=-\tau_2(y)v_x$. (Received October 4, 1943.)

20. Lipman Bers and Abe Gelbart: *On functions satisfying certain systems of partial differential equations.*

This paper continues the study of the class of complex-valued functions $f(z)=u(x, y)+iv(x, y)$ where u and v satisfy the system of differential equations (1) $u_x=\tau_1(y)v_y$, $u_y=\tau_2(y)v_x$, τ_i being positive and analytic. With the aid of the function $E(\alpha, y)$ belonging to the class and corresponding to the analytic function exp (αy) (cf. Quarterly of Applied Mathematics vol. 1 (1943) pp. 168–188), a "Laplace transformation" $f(z)=\int_0^\infty E(-\alpha, z)g(\alpha)d\alpha$ is defined. Under suitable conditions $f(z)$ belongs to the class. If $g(z)\equiv 1$, $f(z)=z^{(-1)}(z)$ corresponds to the analytic function $1/z$. Its nth Σ-derivative corresponds to $1/z^n$, its Σ-integral to log z. Properties of these functions are investigated. (Received October 4, 1943.)

21. D. G. Bourgin and C. W. Mendel: *Orthonormal sequences.*

Let $f(x)\in L_2$ be odd and periodic of period 2π. Suppose $\{f(nx)\}$, $n=1, 2, \cdots$, is an orthonormal sequence on $0\leqq x\leqq 2\pi$. The investigation of functions satisfying these conditions leads to certain natural subclasses. Explicit examples are given other than the trivial solutions $f(x)=\sin kx$. Properties of these solutions are developed and various subsidiary conditions are formulated under which the trivial solutions alone are possible. (Received October 22, 1943.)

22. Alfred Brauer: *On certain limits.*

The following theorem is proved: Let k and l be positive integers and $m=k+l-2$. Suppose that the functions $f(x)$ and $g(x)$ and their first $m+2$ derivatives exist and are bounded in the interval $0<x\leqq h$. Suppose further that $f(x)$ is positive there. Finally suppose that $\lim f(x)=\lim f'(x)=\cdots=\lim f^{(k-1)}(x)=0$, but $\lim f^{(k)}(x)\neq 0$, and $\lim g(x)=\lim g'(x)=\cdots=\lim g^{(l-1)}(x)=0$, but $\lim g^{(l)}(x)\neq 0$. Then $\lim f(x)^{g(x)}=1$. In all limits $x\to +0$. (Received November 19, 1943.)

23. W. B. Caton and Einar Hille: *On the class of functions $H_p(1/2)$.*

One says that $f(z)\in H_p(1/2)$, $1<p\leqq 2$, if $f(z)$ is holomorphic in $R(z)>1/2$ and if $\int_{-\infty}^{\infty}|f(x+iy)|^p dy<M$ for all $x>1/2$. In this note a new proof is given of the well known theorem which states that $f(z) \in H_p(1/2)$ implies $f(z)=\int_0^{\infty}e^{-uz}F(u)du$, $F(u)e^{-u/2} \in L_q(0, \infty)$, $1/p+1/q=1$. The proof follows a general plan outlined in a paper of Hille (Compositio Math. vol. 6 (1938) pp. 93–102). (Received October 19, 1943.)

24. B. H. Colvin: *The expansion problem associated with a third order ordinary differential system of highly irregular type.*

The expansion problem considered is that associated with the third order ordinary differential system $y'''(x)+\lambda^3y(x)=0$, $H_i(y)\equiv h_{i1}(\lambda)y''(a)+h_{i2}(\lambda)y'(a)+h_{i3}(\lambda)y(a)+h_{i4}(\lambda)y''(b)+h_{i5}(\lambda)y'(b)+h_{i6}(\lambda)y(b)=0$ $(i=1, 2, 3)$, in which the boundary conditions are of highly irregular type. The variable x is restricted to the real interval $a\leqq x\leqq b$: λ is an unbounded complex parameter and the coefficients $h_{ij}(\lambda)$ are polynomials in λ with complex coefficients. The general mode of attack is that recently employed by R. E. Langer in connection with highly irregular systems of the second order. By the use of some rather general notions of summability (first introduced by R. E. Langer) an expansion theory associated with such irregular systems is developed for integrable vectors $f(x)$ which fulfill certain conditions customarily imposed in such theories. The third order boundary systems considered include as special cases those discussed previously by J. W. Hopkins and L. E. Ward. The present theory, however, provides summable expansions for a class of functions far wider than that for which they obtained any results. (Received October 30, 1943.)

25. Paul Erdös and S. M. Ulam: *Some combinatorial problems in set theory.* Preliminary report.

The following problems seem of some interest: 1. Given a class K of subsets of a set E of power m, the class closed with respect to the operation of addition of less than m sets belonging to the class, and with respect to complementation, and with this additional property: For every subdivision of E into disjoint sets containing each more than one element, there exists a set in K which contains exactly one element of each of the sets of the subdivision. Must K under these conditions coincide with the class of all subsets of E? If $m=\aleph_0$, the answer is negative; otherwise, unsolved. 2. Let E be a set of power m. Is it possible to define $n<2^m$ completely additive measure functions on the class of all subsets of E so that each subset will be measurable in at least one measure function? 3. If the continuum hypothesis is true, every subset of the continuum can be obtained by Borel operations effected on sets that are additive subgroups of the real numbers. The statement that every subset can be obtained by Borel operations on sets consisting of rationally independent numbers is equivalent to the continuum hypothesis. (Received October 27, 1943.)

26. R. C. James: *Orthogonality and differentiability in normed linear spaces.*

Several definitions can be given of orthogonality in normed linear spaces. In this paper, it is said that an element x is orthogonal to y if and only if $\|x+ky\|\geqq\|x\|$ for all k. Such orthogonality is neither symmetric ($x\perp y$ does not imply $y\perp x$) nor additive ($x\perp y$ and $x\perp z$ do not imply $x\perp y+z$). However, this orthogonality is never vacuous, since for any x and y there exists an a such that $x\perp(ax+y)$. An equivalent definition of orthogonality is: "$x\perp y$ if $\lim_{n\to\infty}\|nx+y\|-\|nx\|=0$," provided $x\neq 0$, and it follows that this orthogonality is additive if and only if the norm is linearly Gateaux differentiable at each nonzero point. Furthermore, if this differential exists at a point x, it is equal to $-a\|x\|$, where $x \perp (ax+y)$. This gives rise to a condition for the existence of a scalar product, and to a means of investigating the effect on a normed linear space of the existence of the differential of the norm. (Received October 29, 1943.)

27. Anne L. Lewis: *Sufficiency proofs for the problem of Bolza in the calculus of variations.*

This paper is mainly concerned with the establishment of sufficient conditions for a strong relative minimum for the general isoperimetric problem of Bolza in nonparametric form. The proof involves an expansion of the Hilbert integral and the use of a so-called quasi Mayer field, which is defined in terms of the variations of the slope functions and multipliers generally associated with a Mayer field. This procedure combines the field theory methods of Hestenes (Trans. Amer. Math. Soc. vol. 42 (1937) pp. 141–153; Duke Math. J. vol. 5 (1939) pp. 309–324) and the expansion methods of Reid (Trans. Amer. Math. Soc. vol. 42 (1937) pp. 183–190; Ann. of Math. (2) vol. 38 (1937) pp. 662–678) and utilizes certain dominant properties of the Weierstrass E-function. There are obtained, moreover, various results concerning the Weierstrass E-function which yield a Lindeberg theorem for the isoperimetric problem considered. (Received October 22, 1943.)

28. A. T. Lonseth: *The propagation of error in linear problems.*

Suppose that A is a bounded linear transformation in Hilbert space H which possesses a bounded inverse A^{-1}, and that C is in H. Then the solution $X=A^{-1}C$ of $AX=C$ is also in H, its norm $N(X)$ satisfying $N(X) \leqq M(A^{-1})N(C)$, where $M(A^{-1})$ is the bound of A^{-1}. The error ξ induced in X by errors α in A and γ in C is considered; it is found that $N(\xi) \leqq M(A^{-1})\{N(\gamma)+M(\alpha)N(X)\}/\{1-M(A^{-1})M(\alpha)\}$, provided that $M(A^{-1})M(\alpha)<1$. (This inequality generalizes readily to linear functional equations in a Banach space.) A case of special interest is that in which $A=I+K$, where I is the identity and $M(K)<1$; then (E. Hilb) A has a unique inverse A^{-1} and $M(A^{-1})<1/\{1-M(K)\}$. In this case bounds are also found for the components of ξ. The inequalities provide limits to the errors incurred in certain approximate methods for solving linear problems: infinite systems of linear equations in infinitely many unknowns (method of segments) and the related linear integral equations of first and second kinds. (Received October 20, 1943.)

29. Morris Marden: *A recurrence formula for the solutions of certain partial differential equations.*

Given the second order linear partial differential equation $L(U)=U_{z\bar{z}}+AU_z+BU_{\bar{z}}+CU=0$, where A, B and C are analytic functions of the two complex variables $z=x+iy=re^{i\theta}$ and $\bar{z}=x-iy=re^{-i\theta}$. As proved by Bergman (Rec. Math. (Mat. Sbornik) N.S. vol. 44 (1937) pp. 1169–1198), there exists a function $E(z, \bar{z}, t)$ such that the operator $P(f)=\int_{-1}^{1}E(z, \bar{z}, t)f[(1/2)z(1-t^2)](1-t^2)^{-1/2}dt$, acting upon an arbitrary analytic function $f(z)$, produces a solution U of the equation $L(U)=0$. As the solutions so produced may be expanded in terms of the particular solutions $U_n=P(z^n)$, the study of the solutions U would be aided by the existence of recurrence relations among the solutions U_n. In this paper it is proved that when $\log E(z, \bar{z}, t)$ is an even or odd polynomial in t, there can be found $q+1$ functions $k_j(r, \theta, n)$ such that $\partial U_n/\partial r=k_0U_n+k_1U_{n+1}+\cdots+k_qU_{n+q}$, where q is the degree of $\log E$ as a polynomial in t if $\log E$ is an odd polynomial in t, but is only half this degree if $\log E$ is an even polynomial in t. The result is a generalization of a recurrence relation for Bessels' functions J_n. (Received October 7, 1943.)

30. K. L. Nielsen: *On operators for linear partial differential equations.*

Bergman (Trans. Amer. Math. Soc. vol. 53 (1943) pp. 130–155) has considered the relation between functions $U(z, \bar{z})$ satisfying certain partial differential equations $L(U)=U_{z\bar{z}}+aU_z+bU_{\bar{z}}+cU=0$ and an associated analytic function $f(z)$ in terms of which the solution of $L(U)=0$ can be expressed. He has introduced an operator $P(f)=\int_{-1}^{+1}E(z, \bar{z}, t)f(z[1-t^2]/2)(1-t^2)^{-1/2}dt$ which transforms the class of analytic functions of one complex variable into a certain class of functions $C(E)$ which represents solutions of $L(U)=0$. The author and Ramsay (Bull. Amer. Math. Soc. vol. 49 (1943) pp. 156–162) have found simple forms of $E=\exp\,[N(z, \bar{z})t^n+M(z, \bar{z})t^m]$ for given partial differential equations. In this paper the case of $E=\exp\,[\sum_{\nu=1}^{n}p_\nu(z)t^\nu]$ is developed and limiting theorems on m and n are found for the above expression of E. (Received October 22, 1943.)

31. W. T. Reid: *Expansion theorems for boundary problems of the calculus of variations.*

In a previous paper (Amer. J. Math. vol. 54 (1932) pp. 769–790) the author treated a two-point boundary problem associated with the general problem of Bolza in the calculus of variations, and established certain expansion theorems in terms of the characteristic solutions of the considered problem. The present paper is concerned with the proof of more refined expansion theorems for such a boundary problem. In particular, the general theorems herein proved provide improvements of the expansion theorems of Kamke (Math. Zeit. vol. 45 (1939) pp. 759–787; vol. 46 (1940) pp. 231–250 and pp. 251–286) for self-adjoint definite problems involving a single differential equation of even order. The paper contains results of the Riesz-Fischer type which include as very special cases the results of Krein (Rec. Math. (Mat. Sbornik) N.S. vol. 2 (1937) pp. 923–933) for a self-adjoint problem involving a single differential equation of even order, and the more extensive results of this type for a self-adjoint problem involving a second order differential equation that have been established by Galbraith and Warschawski (Duke Math. J. vol. 6 (1940) pp. 318–340). (Received October 21, 1943.)

32. A. R. Schweitzer: *On functional equations with solutions containing arbitrary functions.* III.

When the method of solving functional equations outlined in previous reports in this Bulletin (abstracts 49-9-212, 213) is not applicable, the author notes that a procedure of restricting variables, or generalizing functions or variables, frequently leads to equations with solutions containing arbitrary functions. Perhaps the simplest example is offered by the associative equation $\phi\{x, \phi(y, z)\}=\phi\{\phi(x, y), z\}$. The equation $\phi\{x, \phi(y, x)\}=\phi\{\phi(x, y), x\}$ has the solution $\phi(x, y)=\alpha(x+y)$ and the equation $\phi\{x, f(y, z)\}=f\{\phi(x, y), z\}$ has the solution $\phi(x, y)=\alpha(x)\cdot y$, $f(x, y)=x\cdot\beta(y)$. Also the equation $\phi\{x_1, x_2, \cdots, x_n, \phi(y_1, y_2, \cdots, y_{n+1})\}=\phi\{\phi(x_1, x_2, \cdots, x_n, y_1), y_2, y_3, \cdots, y_{n+1}\}$ has the solution $\phi(x_1, x_2, \cdots, x_{n+1})=x_1\cdot\alpha(x_2, x_3, \cdots, x_n)\cdot x_{n+1}$ $(n>1)$. The latter equation has the inverse $f(u, x_2, \cdots, x_n, v)=f(x_1, x_2, \cdots, x_{n+1})$ where $u=f(x_1, t_1, t_2, \cdots, t_n)$ and $v=f(x_{n+1}, t_1, t_2, \cdots, t_n)$. In the above the functions α and β are arbitrary. (Received October 21, 1943.)

33. Wolfgang Sternberg: *On difference equations.*

The author gives a solution of the often treated fundamental difference equation

$F(t+1)-F(t)=\phi(t)$. It is supposed that the given function $\phi(t)$ can be expanded in a Fourier series in every finite interval of t and, for simplicity, that $\phi(t)$ is continuous. The solution is $F(t)=-\phi(t)/2+\int_a^t\phi(\tau)d\tau+2\sum_{k=1}^{\infty}\int_a^t\phi(\tau)\cos 2\pi k(t-\tau)d\tau$ where a is an arbitrary real constant. The above series is not a Fourier series in the usual sense, because the upper limit of the integrals is not constant, but the variable t itself. Some other solutions are given besides the above one. Finally the general linear difference equation of order n with constant coefficients $a_nF(t+n)+\cdots+a_1F(t+1)+a_0F(t)=\phi(t)$ is reduced to n equations of order 1 by the following theorem: If the characteristic equation $P(u)=a_nu^n+\cdots+a_1u+a_0=0$ has n different roots α_i $(i=1, 2, \cdots, n)$ and if $F_i(t)$ is an arbitrary solution of $F_i(t+1)-\alpha_iF_i(t)=\phi(t)/P'(\alpha_i)$, then $F_1(t)+F_2(t)+\cdots+F_n(t)$ is a solution of the above equation of order n. The case of multiple zeros of $P(u)$ can be dealt with in a similar way. (Received October 30, 1943.)

34. W. J. Thron: *Sets of convergence for continued fractions.* Preliminary report.

Let A be a set of complex numbers. Corresponding to every set A, a set $Z(A)$ is defined as follows: $1\in Z$; for every positive integer k, $1+a_1/1+a_2/1+\cdots+a_k/1\in Z$ if $a_i\in A$ $(i=1, 2, \cdots, k)$; the set Z contains no other elements. The following theorem has been established: for every set A the set $Z(A)$ either contains the point $z=0$ or has no point in common with the set $1-Z(A)$ $(v\in 1-Z$ if $1-v\in Z)$. If $0\in Z(A)$ there exists a continued fraction $1+K(a_n/1)$, all of whose elements a_n are in A, which diverges by oscillation. Hence such a set A cannot be a set of convergence, even if convergence is understood to mean convergence in the wider sense. Thus a necessary condition for a set A to be a set of convergence for continued fractions $1+K(a_n/1)$ is that the sets $Z(A)$ and $1-Z(A)$ have no point in common. (Received November 17, 1943.)

35. W. J. Trjitzinsky: *Problems of representation and uniqueness for functions of a complex variable.*

The leading idea in the present extensive work (as in the author's earlier related memoirs in Ann. École Norm. (3) vol. 55 (1938) Fasc. 2 pp. 119–191, and in Acta Math. vol. 70 (1939) pp. 63–163) is that functions are studied which are not necessarily analytic but which at the same time are sufficiently specialized so as to come within the scope of classical analytic tools (like Lebesgue-Stieltjes integration). It appears that conditions of monogeneity in one sense or another (in a rather general sense) over sets in the complex plane, which may be without interior points, are the conditions which give the desired degree of specialization. On the other hand, the classes of functions so obtained are of such great vastness and the possibility of classifying these functions according to various uniqueness properties and subsequently studying them is so wide that there appears to be on hand a very extensive field for new investigation. (Received October 23, 1943.)

36. S. M. Ulam: *Theory of the operation of product of sets.* III. Preliminary Report.

Two subsets A, B of a product set E^n are called product isomorphic (abstract 47-9-406), if there exists a one-to-one transformation $T(x)$ of E into itself such that the transformation T^n: $x_1\rightarrow T(x_1), \cdots, x_n\rightarrow T(x_n)$ maps A into B: $T^n(A)=B$. Starting with a class K of subsets of E one obtains the projective class containing it by considering in the product set E^ω sets of the form $Z_1 \times Z_i \times Z_n$, where Z_i belongs to K, the Borel field over such sets and using the operation of projection and complementation a finite number of times. Necessary and sufficient conditions are found in terms of a special mapping of E into E^ω preserving a given class of sets in E^ω, for the product isomorphism in case of sets A, B in the projective class. The classical theories of Borel sets and of projective sets correspond to the case where K is the sequence of rational intervals. Many results of these theories hold for the case of a general K and thus show a purely combinatorial and not topological origin. This permits the formulation of an analogous theory in the even more general case of projective algebras (abstract 49-5-151). (Received October 22, 1943.)

37. S. E. Warschawski: *On Theodorsen's method of conformal mapping of nearly circular regions.*

The paper deals with the problem of determining the mapping function of a circle onto a nearly circular region. This problem has some practical importance in the theory of airfoils. Let C be a nearly circular closed Jordan curve: $\rho = \rho(\phi)$, $0 \leqq \phi \leqq 2\pi$, (ρ, ϕ polar coordinates) where $(1/\rho)(d\rho/d\phi)$ is continuous and $1 \leqq \rho(\phi) \leqq 1+\epsilon$ for some ϵ, $0<\epsilon<1$. Suppose that $w = f(z)(f(0)=0, f'(0)>0)$ maps the circle $|z| = |re^{i\theta}| < 1$ conformally onto the interior of C. If $\arg f(e^{i\theta}) = \phi(\theta)$, the $\log (f(z)/z)$ for $z = e^{i\theta}$ may be written as $\log \rho[\phi(\theta)] + i(\phi(\theta) - \theta)$. Hence by Fatou's formula, $\phi(\theta) - \theta = (1/2\pi)\int_0^\pi \{\log \rho[\phi(\theta+t)] - \log \rho[\phi(\theta-t)]\} \cot (t/2)dt \equiv H[\phi(\theta)]$, and the function $\phi(\theta)$ may thus be determined by solving this integral equation. Theodorsen (National Advisory Committee for Aeronautics, Report 411, 1932) and Theodorsen and Garrick (National Advisory Committee for Aeronautics, Report 452, 1933) developed a practical method for computing a solution of this equation by successive approximations. In the present paper the theoretical basis of this method is studied. Sufficient conditions for the curve C are established under which the approximations $\phi_0(\theta) \equiv \theta$, $\phi_n(\theta) \equiv H[\phi_{n-1}(\theta)]$ and their derivatives $\phi_n'(\theta)$ converge to $\phi(\theta)$ and $\phi'(\theta)$ respectively, and the errors $|\phi_n(\theta) - \phi(\theta)|$ and $|\phi_n'(\theta) - \phi'(\theta)|$ are estimated. (Received October 27, 1943.)

APPLIED MATHEMATICS

38. L. W. Cohen and S. M. Ulam: *On the algebra of systems of vectors and some problems in kinematics.*

The properties of equivalence for systems of vectors as postulated for the mechanics of rigid bodies are studied in linear spaces. It is proved that any finite system of vectors in n-space is equivalent to a unique system of vectors collinear with the edges of an n-simplex. It is also proved that any such system is equivalent to a system of $[n/2]+1$ vectors. Similar theorems hold for infinitely many vectors and for spaces of infinitely many dimensions. The problem of topological invariants of trajectories of systems of n points with respect to arbitrary motions of the coordinate system is formulated and results are obtained for the case of three points. (Received October 22, 1943.)

39. A. H. Copeland: *The nature of turbulence.*

There are exhibited in this paper a number of flows which consist of series of disturbances distributed temporally but unfortunately not specially at random, and which satisfy the hydrodynamic equations together with appropriate boundary conditions. In conformance with experimental evidence each disturbance becomes dissipated as time increases. The following motions are obtained: flow without obstructions with constant velocity at infinity, flow through a pipe, flow outside a rotating cylinder, flow past a wall, and flow between parallel walls. These solutions may be regarded as mathematical models displaying some but not all of the characteristics of physical turbulent motion. (Received October 30, 1943.)

40. Isaac Opatowski: *Isoperimetric problems in bending of cantilevers.*

The paper considers cantilevers characterized by the fact that any two cross sections are obtainable from each other by a transformation of dilatation. The surface bounding these cantilevers is representable in the parametric form: $y=F(x)f(t)$, $z=G(x)g(t)$; the beam's axis is taken as the x-axis. The load consists of the cantilever's own weight W and a concentrated force P at the free end. The problem of finding cantilevers which have an assigned type of cross section (that is, assigned expressions of f and g), assigned length and weight, and which are of uniform strength (S. Timoshenko, *Strength of materials*, 2nd ed., part 1, p. 209) and have a minimum deflection Y is of an isoperimetric type with an additional condition for F and G in the form of a Volterra integral equation. If W is neglected with respect to P the problem has no meaning because Y becomes proportional to W. However if, besides this, the condition of uniform strength is abandoned, the classical type of a simple isoperimetric problem is obtained. This is a generalization of a result given by Blasius for circular cross section (Zeitschrift für Mathematik und Physik vol. 62 (1914) pp. 182–197). (Received October 21, 1943.)

41. H. E. Salzer: *Supplementary calculation of coefficients for numerical integration with central differences.*

The coefficients M_{2s} which are the $(2s)$th Bernoulli polynomials of order $2s$ for argument equal to s, divided by $(2s)!$, were calculated from M_{22} to M_{40} to 20 significant figures, from M_{42} to M_{52} to 8 significant figures, and finally from M_{54} to M_{62} to 20 decimals. (This extends the earlier calculation which went up to M_{20}, mentioned in an abstract of this Bulletin, 49-9-224.) With these coefficients it is possible to integrate functions that are tabulated at very wide intervals, even when the size of the interval is such that the differences do not fall off. Functions tabulated to an extremely large number of decimal places can be integrated to the same or greater accuracy, since we can use many differences. The ratio of consecutive coefficients, about 1/4, practically annuls the error that accumulates in successive differences when the tabulated function has a certain accuracy. This is an advantage over the Laplace formula. These coefficients are very useful for checking by quadrature the key values of functions whose integrals are known and for computing key values of functions defined by integrals. (Received October 29, 1943.)

42. L. R. Wilcox: *Theory of traffic light networks.* I.

Let points P_{ij} with coordinates (x_i, y_j) $(i=1, 2, \cdots, m; j=1, 2, \cdots, n)$ be given, and let a matrix function $A(t)=(a_{ij}(t))$ be given, where each $a_{ij}(t)$ is a step function with values 0, 1. The combined system $S=(P_{ij}, A(t))$ is called a rectangular system of traffic lights. Motivation of the definition lies in the obvious connection with ordinary systems of traffic lights, where $a_{ij}(t)=1$ (0) means that the light of P_{ij} at the time

t shows green (red) for the horizontal line through P_i, and red (green) for the vertical. Various questions are asked of S, notably those concerning the possibility of non-stop travel along all horizontal and vertical paths through the point network. Those systems S admitting "through" speeds are determined and for each such S all through speeds are found. Such problems are solved by the theory of simultaneous congruences. The results are applied to the determination of the synchronization for the "best" flow of traffic. Consideration of "almost" through speeds and more general problems in traffic control is deferred to a later paper. (Received October 19, 1943.)

43. František Wolf: *On a problem of L. M. K. Boelter. Viscous fluid forced through a heated vertical pipe.*

The temperature at the wall is prescribed to be $t^\circ = \lambda \coth mz + \mu$, independent of θ. The density and viscosity are analytic functions of t. There are four differential equations of which three are of the second order. Solving Stokes' hydrodynamical equations for a cylinder, we obtain a set of integro-differential equations which can be used for successive approximations. The chief difficulty is in determining the boundary value of the pressure which, generally, is determined by an integral equation. After some preliminary changes of the differential system, the Laplace transformation can be applied. This leads to an algebraic equation for the transform of the boundary value of the pressure. A later paper will be devoted to the study of the convergence of the successive approximations and to the problem of the existence of the solution. (Received October 28, 1943.)

Ergodic Theory

44. P. R. Halmos: *In general a measure preserving transformation is mixing.*

The first proof is given of the old standing conjecture announced in the title. "In general" means of course that the exceptional set is of the first category in one of the usual natural topologies (the strong neighborhood topology) for measure preserving transformations. The principal new and quite surprising fact used in the proof is that for any almost nowhere periodic transformation T the set of all conjugates of T, that is, the set of all STS^{-1}, is everywhere dense. (Received October 26, 1943.)

Geometry

45. V. G. Grove: *The transformation of Čech.*

The purpose of this paper is to give a simple geometric construction of the general transformation of Čech. This is accomplished by first constructing a two parameter family of quadrics having second order contact with a surface and associated in a simple manner with a conjugate net on the surface. The polar plane of a point on the tangent to a curve of the net with respect to a quadric of the family is related to that point by a general transformation of Čech. Geometrical characterizations are made for several particular transformations of Čech. (Received November 19, 1943.)

46. Janet MacDonald: *Conjugate nets in asymptotic parameters.*

This paper presents some contributions to the projective differential geometry of conjugate nets in asymptotic parameters on an analytic nonruled surface in ordinary space. The equation of the bundle of quadrics each of which has contact of at least the third order with both curves of a conjugate net at a point is deduced, and the cones in the bundle and the intersections of the bundle with the tangent plane are studied. Certain polar relations with respect to the quadrics of the bundle are presented. Certain loci and envelopes are investigated at a point of the surface in connection with a pencil of conjugate nets. Among these are the principal cubic, which is the locus of the principal points, and the principal conic, which is the envelope of the principal joins. Davis (*Contributions to the theory of conjugate nets*, doctoral dissertation, Chicago, 1932) defined and studied several canonical configurations, considering the conjugate net as parametric. In the present paper a study is made of Davis's canonical configurations in asymptotic parameters. (Received October 21, 1943.)

47. E. J. Purcell: *Variety congruences of order one in n-dimensional space.*

A variety congruence of order one in $[n]$ is an algebraic ∞^k-system of varieties, each of dimension $n-k$ and order h, in n-dimensional projective space, such that through a generic point of $[n]$ one and only one V_{n-k}^{h} of the system passes (k any positive integer not greater than n, and h any positive integer). The results of very many writers on Cremona transformations, Cremona involutions, $(1, m)$ correspondences, and line or curve congruences of order one can be obtained by specializing this paper. (Received October 7, 1943.)

TOPOLOGY

48. K. T. Millsaps: *A note on generalized Hilbert space.*

By analyzing an example formulated by A. Tychonoff, Math. Ann. vol. 111 (1935) p. 768, the spaces H^p, $0 \leq p \leq \infty$, are defined in a manner analogous to that for classical Hilbert space; some basic properties such as linearity, necessary and sufficient conditions for normability, completeness, separability, and sufficient conditions for local non-convexness are proved. (Received October 30, 1943.)

49. R. L. Moore: *Concerning webs in the plane.*

Among other things it is shown that if a compact plane continuum contains a web it is one. (Received November 25, 1943.)

50. M. E. Shanks: *Monotone decompositions of continua.*

In this paper the author considers the lattice $\mathfrak{D}_m(X)$ of all monotone upper semi-continuous decompositions of the compactum X. This lattice is well suited for the study of the structure of continua. Two continua X and Y are homeomorphic if and only if there is an isomorphism carrying $\mathfrak{D}_m(X)$ onto $\mathfrak{D}_m(Y)$ which makes simple decompositions correspond to simple decompositions. By means of a factorization theorem it is shown that if X is a dendrite or a linear graph then $\mathfrak{D}_m(X)$ is isomorphic to $\mathfrak{D}_m(A)$, where A is an arc. Spaces for which $\mathfrak{D}_m(X)$ is isomorphic to $\mathfrak{D}_m(A)$ are hereditarily locally connected. A class of continua called generalized dendrites is defined and characterized as those continua for which $\mathfrak{D}_m(X)$ is a sublattice of the lattice of all upper semi-continuous decompositions of X. Both dendrites and Knaster continua are generalized dendrites, and both make $\mathfrak{D}_m(X)$ distributive. A characterization of dendrites is obtained. (Received October 22, 1943.)

51. G. E. Schweigert: *Periodic analysis for onto-homeomorphisms of semi-locally-connected continua.* Preliminary Report.

$T(S)=S$ denotes any homeomorphism of a semi-locally-connected continuum. The least (invariant) A-set A which contains all cyclic elements of finite period is non-vacuous. Moreover $T(A)=A$ is componentwise periodic, that is, components of the complements of invariant A-sets in A have finite periods. It is shown that the action of $T(A)=A$ resembles that of an elementwise periodic homeomorphism (each cyclic element has finite period) provided certain free cyclic chains are used instead of cyclic elements. This involves a classification of the orbits of cyclic elements E with infinite period by means of the integer n for which the cyclic chain $C(E, T^n(E))$ has no invariant cyclic element (a property first studied by Ayres). It is also shown that the orbits of the components of $S-A$ are influenced by the periodicity in A and hence a pattern is formed for the action of $T(S)=S$ in the large. (Received October 21, 1943.)

52. G. T. Whyburn: *Mapping classes for locally connected continua.*

If A and B are locally connected continua, conditions are developed under which the limit mapping of a uniformly convergent sequence of continuous transformations of A onto B will be monotone, interior or quasi-monotone respectively. In particular it is shown that the limit mapping of any uniformly convergent sequence of quasi-monotone mappings of A onto B is itself quasi-monotone. Thus the class of all quasi-monotone mappings of A onto B is closed in the space B^A of all mappings of A onto B. It results from this that the limit of a uniformly convergent sequence of interior mappings of A onto B will be interior provided it is light. (Received October 22, 1943.)

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WISHART, J. See PEARSON, E. S.

TWO WORKS ON ITERATION AND RELATED QUESTIONS

J. HADAMARD

I want to give a brief account of results which have been communicated to me and which have been obtained by two young geometers, Eri Jabotinsky and Michel Lüntz. Mr. Lüntz is unhappily in a concentration camp in France. I must add that this is one of the reasons for the fact, for which I must apologize, that I am writing this exposé only now, although I have had both works in my possession for several months. The impossibility of communicating with Mr. Lüntz and therefore of asking him for any explanation made the examination of his paper especially difficult.

As is well known, the problem of iteration, a function $f(x)$ being given, consists in finding a one-parameter family of functions $f_n(x) = f(n,x)$ such that, for $n=1$, we have

$$f(1, x) = f(x) \tag{1}$$

and, moreover, for any m, n,

$$f_m[f_n(x)] = f_{m+n}(x); \tag{2}$$

also

$$f(0, x) = x \tag{3}$$

which follows from (2) as is seen by taking $m=0$.

The question is connected with Abel's functional equation

$$\phi[f(x)] = 1 + \phi(x) \tag{A}$$

because if ϕ is a solution of that equation a solution of (1), (2) will be given by

$$\phi[f_n(x)] = n + \phi(x). \tag{4}$$

Instead of (A), one can introduce Schroeder's equation

$$\psi[f(x)] = k\psi(x) \tag{A$'$}$$

(k a constant) in which ψ is connected with the unknown ϕ of (A) by $\psi = k^\phi$, and with the help of which the solution would be expressed by

$$\psi[(f_n(x)] = k^n\psi(x).$$

The fact that every solution of (A) gives a solution of the iteration

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problem is sufficient to show that the latter is indeterminate unless proper conditions are added: for, if $\phi(x)$ is a solution of (A), so is also $\phi(x)+\Pi[\phi(x)]$ where Π is any function admitting of the period 1.

The iteration problem has been considered by several authors. The most recent work, as far as I know, is due to Paul Lévy (C. R. Acad. Sci. Paris vol. 184 p. 500 and Annali di Matematica (4) vol. 5 (1928) pp. 270–296, esp. p. 287 ff.). Additional conditions being necessary (as a matter of fact, conditions of regularity), Paul Lévy imposes such conditions either (1) at infinity—a difficult subject, which we shall leave aside—or (2) in the neighborhood of a determinate value, which the successive iteratives of f are assumed to approach. In the latter case, the one which will interest us, he uses the classic studies of Koenigs (Ann. École Norm. 1884).

One feature of the problem is not considered by the above mentioned authors, namely that, on account of (2), the set of the various transformations $x \rightarrow f(n, x)$ is a one-parameter group, which can be defined by its infinitesimal transformation $\delta x=\xi(x)\delta t$. The question has been faced from this point of view in a note of Bouton[1] for the case in which the transformations are assumed to be tangent to the identical one. Moreover, that group aspect and the infinitesimal transformation have also been introduced in the fundamental memoir of G. D. Birkhoff,[2] though without playing an essential role.

Jabotinsky begins by considering not one problem of iteration, but the general class of them, that is, the equations (2) and (3) without immediate consideration of any special function $f(x)$ nor, therefore, of equation (1). Now all these problems satisfy a common partial differential equation. For, writing (2) in the form

$$f[(m-n), f(n, x)] = f(m, x)$$

and expressing the fact that the left-hand member does not depend on n, Jabotinsky easily obtains the partial differential equation

$$\text{(E)} \qquad (\partial^2 f/\partial n^2)(\partial f/\partial x) - (\partial^2 f/\partial n \partial x)(\partial f/\partial n) = 0.$$

Conversely, this equation admits of the two intermediate integrals

$$\text{(5)} \qquad (\partial f/\partial n):(\partial f/\partial x) = \xi(x),$$

$$\text{(6)} \qquad \partial f/\partial n = \eta(f)$$

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[1] Bull. Amer. Math. Soc. vol. 23 (1917) p. 75: A scanty note, the development of which has not been published as far as we know.

[2] Acta Math. vol. 43 (1922) p. 1. As is well known, that memoir deals with transformations in two variables, the existence of an integral invariant being assumed. See also the paper of Daniel C. Lewis, Duke Math. J. vol. 5 (1939) p. 794 in connection with G. D. Birkhoff's ideas.

(ξ, η, arbitrary functions) so that its general solution is

$$f(n, x) = \chi[n + \phi(x)] \qquad \left(\phi(x) = \int dx/\xi(x), \chi \text{ arbitrary}\right). \tag{7}$$

If we add condition (3), we see that χ must be the inverse function of ϕ, so that the above formula brings us back to (A). Thus we have a simple proof of the fact that not only every solution of (A) gives us a solution of (2), (3), but that, conversely,[3] every solution of (2) corresponds to a solution of (A).

The partial differential equation (E) is common to every problem of the kind which interests us, whatever the given function $f(x)$ may be.

In order to integrate that equation, (3) gives the condition $f(0, x) = x$; and, if we now introduce (1), we see that we are also given the values of $f(n, x)$ for $n = 1$. It is a boundary problem of an unusual form; and, indeed, we know that our problem is not determinate if we do not add complementary conditions of regularity.

As was already done in the aforesaid works of Koenigs and Paul Lévy, Jabotinsky investigates what will happen in the neighborhood of a "double point," that is of a value of x—say $x = 0$—such that

$$f(x) = x, \tag{8}$$

and he limits himself to iterates $f(n, x)$ of $f(x)$ such that every $f(n, 0)$ is also equal to 0, a hypothesis which, as we shall see, hardly diminishes generality. Moreover, he assumes f to be expansible in powers of x and tries to find a corresponding expansion for f_n, say

$$f_n(x) = A_1(n)x + A_2(n)x^2 + \cdots. \tag{9}$$

This is what Paul Lévy has carried out in his note in the C. R. Acad. Sci. Paris for the case of $f(x) = e^x - 1$ (of which Jabotinsky was not informed). For that purpose, Jabotinsky substitutes (9) into (E) or, preferably, into the intermediate integral (5). If $\xi(x) = \alpha_1 x + \alpha_2 x^2 + \cdots$ were known, this would give, for the $A_i(n)$, the successive linear differential equations of the first order

$$\begin{aligned} A_1'(n) &= \alpha_1 A_1(n), \\ A_2'(n) &= 2\alpha_1 A_2(n) + \alpha_2 A_1(n), \\ A_3'(n) &= 3\alpha_1 A_3(n) + 2\alpha_2 A_2(n) + \alpha_3 A_1(n), \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots \end{aligned} \tag{10}$$

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[3] Around a double point (which case we consider below), it is necessary to replace (A) by Schroeder's equation (A′), as does Koenig in the above cited memoir.

to be integrated with the initial conditions $A_1(0)=1$, $A_i(0)=0$ for $i \geqq 2$—an elementary integration.

Now, on account of (3), $\xi(x)$ is nothing else than the derivative of $f(n, x)$ with respect to n for $n=0$, so that if ξ were known, this would come back to solving Cauchy's problem for (E). From the point of view of group theory, *$\xi(x)$ corresponds to the infinitesimal transformation of the group which interests us.*[4]

As a matter of fact, in our case the α's are auxiliary unknowns; they are to be determined by the knowledge of the values of the A's for $n=1$, which proceeds without any difficulty, for each i, after the integration of the successive equations (10). If $\alpha_1 \neq 0$, that is if $a_1 = A_1(1) \neq 1$, the expressions thus obtained obviously contain n through polynomials in powers of $e^{\alpha_1 n} = a_1^n$, so that they are quite similar to those which G. D. Birkhoff has constructed in the case which he has investigated.

The case $a_1=1$ is comparable to Birkhoff's case II$''$, the successive unknowns A in (10) being polynomials in n.

But, as in Birkhoff's case, there remains the question whether the A_i' thus calculated would give a convergent series for (9); and, precisely, Birkhoff has shown, by a remarkable example (see §31, p. 55 of his memoir), that the contrary case may occur in the problem which he has treated. Whether the same fact is possible for the transformations in one variable remains to be decided.

Lüntz's starting point is slightly different: he investigates "mutually reversible" functions, two functions f and g being said to be reversible to each other[5] if

$$f[g(x)] = g[f(x)]. \tag{11}$$

He points out that this is a kind of generalization of fractional iteration; and, indeed, we see at once by (2) that any two of the f_n are reversible to each other. Conversely, will the investigation of functions g reversible to a given function f always lead to the iterates of f?

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[4] If the transformation $x \rightarrow f(x)$ were known to belong to a certain group of Lie, G_r, in a finite number r of parameters, our question would be considered as being treated by Élie Cartan in his memoir on *Geométrie des groupes de transformations* (Annali di Matematica (9) vol. 6 (1927) p. 1). In Cartan's terminology, a one-parameter group in G_r is the analogue of a geodesic in the G_r-space, which would be defined by its origin (the identical transformation) and its initial direction (defined by the corresponding infinitesimal transformation); our problem would correspond to drawing a geodesic through two given points. The solution is elementary in Cartan's case, but of a quite different difficulty when no G_r containing the given transformation is known.

[5] Lüntz is right in not using the word "permutable," which is used by Volterra in the *Functional calculus* with a different meaning. On the contrary, the transformations $x \rightarrow f(x)$ and $x \rightarrow g(x)$ are said to be permutable.

We shall presently see that, under the above postulated regularity conditions, such will be the case. But it must be noticed that this new problem is, from a certain point of view, simpler than the former since, instead of a family of functions, we investigate a single function g.

As was done above, Lüntz also starts from a double point x_0 of f, say $x_0=0$. Now, taking $x=x_0=0$ in the fundamental relation (11), we see that $g(x_0)$ must be a double point of f also, so that Lüntz is led to admit that $g(x_0)=g(0)$ is nothing else than x_0 itself.

Lüntz also remarks that if g_1 and g_2 are both reversible to f, so is also $g_1[g_2(x)]$, so that (a well known fact in the transformation theory) the g's are the elements of a group. He, moreover, supposes that such functions g_1 and g_2 are also reversible to each other, which we shall see to be the case.[6]

Again, taking $x_0=0$ and representing f and g by expansions in powers of x,

$$f(x) = a_1x + a_2x^2 + \cdots, \tag{12}$$

$$g(x) = b_1x + b_2x^2 + \cdots, \tag{13}$$

(11) gives the relations

$$\begin{aligned} &a_1b_1 = b_1a_1, \qquad a_1b_2 + a_2b_1^2 = b_1a_2 + b_2a_1^2,\\ &a_1b_3 + 2a_2b_1b_2 + a_3b_1^3 = b_1a_3 + 2b_2a_1a_2 + b_1a_1^3,\\ &a_1b_4 + a_2(2b_1b_3 + b_2^2) + 3a_3b_1^2b_2 + a_4b_1^4\\ &\qquad\qquad = b_1a_4 + b_2(2a_1a_3 + a_2^2) + 3b_3a_1^2a_2 + b_4a_1^4\\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \end{aligned} \tag{14}$$

The first one being an identity, we see that if a_1 is neither equal to 0 (a case which the author excludes) nor to 1, the coefficient b_1 is arbitrary and may be considered as an arbitrary parameter λ, by means of which we have the general form for g,

$$\begin{aligned} g(x) &= g_\lambda(x)\\ &= \lambda x + (\lambda - \lambda^2)\{\beta_2x^2 + [\beta_3(1+\lambda) + \beta_2^2(1-\lambda)]x^3 + \cdots\}. \end{aligned}$$

The one parameter family represented by the above expression for g can not be distinct from the group (9), as every iterate of f necessarily satisfies (11)

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[6] Lüntz notices the consequence that if $f(x)$ is odd—that is permutable with $x \to (-x)$—so is also $g(x)$.

Moreover, the above expression of the unknown is not written in terms of the a's but, as previously, implies an auxiliary set of parameters β, the meaning of which would not be obvious by itself, but appears by comparison with Jabotinsky's result. Indeed, giving the terms of the first degree in $\lambda-1$, they are, but for numerical factors, nothing else than the α in (10), so that the above formula leads us back to Jabotinsky's calculation.

Again, these parameters are to be found in terms of the a's, and the convergence question is left aside.

If we assume a_1 to be equal to 1 and if $a_2 \neq 0$, the coefficient b_1 will no longer be arbitrary: it must be equal either to 0 or to 1, so that the expansion of g will be ($b_1=0$ being again excluded) of the form $x+\cdots$, and will again include an arbitrary parameter λ, namely the coefficient b_2 of x^2.

The third, fourth, $\cdots$ equations (14) allow us to calculate the successive b's. Again Lüntz is led to introduce a set of parameters which coincide with the coefficients in the infinitesimal transformation of the group and, therefore, to write the same formula as would be found by Jabotinsky's method.

Consequently, we see that for $a_1 \neq 0$, 1, as for $a_2 \neq 0$, the family of functions reversible to f and satisfying our regularity conditions is not distinct from the family of iterates of f.

These are the results obtained by Jabotinsky and Lüntz; they coincide with each other, at least if we do not have $a_1=0$ or $a_1=1$, $a_2=0$, though they are deduced from rather different points of view.

This coincidence suggests several remarks, some of which I shall point out now.

I. In what precedes, we have started from the hypothesis that a double point $x=x_0$ of $f(x)$ is common to all iterates $f(n, x)$ or to all the functions $g(x)$ reversible to f. As we have seen, Lüntz makes the important remark that, on account of the definition (11) of reversible functions, the transform $g(x_0)$ is also a double point of f. He leaves, however, one question open, namely whether this transform is x_0 itself or another solution of (8). But, as a matter of fact, this question can be answered easily if the function g is assumed to belong to a continuous family of functions all of which are reversible to f, one of them being $g(x)=x$; for, substituting x_0 in each of these functions, we obtain results which, varying continuously, can not be different from x_0 itself, at least if the latter is an isolated zero of $f(x)-x$.

This conclusion immediately applies to the iteration problem, as any solution of it is a solution of Lüntz's problem, so that at least some information on this subject is obtained by Lüntz's way of attacking it.

II. Is the above assumption—namely that either $a_1 \neq 0$, 1 or $a_1 = 1$, $a_2 \neq 0$—essential for the general behaviour of the results and especially for the equivalence of those problems?

This would seem to be presumable a priori: for it is evident that if the function $f(x)$ would reduce to x itself, Lüntz's problem would become fully indeterminate, as the function $f(x) = x$ is reversible to every other one.

A most remarkable fact is that this presumption is not verified. Let the first term in (12) be reduced to x, the first following term which does not vanish being of degree p, so that

$$f(x) = x + a_p x^p + \cdots \qquad (p \geqq 2,\ a_p \neq 0).$$

Equating the coefficients of x^p in both members of (11), we immediately see that $b_1 = b_1^p$. Let us take, therefore,[7] $b_1 = 1$, so that the expansion of $g(x)$ is also of the form

$$g(x) = x + b_1 x^r = \cdots \qquad (r \geqq 2,\ b_r \neq 0).$$

If so, all terms issuing from the first term of (12) or from the first term of (13) annul each other. After them, the terms of lowest degree are in x^{p+r-1}, with coefficients respectively equal to pa_pb_r and ra_pb_r in both members, so that r can not be different from p. Then, b's with suffixes greater than p would be determined by equations some of which (except for $p = 2$) will reduce to the form $a_pb_{p+q} = b_pa_{p+q}$ while, for $p = 2$, we shall have

$$a_2b_3 - a_3b_2 = a_2b_2(b_2 - a_2).$$

We see that, in every case, the functions reversible to $f(x)$ and regular at $x = 0$ constitute a one-parameter family, which necessarily coincides with the iterates of f.

We also see that Lüntz was right in supposing that two functions reversible to the same third one are reversible to each other. This, as is classic in the group theory, carries the consequence that the reversibility conditions can be expressed by the existence of *invariants*. Let c_1 be chosen once for all, different from 0 and 1: we can consider as a "reduced" form of $g(x)$ the one which has c_1 as coefficient of x. Every other coefficient in that reduced expansion will be determined by these conditions and, on the other hand, we just saw that the reduced g will be reversible not only to f, but also to any other function (13) reversible to f, so that such successive coefficients of the reduced expansion, when calculated in terms of the a's, must have the same values as the ones similarly calculated in terms of the b's: they give invariants (which, obviously, only seemingly depend on the special value of c_1). The first of them are

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[7] Provisionally, we still exclude, like Lüntz, the value zero and, to simplify discussion, do not consider roots of unity. At any rate, in iteration problems it is natural (cf. Birkhoff, loc. cit. p. 9) to eliminate negative first coefficients.

$$j_2 = a_2:(a_1 - a_1^2), \qquad j_3 = (a_3 - 2a_2 j_2):(a_1 - a_1^3),$$
$$j_4 = \left\{a_4 - \left[(3a_1 + 5)j_3 + 5j_2^2\right]a_2\right\}:(a_1 - a_1^4),$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

Such invariants are evidently related to the coefficients of the infinitesimal transformation of the group—a connection which, however, we shall not investigate.

In the case, considered in II, of $a_1 = 1$, the first invariant, for $p = 2$, is

$$a_3/a_2 - a_2 \tag{15}$$

while, for $p \geqq 2$, one or several invariants will be of the form $a_{p+q}:a_p$, after which there will come one analogous to (15).

III. The transformations $S:x \to f(x)$, $T:x \to g(x)$, where f and g are connected by Lüntz's relation (11), being permutable to each other, the transform of T by S coincides with T. In other words, the curve $y = g(x)$ is invariant by the point transformation $X = f(x)$, $Y = f(y)$. The general problem of invariant points has been studied several times, especially by Poincaré, Samuel Lattès,[8] G. D. Birkhoff (loc. cit.) and the present author; but the present transformation behaves in an abnormal manner from this point of view, as there exist an infinite number of invariant curves, namely every nth iterate $y = f_n(x)$, such iterates existing at least for every integral n.

IV. The problem of reversible functions being connected with the iteration problem, we need not wonder at seeing it connected with Abel's equation (A). As a matter of fact, this connection is a very simple one. Let ϕ be any solution of (A) and let us apply the operation ϕ to both members of (11). We immediately see that the function

$$\Phi(x) = \phi[g(x)] \tag{16}$$

is again a solution of (A). Conversely, if ϕ and Φ be two solutions of

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[8] Poincaré, J. Math. Pures Appl. (4) vol. 1 (1885); Hadamard, Bull. Soc. Math. France vol. 29 (1901); Samuel Lattès, Annali di Matematica (3) vol. 13 (1907).

(A), the function g defined by (16)—in other words, $g(x)=\phi^{-1}[\Phi(x)]$—is reversible to f. Any determinate function $g(x)$ permutable with $f(x)$ can be obtained in an infinite number of ways by this procedure.

V. We have noticed—see footnote (4)—that if we knew a finite Lie group G containing the transformation $x\rightarrow f(x)$, we could deduce therefrom the one-parameter group (9) containing that same transformation. Now, in G. D. Birkhoff's example, the one-parameter group analogous to (9) does not exist (otherwise than formally). This would suggest that the corresponding transformation is contained in no finite Lie group. However, there is the objection that the deduction of (9) from G, by Elie Cartan's method, might be valid only in a local domain, Birkhoff's transformation lying beyond that domain.

VI. The case $a_1=0$, which Lüntz has explicitly excluded, is also implicitly excluded from Jabotinsky's study. Indeed, the impossibility of any regular solution of the form (9) at once appears from the first equation (10); as for any solution of that equation, the conditions $A_1(0)=1$ and $A_1(1)=0$ are contradictory. Therefore, no function which is regular around $x=0$ and whose expansion begins by a term of degree 2 or higher can belong to a regular continuous one-parameter group of Lie; nor can it, as we saw, belong to a regular finite group in any number of parameters.

Now, such a function f of that kind being given, let us consider the reversibility problem. In the expansion (13) of a function g reversible to f, we shall have, as previously, either

(1) $b_1=1$: one easily sees that this gives nothing else than $g(x)$ identically equal to x; or

(2) $b_1=0$: corresponding solutions are evidently $g(x)$ equal to $f(x)$ itself or to one of its iterates.

But there may be solutions other than these trivial ones, for the two functions $f(x)=x^m$ and $g(x)=x^n$ are obviously reversible. From those, or more generally from $f(x)=ax^m$ and $g(x)=bx^n$ (which are reversible if $ab^m=ba^n$), we obviously could deduce other ones seemingly, but not essentially, distinct from them by applying to $x, f(x)$ and $g(x)$ a common point transformation, regular and one-one around $x=0$. Whether this would afford the most general solution relating to our present case, $a_1=0$, remains to be investigated.

Another curious question would arise if equation (8) were to admit two different roots x_0, x_1 around which expansions were to behave differently from the expansions considered above—for example $f'(x_0)=0$, $f'(x_1)=0$—giving rise possibly to special kinds of singularities.

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ON THE EXTENSION OF DIFFERENTIABLE FUNCTIONS

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The author has shown previously how to extend the definition of a function of class C^m defined in a closed set A so it will be of class C^m throughout space (see [1]).[1] Here we shall prove a uniformity property: If the function and its derivatives are sufficiently small in A, then they may be made small throughout space. Besides being bounded, we assume that A has the following property:

(P) There is a number ω such that any two points x and y of A are joined by an arc in A of length less than or equal to ωr_{xy} (r_{xy} being the distance between x and y).

This property was made use of in [2]; its necessity in the theorem is shown by two examples below.

A second theorem removes the boundedness condition in the first theorem, and weakens the hypothesis (P); its proof makes use of the proof of the first theorem. We remark that in each theorem, as in [1], the extended function is a linear functional of its values in A.

The proof of Theorem I is obtained by examining the proof in [1]; hence we assume that the reader has this paper before him, and we shall follow its notations closely.

THEOREM I. *Let A be a bounded closed set in n-space E with the property* (P), *and let m be a positive integer. Then there is a number α with the following property. Let $f(x)$ be any function of class C^m in A, with derivatives $f_k(x)$ $(\sigma_k = k_1 + \cdots + k_n \leqq m)$. Suppose*

$$|f_k(x)| < \eta \qquad (x \in A, \sigma_k \leqq m).$$

Then $f(x)$ may be extended throughout E so that

$$|f_k(x)| < \alpha\eta \qquad (x \in E, \sigma_k \leqq m).$$

Let d be the diameter of A, or 1 if this is larger, and let R be a spherical region of radius $2d$ with its center at a point of A. Set $f(x) = 0$ in $E - R$. Then the extension of f in $R - A$ given in [1] will be shown to have the property, using

$$\alpha = 2n(m!)^n(m+1)^{3n}(433n^{1/2}d\omega)^m cN,$$

where N and c are as given in [1, §§11, 12]. Note that $433 = 4 \cdot 108 + 1$.

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[1] Numbers in brackets refer to the references cited at the end of this paper.

Set $B=A\cup(\bar{R}-R)$. We show first that for any points x', x'' of B,

$$|R_k(x';x'')| < \beta r_{x'x''}^{m-\sigma_k}\eta, \qquad \beta = 2n(m+1)^n\omega^m.$$

Suppose first that x' and x'' are in A. Let C be a curve in A joining them, of length less than or equal to $\omega r_{x'x''}$. The inequality is then a consequence of [2, Lemma 3]. Suppose next that one of the points is in A, and the other is in $\bar{R}-R$ (the case that both are in $\bar{R}-R$ is trivial). By [1, (3.1)], since $r_{x'x''}\geqq d\geqq 1$,

$$|R_k(x';x'')| \leqq \eta + \sum_{\sigma_l\leqq m-\sigma_k} \eta r_{x'x''}^{\sigma_l} \leqq r_{x'x''}^{m-\sigma_k}\eta\left[1+\sum_{\sigma_l\leqq m-\sigma_k} 1\right]$$

$$\leqq (m+1)^n r_{x'x''}^{m-\sigma_k}\eta.$$

Now take any x in $R-B$. Let $\delta^*/4$ be the distance from x to B, and let x^* be a point of B distant $\delta^*/4$ from x. Say x is in the cube C of the set of cubes K_s; let $I_{\lambda_1},\cdots,I_{\lambda_t}$ be those I_λ with points in C (see [1, §11]). Now y^ν is the center of I_ν, and x^ν is a nearest point of B to y^ν. As noted in [1, (9.1)], $r_{y^\nu x^*}$ and $r_{y^\nu x^\nu}$ each lie between $\delta^*/8$ and $\delta^*/2$. Since $r_{xy^\nu}<\delta^*/2$, we have

$$r_{x^\nu x^*} < \delta^*, \qquad r_{xx^\nu} < \delta^*.$$

The definition of ζ in [1, §11] together with [1, (6.3)] gives

$$\zeta_{\nu;k}(x) = \psi_k(x;x^\nu) - \psi_k(x;x^*) = \sum_{\sigma_l\leqq m-\sigma_k} \frac{R_{k+l}(x^\nu;x^*)}{l!}(x-x^\nu)^l.$$

Hence

$$|\zeta_{\nu;k}(x)| < (m+1)^n\beta r_{x^\nu x^*}^{m-\sigma_k-\sigma_l} r_{xx^\nu}^{\sigma_l}\eta < (m+1)^n\beta\delta^{*m-\sigma_k}\eta.$$

Following [1, §11] still, we find

$$|D_kg(x)-\psi_k(x;x^*)| < c\sum_{\sigma_l\leqq m-\sigma_k}(m!)^n 2^{s\sigma_l}N(m+1)^n\beta\delta^{*m-\sigma_k+\sigma_l}\eta.$$

As in [1], $2^s<108n^{1/2}/\delta^*$; hence

$$|D_kg(x)-\psi_k(x;x^*)| < c(m!)^nN(m+1)^{2n}(108n^{1/2})^m\beta\delta^{*m-\sigma_k}\eta.$$

Moreover, since $r_{xx^*}<3d$, [1, (6.1)] gives

$$|\psi_k(x;x^*)| < 3^m(m+1)^nd^m\eta.$$

Since $\delta^*\leqq 4d$ and $f(x)=g(x)$ in $R-B$, the theorem follows.

We turn now to the second theorem. We shall say A *satisfies* (P) *locally* if for each $x\in A$ there is a neighborhood U of x and a number

ω such that any two points y and z of $A\cap U$ are joined by an arc in A of length not greater than ωr_{zy}.

THEOREM II. *Let A be a closed subset of an open set R in E, satisfying (P) locally, and let m be a positive integer. Then for any continuous function $\epsilon(x)$ defined and greater than 0 in R there is a continuous function $\delta(x)$ defined and greater than 0 in A with the following property. Let $f(x)$ be any function of class C^m in A, such that*

$$|f_k(x)| < \delta(x) \qquad (x \in A,\ \sigma_k \leqq m).$$

Then $f(x)$ may be extended throughout R so that

$$|f_k(x)| < \epsilon(x) \qquad (x \in R,\ \sigma_k \leqq m).$$

REMARKS. The preceding theorem is easily seen to be a consequence of this one. The present theorem holds if E is replaced by a differentiable manifold M, in which a fixed set of coordinate systems (each one intersecting but a finite number of others) is used to measure the size of derivatives. To show this, we imbed M in a Euclidean space E' (see [3, Theorem 1]), giving $A\subset R\subset R'\subset E'$ (R' open in E'; we let R' contain no points of the limit set of M), extend f throughout R' (see the proof of [3, Lemma 4]), and consider its values in R.

To prove the theorem, we begin by choosing spherical regions $U_1, U_2, \cdots$, each $\overline{U}_i\subset R$, with the following properties:

(a) Each U_i is in a neighborhood U as described above.

(b) Each $\overline{U}_i$ intersects but a finite number of other $\overline{U}_j$.

(c) If U_i is of radius ρ_i, and U_i' is the concentric region of radius $\rho_i/2$, then $R=\sum U_i'$.

Let $\psi^i(x)$ be a function of class C^m in E such that

$$\psi^i(x) > 0 \qquad (x \in U_i'),$$
$$\psi^i(x) = 0 \qquad (x \in E - U_i').$$

Set

$$\phi^i(x) = \psi^i(x)/\sum \psi^j(x) \qquad (x \in R);$$

then $\phi^i(x)$ is of class C^m in R, and

$$\phi^i(x) = 0 \qquad (x \in R - U_i'),$$
$$\sum \phi^i(x) = 1 \qquad (x \in R).$$

The extension of $f(x)$ is defined as follows. Set

$$f^i(x) = \phi^i(x)f(x) \qquad (x \in A),$$
$$f^i(x) = 0 \qquad (x \in R - U_i).$$

Then f^i is of class C^m in $A\cup(R-U_i)$. Extend it as in [1] (using a fixed subdivision of $U_i - A$; we could set $f^i(x) = 0$ in $E - R$) to be of class C^m in R (or E). (Note that if $A \cap U_i' = 0$, then $f^i(x) = 0$, $x \in R$.) Set

$$f(x) = \sum f^i(x) \qquad (x \in R).$$

Then f is an extension of class C^m of its values in A. We must show that it satisfies the condition of smallness.

Choose $a_i \geqq 1$ so that

$$\left| \overset{i}{\phi}_k(x) \right| \leqq a_i \qquad (x \in R,\ \sigma_k \leqq m),$$

then if $|f_k(x)| < \eta$ $(x \in A \cap U_i')$,

$$\left| \overset{i}{f}_k(x) \right| = \left| \sum_{\sigma_l \leqq \sigma_k} \overset{i}{\phi}_l(x) f_{k-l}(x) \right| \leqq (m+1)^n a_i \eta \qquad (x \in A).$$

By the choice of U_i, there is an ω_i such that any x' and x'' in $A \cap U_i$ are joined by an arc in A of length not greater than $\omega_i r_{x'x''}$. Set $\sigma_i = \max(1, 2/\rho_i)$. If $\overset{i}{R}_k$ is the remainder for $\overset{i}{f}_k$, we shall show that for any x' and x'' in $A \cup (R - U_i)$,

$$\left| \overset{i}{R}_k(x'; x'') \right| < 2n(m+1)^{2n} \omega_i^m a_i \sigma_i^m r_{x'x''}^{m-\sigma_k} \eta.$$

If x' and x'' are both in U_i, we apply [2, Lemma 3]. If $x' \in R - U_i$ and $x'' \in U_i'$, or vice versa, then $r_{x'x''} \geqq \rho_i/2$, and the proof in the preceding theorem applies; we consider separately the cases $\rho_i/2 \geqq 1$, $\rho_i/2 < 1$, using $r_{x'x''} \geqq 1$ and $\sigma_i r_{x'x''} \geqq 1$ respectively. If $x' \in R - U_i$ and $x'' \in R - U_i'$, or vice versa, $\overset{i}{R}_k = 0$, since $\overset{i}{\phi}_l(x') = \overset{i}{\phi}_l(x'') = 0$. The proof of the preceding theorem now shows that for some α_i, if

$$\left| f_k(x') \right| < \eta \qquad (x' \in A \cap U_i',\ \sigma_k \leqq m),$$

then

$$\left| \overset{i}{f}_k(x) \right| < \alpha_i \eta \qquad (x \in R,\ \sigma_k \leqq m).$$

(We may set $\alpha_i = 1$ if $A \cap U_i' = 0$.)

Given $\epsilon(x)$, we determine $\delta(x)$ as follows. For each $x \in R$ there is a set of numbers $\lambda_1, \cdots, \lambda_s$, $s = s(x)$, such that

$$x \in \text{each } U'_{\lambda_j}, \qquad x \in \text{no other } U_i'.$$

Because of (b), s is finite. Set $\alpha(x) = \alpha_{\lambda_1} + \cdots + \alpha_{\lambda_s}$.
We can clearly choose a continuous function $\beta(x)$ in R such that

$$\alpha(x) < \beta(x) \qquad (x \in R).$$

We may now choose a continuous function $\delta(x') > 0$ in A such that for any $x' \in A$, if $U'_{\mu_1}, \cdots, U'_{\mu_t}$ are the U'_j containing x', then

$$\delta(x') \leqq \epsilon(x)/\beta(x) \quad (x \in U'_{\mu_1} \cup \cdots \cup U'_{\mu_t}).$$

Now take any f of class C^m in A, with $|f_k(x)| < \delta(x)$ $(x \in A, \sigma_k \leqq m)$; the extension of f through R has been defined. Take any $x \in R$; define $\lambda_1, \cdots, \lambda_s$ as above. Then

$$|f_k(x')| < \delta(x') \leqq \epsilon(x)/\beta(x) \qquad (x' \in A \cap U'_{\lambda_j}, \sigma_k \leqq m),$$

and hence

$$|f_k^{\lambda_j}(x)| < \alpha_{\lambda_j}\epsilon(x)/\beta(x) \qquad (\sigma_k \leqq m).$$

Since $f_k(x) = f_k^{\lambda_1}(x) + \cdots + f_k^{\lambda_s}(x)$,

$$|f_k(x)| < \alpha(x)\epsilon(x)/\beta(x) < \epsilon(x)$$

for $\sigma_k \leqq m$, which completes the proof.

EXAMPLES. (1) Let A consist of a point, together with a sequence of points approaching it. Letting $f(x) = 1$ at a finite number of points of the sequence, and $f(x) = 0$ in the rest of A shows (with $m = 1$) that the theorem fails here.

(2) Let A be the closed region of the plane defined by (a) $x^2 + y^2 \leqq 1$, and (b) either $x \leqq 0$ or $|y| \geqq x^{3/2}$. Let $f(x, y) = 0$ if $x \leqq 0$, and set

$$f(x, y) = \begin{cases} \gamma x^2/(1 + \gamma^2 x^2) & \text{if} \quad x \geqq 0, \ y > 0, \\ -\gamma x^2/(1 + \gamma^2 x^2) & \text{if} \quad x \geqq 0, \ y < 0. \end{cases}$$

We see easily that f is of class C^1 in A. (It would not be if, in (b), we used $|y| \geqq x^2$.) The maximum $\partial f/\partial x$ occurs at $x = 1/(3^{1/2}\gamma)$, and has the value $9/(8 \cdot 3^{1/2})$. Set

$$p = (1/3^{1/2}\gamma, \ 1/3^{3/4}\gamma^{3/2}), \qquad q = (1/3^{1/2}\gamma, \ -1/3^{3/4}\gamma^{3/2}).$$

Then

$$\frac{f(p) - f(q)}{r_{pq}} = \frac{2\gamma/3\gamma^2}{1 + \gamma^2/3\gamma^2} \div \frac{2}{3^{3/4}\gamma^{3/2}} = \frac{3^{3/4}}{4}\gamma^{1/2}.$$

Hence, in any extension of f through the plane, we must have $|\partial f/\partial y| \geqq 3^{3/4}\gamma^{1/2}/4$ at some point (between p and q); yet $|f|$, $|\partial f/\partial x|$ and $|\partial f/\partial y|$ are uniformly bounded for all $\gamma > 1$. Taking γ arbitrarily large shows that the conclusion of the theorem does not hold.

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THE SYMMETRIC JOIN OF A COMPLEX

C. E. CLARK

1. **The definition of J.** Let K be a polyhedron. With each pair of distinct points p, q of K we associate a closed line segment pq. No distinction is made between p and q and the corresponding end points of pq. The length of pq is a continuous function of p and q, and the length approaches zero if p and q approach a common limit. Distinct segments do not intersect except at a common end point. The points of these segments with their obvious natural topology make up J, the symmetric join of K. This space arises in [4][1] in connection with the problem of finding the chords of a manifold that are orthogonal to the manifold.

2. **The subdivision of J.** Let the mid-point of pq be denoted by $\Lambda p \times q = \Lambda q \times p$, and let $p = \Lambda p \times p$. These points $\Lambda p \times q$ make up the symmetric product S of K. Let the mid-point of the segment from p to $\Lambda p \times q$ be denoted by $p \times q$, and let $p = p \times p$. These points $p \times q$ make up the topological product $P = K \times K$. Consider the closed segment of pq from $p \times q$ to $q \times p$, it being understood that this segment is the point p when $p = q$. All such segments form the "neighborhood" N_S. Clearly N_S can be homotopically deformed in N_S along the segments pq upon S with S remaining pointwise invariant. Finally consider the closed segment of pq from p to $p \times q$, it being understood that this segment is the point p when $p = q$. All such segments form the "neighborhood" N_K. Clearly N_K can be homotopically deformed in N_K along the segments pq upon K with K remaining pointwise invariant.

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[1] Numbers in brackets refer to the References at the end of the paper.

There is a natural cell division of P whose oriented cells are $\sigma\times\tau$ for all oriented simplexes σ, τ of K.[2] Let $|\sigma\times\tau|$ denote the carrier of $\sigma\times\tau$, that is, the point set of J associated with $\sigma\times\tau$. The rays pq determine a mapping Λ of P on S. The cells $\Lambda|\sigma\times\tau| = \Lambda|\tau\times\sigma|$ form a natural cell division of S.[3] With this division Λ is a cell mapping. Furthermore $\Lambda\sigma\times\tau$ is an orientation of $\Lambda|\sigma\times\tau|$ such that

$$\Lambda\sigma^s \times \tau^t = (-1)^{st}\Lambda\tau^t \times \sigma^s, \tag{1}$$

the superscripts denoting dimension.[3]

Theorem 1. *There is a simplicial division of J such that K, S, P, N_S, and N_K carry subcomplexes of the division, the first of which is a subdivision of the given polyhedron K, and the second and third of which are subdivisions of the above natural cell divisions of S and P.*

Proof. Let $[\sigma, \tau]$ be the point set made up of the points of all segments pq with $p\in\sigma$ and $q\in\tau$. If σ and τ have no common vertex, $[\sigma, \tau] = (\sigma, \tau)$, the join[4] of σ and τ. In this case S and P separate $[\sigma, \tau]$ into four $(s+t+1)$-cells, where s and t are the dimensions of σ and τ, respectively.

Consider $[\sigma, \sigma]$. As an auxiliary set we construct the join (σ, σ') with σ' homeomorphic to σ. Each pair of points, one in σ and one in σ', determines their join in (σ, σ'). This join of the two points is called a ray of (σ, σ'). The mid-points of the rays form a set homeomorphic to $\sigma\times\sigma'$ or $\sigma\times\sigma$. This product separates (σ, σ') into two cells. We discard the one of these cells that contains σ' and retain C, the closure of the other cell which contains σ, $\sigma\times\sigma$, and rays connecting these two sets. We subdivide $\sigma\times\sigma$ into a simplicial complex in such a way that the set of points $p\times p$, all p in σ, carries a subcomplex (that such a division exists is proved in [5]). Each simplex of this division is extended to a cell of C by adding to the simplex the points of all rays of C with one end point in the simplex (the set so obtained is a cell because it is obtained from a prism whose bases are simplexes by simple identifications in one base of the prism). These cells form a division of C into a cell complex. The rays of C that join p and $p\times p$, all p in σ, carry a subcomplex of this cell division. Let each such ray be reduced to a point by identification of all points of the ray. It is seen that the transform of C under this identification is a cell complex which is a division of $[\sigma, \sigma]\cap N_K$.

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[2] The properties of P which we use can be found in [3].

[3] See [5].

[4] A discussion of the join of two complexes is found in [3]. Also the properties of the join which we assume are presented in [1].

To obtain a division of $[\sigma, \sigma] \cap N_S$ we consider the prism D defined as the product of $\sigma \times \sigma$ and a 1-simplex. We use the same simplicial division of $\sigma \times \sigma$ used in the preceding paragraph. This division is extended to a cell division of D by adding to each simplex the product of the simplex and the 1-simplex used to define D. One base of the prism D is reduced to the symmetric product of σ by the identification of $p \times q$ and $q \times p$ for all p, q in σ. If our original simplicial division of $\sigma \times \sigma$ is properly determined (as in [5]), these identifications transform D into a cell complex E.[3] Next the rays of E which are images of the rays of D with end points $p \times p$, all p in σ, are reduced each to a point by identification. It is seen that the resulting space is a cell complex which is a division of $[\sigma, \sigma] \cap N_S$.

In the same way we subdivide $[\sigma, \tau]$ where σ and τ have some but not all of their vertices in common. It is seen that Theorem 1 is true.

3. Homologies in P and S. Let z_i, $i=1, 2, \cdots$, be the cycles of a canonical basis[5] for the chains of all dimensions of K, and let f_i be the non-cyclic chains of this canonical basis. The range of the subscript of f is a subset of the range of the subscript of z. For each f there is a boundary relation $Ff_i^{r+1} = e_i \overset{r}{z}_i$, $e_i \geqq 1$. With each z_i that is free there is associated an e_i equal to zero.

A homology base for P is given by the cycles $z_i \times z_j$ and $(1/(e_i, e_j))F(f_i \times f_j)$, the denominator of the coefficient denoting the greatest common divisor of e_i and e_j. A complete set of homology relations for the cycles of this base is given by the two sets

$$(e_i, e_j) z_i \times z_j \sim 0, \qquad F(f_i \times f_j) \sim 0, \tag{2}$$

with the understanding that $(a, 0) = a$.

We next obtain a homology base for S. We consider

$$\Delta \overset{r}{z}_i \times \overset{s}{z}_j = (-1)^{rs} \Delta \overset{s}{z}_j \times \overset{r}{z}_i, \qquad i \neq j, \tag{3}$$

the equality holding because of (1). The dimension superscripts in (3) could be r_i and r_j rather than r and s, but we use the less clear notation to simplify the printing. We also consider

$$\Delta \overset{r}{z}_i \times \overset{r}{z}_i, \qquad r \text{ even}, \tag{4}$$

$$\Delta \frac{1}{(e_i, e_j)} F(f_i^{r+1} \times f_j^{s+1}) = (-1)^{(r+1)(s+1)} \Delta \frac{1}{(e_i, e_j)} F(f_j^{s+1} \times f_i^{r+1}), \qquad i \neq j, \tag{5}$$

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[5] See [3, p. 104].

and

$$\frac{1}{2}\,\Delta\,\frac{1}{e_i}F(\overset{r+1}{f_i}\times\overset{r+1}{f_i}), \qquad r \text{ odd}. \tag{6}$$

Using [6, p. 22, line 15] and the method of [2, §4] we infer that the cycles (6) are integral, the orders of (3), (4), and (5) are (e_i, e_j) while (6) have the orders $2e_i$, the cycles (3), (4), (5), and (6) are a homology base for S, and the orders of these cycles are a complete set of homology relations in S.

4. **The sum cycles and their group** S. We shall apply the theory of the sum of two complexes to $J=N_S+N_K$. To prepare for this we define a sum cycle to be a cycle that is expressible as the sum of two cycles, one in N_S and the other in N_K. The group of the sum cycles modulo those that bound in J is called S. This group S is known to be a subgroup of the Betti group of J. Here, as throughout the paper, unless otherwise stated, we do not define a group for each dimension but combine all elements of all dimensions into a single group.

THEOREM 2. *The group S is generated by a free 0-dimensional element and a set of elements of order 2, each containing one and only one of the cycles* (6).

PROOF. Any cycle of K can be deformed in J onto a point p of K along the segments pq. Since N_K is a deformation retract of K, the group of the cycles of N_K modulo those that bound in J has a single nonzero 0-dimensional element as generator.

A cycle of S which bounds in J is homologous in N_S to a cycle of P. Indeed if $z=FC$, z in S, C in J, and C simplicial, then C can be expressed as the sum of two chains, one in N_S and the other in N_K. The boundary of the first of these two chains is z minus a cycle of P.

Any cycle z of P is homologous in N_S to Δz. Hence using (3), (4), (5), and (6) we obtain Theorem 2.

5. **The seams and their group** Γ. The seams are defined to be the cycles of P that bound both in N_S and N_K. We note that if z is a cycle of P and $z\sim 0$ in N_S, then since z and Δz are homotopic, $\Delta z\sim 0$ in N_S. But N_S was shown in §1 to be a neighborhood retract of S. Hence $\Delta z\sim 0$ in S. From this observation, from (1), and from §3 we deduce that the cycles of P that bound in N_S are the cycles homologous in P to all linear combinations of

$$(\overset{r}{z_i}\times\overset{s}{z_j})-(-1)^{rs}(\overset{s}{z_j}\times\overset{r}{z_i}), \qquad i\neq j, \tag{7}$$

$$(8) \qquad z_i^r \times z_i^r, \qquad r \text{ odd},$$

$$(9) \qquad \frac{1}{(e_i, e_j)}\left[F(f_i^{r+1} \times f_j^{s+1}) - (-1)^{(r+1)(s+1)}F(f_j^{s+1} \times f_i^{r+1})\right], \qquad i \neq j,$$

and

$$(10) \qquad \frac{1}{e_i}F(f_i^{r+1} \times f_i^{r+1}), \qquad r \text{ even}.$$

To find the cycles of P that bound in N_K we consider the transformation $M(p \times q) = p$. We have $M|\sigma \times \tau| = |\sigma|$, the notation being that of §2. Also for any set $A \subset P$ the discussion of §2 implies that A and $M(A)$ are homotopic in N_K. Hence $z_i^r \times z_j^s$ is homologous in N_K to $M(z_i^r \times z_j^s)$ which is carried by $|z_i^r|$. Hence if $s > 0$, $z_i^r \times z_j^s$ bounds in N_K because it can be homotopically deformed into a set of dimension less than the dimension of the cycle.

Suppose $s = 0$. We assume that z_j^0 is equal to 1 at exactly one vertex of K and is equal to 0 at all other vertices. Then $M(z_i^r \times z_j^0) = z_i^r$ because $z_i^r \times z_j^0$ is obtained by sliding z_i^r along the rays of a cone.

If in f_j^{s+1} the exponent s is greater than 0, $(1/(e_i, e_j))F(f_i^{r+1} \times f_j^{s+1}) \sim 0$ in N_K because it is homotopic to a cycle carried by $|f_i^{r+1}|$, and the dimension of the cycle is $r+s+1$ which is greater than $r+1$. If $s = 0$, the same homology holds because $e_j = 1$.

We have shown that the cycles of P that bound in N_K are the cycles homologous in P to all linear combinations of

$$z_i^r \times z_j^s, \qquad s > 0,$$

$$z_i^r \times z_1^0 - z_i^r \times z_j^0, \qquad i \neq 1,$$

and

$$\frac{1}{(e_i, e_j)}F(f_i^{r+1} \times f_j^{s+1}).$$

It is easily seen that if one cycle of a homology class of the Betti group of P is a seam, then all cycles of the class are seams. Furthermore such classes form a subgroup of the Betti group. This subgroup of the Betti group of P made up of the homology classes whose cycles are seams is called Γ. Combining the results of this section we obtain the following theorem.

THEOREM 3. *The subgroup* Γ *is generated by the homology classes that contain* (8), (9), (10),

$$(11) \qquad (7) \text{ with } r > 0 \text{ and } s > 0,$$

and

$$(12) \qquad (z_i^r \times z_1^0) - (z_i^r \times z_j^0) - (z_1^0 \times z_i^r) + (z_j^0 \times z_i^r), \qquad i \neq 1.$$

6. **The Betti group $B(J)$ of J.** The theory of the sum of two complexes[6] gives the isomorphism $B(J)/S \cong \Gamma$. However we shall obtain a sharper result.

For each seam z we shall define a pair of singular chains c of N_S and d of N_K such that $Fc = Fd = z$. To define c we deform z along the segments pq from P to S sweeping out the continuous image of a prism whose bases are $|z|$ and $|\Delta z|$. This deformation gives rise to a singular chain c whose boundary[7] is $z - \Delta z$. But since z is a seam it follows from (1), (8), (9), (10), (11), and (12) that $\Delta z = 0$. In a similar way we define the chain d of N_K.

Let $H(z)$ be the homology class of $B(J)$ which contains $c - d$. We know that any element of the Betti group $B(J)$ is expressible as the sum of an element of S and an $H(z)$, z a seam.[6] Hence to characterize $B(J)$ we need only find the relations that involve the $H(z)$. We know[6] that any homology among the $H(z)$ implies the same homology in P among the z. On the other hand, $z \sim 0$ in P implies[6] that $H(z)$ is in S. We shall examine all relations $z \sim 0$ in P, and find the corresponding relations among the $H(z)$. The relations to examine are (2) applied to (8), (9), (10), (11), and (12).

First we consider $(e_i, e_j)(11) \sim 0$ in P. We consider

$$\begin{aligned} C = {} & \alpha\big[(f_i^{r+1} \times z_j^{s}) - (-1)^{s(r+1)}(z_j^{s} \times f_i^{r+1})\big] \\ & + \beta\big[(-1)^{r}(z_i^{r} \times f_j^{s+1}) - (-1)^{rs}(f_j^{s+1} \times z_i^{r})\big], \end{aligned} \tag{13}$$

with α, β integers such that $\alpha e_i + \beta e_j = (e_i, e_j)$. Since we find in [3, p. 138, (5.5)] that

$$F(x^p \times y) = (Fx^p \times y) + (-1)^p (x^p \times Fy), \tag{14}$$

it follows that $FC = (e_i, e_j)(11)$. Next we deform C along the segments pq from P to S sweeping out the continuous image of a prism whose bases are C and ΔC. This deformation gives rise to a singular chain[7] whose boundary is $\Delta C - C - (e_i, e_j)c$ because, as defined above, $(e_i, e_j)c$ is swept out by $FC = (e_i, e_j)z$. Using (1) we calculate that $\Delta C = 0$. Hence $(e_i, e_j)c + C$ bounds in N_S. In the same way it is seen that $(e_i, e_j)d + C$ bounds in N_K. Hence $(e_i, e_j)(c - d) \sim 0$ in J and $(e_i, e_j)H(z) = 0$.

In the same way we show that $e_i H(12) = (e_i, e_j)H(9) = e_i H(10) = 0$. The argument is as above with (13) replaced by

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[6] See [3, p. 267, 18].

[7] See [3, p. 307, (21.2)].

$$C = (f_i^{r+1} \times z_1^0) - (f_i^{r+1} \times z_j^0) - (z_1^0 \times f_i^{r+1}) + (z_j^0 \times f_i^{r+1}),$$

$$C = (f_i^{r+1} \times f_j^{s+1}) - (-1)^{(r+1)(s+1)}(f_j^{s+1} \times f_i^{r+1}),$$

and

$$C = (f_i^{r+1} \times f_i^{r+1}),$$

respectively.

Finally we consider $e_i(z_i^r \times z_i^r) \sim 0$ in P, r odd. Take $C = (f_i^{r+1} \times z_i^r)$. Then $FC = e_i(z_i^r \times z_i^r)$. Using (14), (1), and the fact that r is odd, we compute that

$$\Delta F(f_i^{r+1} \times f_i^{r+1}) = 2\Delta(f_i^{r+1} \times z_i^r).$$

Hence

$$e_i H(z_i^r \times z_i^r) \supset (1/2)\Delta F(f_i^{r+1} \times f_i^{r+1}).$$

This implies that the elements of S with dimension greater than zero are in the subgroup of $B(J)$ generated by the $H(z)$, z a seam. Also $H(z_i^r \times z_i^r)$, r odd, is of order $2e_i$. Our discussion implies the following:

THEOREM 4. *The Betti group* $B(J)$ *is generated by a nonzero 0-dimensional element,* $H(8)$, $H(9)$, $H(10)$, $H(11)$, *and* $H(12)$; *furthermore a complete set of relations for these generators consists of* $2e_iH(8) = (e_i, e_j)H(9) = e_iH(10) = (e_i, e_j)H(11) = e_iH(12) = 0$.

We shall conclude with two corollaries of Theorem 4. Let K_a be K augmented as defined in [3, p. 130]. Let $B(K_a)$ be generated by U_i, $i = 1, 2, \cdots$, and u_k, $k = 1, 2, \cdots$, where the U_i are free and independent, u_k is of order e_k, and the e_k are the torsion coefficients. Let the dimensions of U_i and u_k be r_i and r_k, respectively. Let B^n denote the n-dimensional Betti group of J.

COROLLARY 1. *To each pair* U_i *and* U_j, $i \neq j$, $r_i + r_j = n-1$, *there corresponds a free element of* B^n; *to each* U_i, $r_i = (n-1)/2$, r_i *odd, there corresponds a free element of* B^n; *to each pair* u_k *and* u_l, $k \neq l$, $r_k + r_l$ $=$ *either* $n-1$ *or* $n-2$, *there corresponds an element of* B^n *of order* (e_k, e_l); *to each* u_k, $r_k = (n-1)/2$, r_k *odd, there corresponds an element of* B^n *of order* $2e_k$; *and to each* u_k, $r_k = (n-2)/2$, r_k *even, there corresponds an element of* B^n *of order* e_k; *the elements mentioned generate* B^n, *and their orders are a complete set of relations for these generators; finally* B^0 *is a free cyclic group of rank* 1.

Let R^n and $R^n(J)$ denote the n-dimensional Betti numbers of K_a and J, respectively.

COROLLARY 2. *We have*

$$R^n(J) = \sum_i R^i R^{n-i-1} + A^n,$$

with the summation over all integers i from 0 *to the greatest integer less than* $(n-1)/2$ *and*

$$A^n = \begin{cases} R^{(n-1)/2}, & (n-1)/2 \text{ an odd integer,} \\ 0, & (n-1)/2 \text{ not an odd integer.} \end{cases}$$

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THE PROBABILITY THAT A DETERMINANT BE CONGRUENT TO a (MOD m)

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The problem treated is the evaluation of $P_n(a, m)$, the probability that a determinant of order n having integral elements be congruent to a modulo m. By "probability" is meant the fraction obtained by dividing the number of favorable cases by the number of possible cases: let each element of the determinant range over the values $1, 2, \cdots, m$; among the m^{n^2} possible determinants let g be the number which are congruent to a modulo m; then $P_n(a, m) = g/m^{n^2}$.

This problem has been investigated by Jordan,[1] whose solution involves the function

$$S_n(p, \lambda) = \sum p^{-(\lambda_1+\lambda_2+\cdots+\lambda_{n-1})} \qquad (n \geqq 2), \tag{1}$$

where the sum ranges over all values satisfying the inequality

$$0 \leqq \lambda_1 \leqq \lambda_2 \leqq \cdots \leqq \lambda_{n-1} \leqq \lambda.$$

We use here a different method and obtain results more explicit than (1). Our results can be obtained from Jordan's, but it is as convenient to derive them independently.

It will be convenient to use a result stated by Hull,[2] which in our notation can be written

$$P_n(a, m_1)P_n(a, m_2) = P_n(a, m_1m_2) \quad \text{if} \quad (m_1, m_2) = 1. \tag{2}$$

We shall prove

$$P_n(a, m) = P_n(q, m) \quad \text{where} \quad q = (a, m), \tag{3}$$

so that our problem has been reduced to the determination of $P_n(p^\alpha, p^k)$, p being a prime. This can be evaluated by means of

$$P_n(p^\alpha, p^k) = \{\phi(p^{k-\alpha})\}^{-1}\{P_n(0, p^\alpha) - P_n(0, p^{\alpha+1})\} \qquad (0 \leqq \alpha < k), \tag{4}$$

and

$$P_n(0, p^k) = 1 - \prod_{r=k}^{k+n-1} (1 - p^{-r}) \qquad (k \geqq 1), \tag{5}$$

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[1] C. Jordan, *Sur le nombre des solutions de la congruence* $|a_{ik}| \equiv A$ (mod M), J. Math. Pures Appl. (6) vol. 7 (1911) pp. 409–416.

[2] Ralph Hull, *Congruences involving kth powers*, Trans. Amer. Math. Soc. vol. 34 (1932) p. 910.

the Euler ϕ-function being involved in (4). We now prove these results.

1. **Two lemmas.** The first is due to C. Jordan.[3] Let $N_n(d, m)$ represent the number of different sets $a_1, a_2, \cdots, a_n$ of n positive integers not greater than m such that $(a_1, a_2, \cdots, a_n, m)=d$, d being any divisor of m. This, with $d=1$, is Jordan's generalization of the Euler ϕ-function.

LEMMA 1.

$$\begin{aligned} N_n(d, m) &= N_n(1, m/d); \\ N_n(1, ab) &= N_n(1, a)N_n(1, b) \quad \textit{if} \quad (a, b) = 1; \\ N_n(1, p^k) &= (p^k)^n - (p^{k-1})^n \quad \textit{for} \quad k \geqq 1, \quad p \textit{ any prime.} \end{aligned}$$

The first equation reduces our function to that of Jordan, and the other equations are his.

LEMMA 2. *Let $a_1, a_2, \cdots, a_n$ be any integers such that $(a_1, a_2, \cdots, a_n, m)=1$. Then integral solutions λ_i can be found for the congruence*

$$\sum_{i=1}^{n} a_i\lambda_i \equiv 1 \pmod{m}, \tag{6}$$

such that λ_1 is prime to m.

Let $x_1, x_2, \cdots, x_n$ be any solution of the congruence (6). Let c be defined by $c=(a_2, a_3, \cdots, a_n, m)$, so that we have $(x_1, c, m)=1$. Also there exist integers $k_2, k_3, \cdots, k_n$ such that

$$\sum_{i=2}^{n} k_i a_i \equiv c \pmod{m}.$$

Let b be the product of those primes which divide m but not x_1. Then we set

$$\lambda_1 = x_1 + \sum_{i=2}^{n} bk_i a_i, \qquad \lambda_j = x_j - bk_j a_1 \qquad (j = 2, 3, \cdots, n),$$

noting that these give a solution of (6). We have $\lambda_1 \equiv x_1+bc \pmod{m}$. Let p be any prime factor of m. If $p|x_1$, then $p\nmid b$ and $p\nmid c$. If $p\nmid x_1$, then $p|b$. Hence λ_1 is prime to m.

2. **A recursion formula.** Consider now the determinant of order n

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[3] *Traité des substitutions*, Paris, 1870, pp. 95–97, or L. E. Dickson, *History of the theory of numbers*, vol. 1, p. 147.

with arbitrary integral elements modulo m. Let $a_1, a_2, \cdots, a_n$ and $b_1, b_2, \cdots, b_n$ designate the elements of the first two rows, respectively. We shall assume that the g.c.d. of the elements of the first row and m has been factored and placed in front of the determinant, so that $(a_1, a_2, \cdots, a_n, m)=1$. Using Lemma 2 we choose $\lambda_1, \lambda_2, \cdots, \lambda_n$ so that (6) holds with $(\lambda_1, m)=1$. We now apply the following transformation to the determinant. Multiply the first column by λ_1. This of course multiplies the value of the determinant by λ_1, which is immaterial since in this section we are concerned with divisibility by m, that is, with $P_n(0, m)$. Next we add to the elements of the first column the elements of the succeeding columns multiplied by $\lambda_2, \lambda_3, \cdots, \lambda_n$, respectively. Having obtained 1 in the first place, we use it to eliminate all succeeding elements of the first row (by subtracting from the ith column the elements of the first column multiplied by a_i, for $i=2, 3, \cdots, n$), so that the first row is now $1, 0, 0, \cdots, 0$. Suppose the second row has become $c_1, c_2, \cdots, c_n$.

We propose to show that the transformation thus effected on the second row (and hence on all succeeding rows) is unique in the following sense: for any fixed set of a's in the first row and any set of c's, there is a unique set of b's which is transformed into the set of c's. The b's and c's are connected by the equations

$$c_1 = \sum_{j=1}^{n} \lambda_j b_j, \qquad c_i = b_i - a_i c_1 \qquad (i = 2, 3, \cdots, n).$$

The determinant of the transformation has the value λ_1 (prime to m), and the result follows.

Now the value modulo m of the transformed determinant is deduced from the minor of the leading element, and since some divisor (say d) of m was removed from the elements of the first row, we are interested in the probability $P_{n-1}(0, m/d)$ that m/d divide this principal minor. There are $N_n(d, m)$ arrangements of the first row having d for the g.c.d. of the elements and m, and for each of these arrangements there are m^{n^2-n} possible arrangements for all the other rows (m possibilities for each element). Among these the number of favorable cases, that is, determinants divisible by m, is

$$m^{n^2-n} N_n(d, m) P_{n-1}(0, m/d).$$

By considering all possible divisors d of m we get the total number of favorable cases. But this total number can also be obtained by multiplying the number m^{n^2} of possible determinants by $P_n(0, m)$. Hence we have

$$m^{n^2}P_n(0, m) = \sum_{d|m} m^{n^2-n}N_n(d, m)P_{n-1}(0, m/d).$$

Using the first equation in Lemma 1 and the fact that m/d ranges over the divisors of m as d does, and dividing by m^{n^2-n}, we obtain

$$m^nP_n(0, m) = \sum_{d|m} N_n(1, d)P_{n-1}(0, d) \qquad (n > 1, m \geqq 1). \tag{7}$$

If m is a power of a prime, say p^k, we have

$$p^{kn}P_n(0, p^k) = \sum_{\alpha=0}^{k} N_n(1, p^\alpha)P_{n-1}(0, p^\alpha). \tag{8}$$

3. **The formula for** $P_n(0, p^k)$. Subtracting equation (8) from the corresponding equation with k replaced by $k+1$, and using the last formula in Lemma 1, we obtain the recursion formula

$$p^{(k+1)n}P_n(0, p^{k+1}) = p^{kn}P_n(0, p^k) + (p^{(k+1)n} - p^{kn})P_{n-1}(0, p^{k+1}).$$

It is convenient here to set $Q_n^k = 1 - P_n(0, p^k)$ and $q = p^{-1}$. Making these substitutions we have

$$Q_n^{k+1} = q^n Q_n^k + (1 - q^n) Q_{n-1}^{k+1}. \tag{9}$$

We shall now prove

$$Q_n^k = \prod_{r=k}^{k+n-1} (1 - q^r) \qquad (k = 0, 1, 2, \cdots ; n = 1, 2, 3, \cdots). \tag{10}$$

This result is trivial for $k = 0$ and all n, and for $n = 1$ and all k. Assume it true for $k \leqq K$ and all n, and for $k = K+1$ and $n < N$. To complete the induction we must prove it for $k = K+1$ and $n = N$. By (9) and the induction hypotheses we have

$$\begin{aligned} Q_N^{K+1} &= q^N \prod_{r=K}^{K+N-1} (1 - q^r) + (1 - q^N) \prod_{r=K+1}^{K+N-1} (1 - q^r) \\ &= \prod_{r=K+1}^{K+N} (1 - q^r). \end{aligned}$$

This proves (10) and equation (5) follows.

4. **Proofs of** (3) **and** (4). In the light of (2), we need demonstrate (3) for $m = p^k$ only. We wish to prove, then, that the number of determinants congruent to p^α (mod p^k) equals the number congruent to cp^α, where $(c, p) = 1$ and $\alpha < k$. Now any one of the former type gives one of the latter if the first row (say) is multiplied by c, and con-

versely if the first row is multiplied by the inverse of c (mod p^k). This inverse exists, and the correspondence is one-to-one, because c is prime to p. This proves (3).

The sum of the probabilities $P_n(ap^\alpha, p^k)$, where a runs through the values $1, 2, \cdots, p^{k-\alpha}$, is clearly the probability that a determinant be divisible by p^α. The terms of this sum can be simplified and collected by use of (3), and we have

$$P_n(0, p^\alpha) = \sum_{r=0}^{k-\alpha} \phi(p^{k-\alpha-r}) P_n(p^{\alpha+r}, p^k). \tag{11}$$

Replacing α by $\alpha+1$, and subtracting the resulting equation from (11), we arrive at (4).

PURDUE UNIVERSITY

ON THE NOTION OF THE RING OF QUOTIENTS OF A PRIME IDEAL

CLAUDE CHEVALLEY

Let $\mathfrak{o}$ be a domain of integrity (that is, a ring with unit element and with no zero divisor not equal to 0), and let $\mathfrak{u}$ be a prime ideal in $\mathfrak{o}$. We can construct two auxiliary rings associated with $\mathfrak{u}$: the factor ring $\mathfrak{o}/\mathfrak{u}$, composed of the residue classes of elements of $\mathfrak{o}$ modulo $\mathfrak{u}$, and the ring of quotients $\mathfrak{o}_\mathfrak{u}$, composed of the fractions whose numerator and denominator belong to $\mathfrak{o}$, but whose denominators do not belong to $\mathfrak{u}$. These constructions are of paramount importance in algebraic geometry; if $\mathfrak{o}$ is the ring of a variety V, there corresponds to $\mathfrak{u}$ a subvariety U of V; $\mathfrak{o}/\mathfrak{u}$ is the ring of U, whereas the ring $\mathfrak{o}_\mathfrak{u}$ is the proper algebraic tool to investigate the neighborhood of U with respect to V.

Now, the local theory of algebraic varieties involves the consideration of rings which are not domains of integrity (this, because the completion of a local ring may introduce zero divisors). Let then $\mathfrak{o}$ be any commutative ring with unit element, and let again $\mathfrak{u}$ be a prime ideal in $\mathfrak{o}$. We may define the factor ring $\mathfrak{o}/\mathfrak{u}$ exactly in the same way as above, but we cannot so easily generalize the notion of the ring of quotients $\mathfrak{o}_\mathfrak{u}$. If there exist zero divisors outside $\mathfrak{u}$, these zero divisors cannot be used as denominators of fractions, which shows that the definition of $\mathfrak{o}_\mathfrak{u}$ cannot be extended verbatim. If we

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consider only fractions whose denominators are not zero divisors and do not belong to $\mathfrak{u}$, we obtain a ring $\mathfrak{o}'$; but $\mathfrak{o}'$ fails in general to have the essential property of a ring of quotients, namely, of being a local ring in the sense of Krull (that is, the non-units in $\mathfrak{o}'$ will not form an ideal). The object of this note is to construct a ring for which the essential properties of rings of quotients are preserved.

Throughout this paper we shall denote by $\mathfrak{o}$ a Noetherian ring (that is, a ring in which the maximal condition for ideals is satisfied) with a unit element. Generalizing the problem of defining the ring of quotients of a prime ideal, we take any multiplicatively closed[1] subset S of $\mathfrak{o}$ which does not contain 0 (a set is said to be multiplicatively closed if the product of any two elements of the set belongs to the set; if we are concerned with a prime ideal $\mathfrak{u}$ in $\mathfrak{o}$, we take S to be the complement of $\mathfrak{u}$ in $\mathfrak{o}$). There exists at least one primary ideal which does not meet S (otherwise, 0 would belong to S as we see at once by representing the zero ideal as an intersection of primary ideals). We shall denote by $\mathfrak{s}$ the intersection of all primary ideals in $\mathfrak{o}$ which do not meet S.

PROPOSITION 1. *Let* $\{0\}=\mathfrak{q}_1\cap\cdots\cap\mathfrak{q}_h$ *be an irredundant representation of the zero ideal in* $\mathfrak{o}$ *as an intersection of primary ideals, and let* $\mathfrak{p}_i (1\leqq i\leqq h)$ *be the associated prime ideal of* $\mathfrak{q}_i$. *Assume that* $\mathfrak{p}_i\cap S=\emptyset$ *for* $i\leqq g$, *but not for* $i>g$. *Then* $\mathfrak{s}=\mathfrak{q}_1\cap\cdots\cap\mathfrak{q}_g$.

It is clear that $\mathfrak{s}\subset\mathfrak{q}'=\mathfrak{q}_1\cap\cdots\cap\mathfrak{q}_g$. Let $\mathfrak{d}$ be any primary ideal which does not meet S; we shall prove that $\mathfrak{q}'\subset\mathfrak{d}$. Let $\mathfrak{q}''$ be the ideal $\mathfrak{q}_{g+1}\cap\cdots\cap\mathfrak{q}_h$. We have $\{0\}=\mathfrak{q}'\cap\mathfrak{q}''=\mathfrak{q}'\mathfrak{q}''\subset\mathfrak{d}$, whence $\mathfrak{q}'\subset\mathfrak{d}:\mathfrak{q}''$. Let $\mathfrak{u}$ be the associated prime ideal of $\mathfrak{d}$; since $\mathfrak{d}$ contains some power of $\mathfrak{u}$, it follows from the multiplicatively closed character of S that $\mathfrak{u}$ does not meet S. If $i>g$, the ideal $\mathfrak{p}_i$ meets S and is therefore not contained in $\mathfrak{u}$. It follows[2] that $\mathfrak{d}:\mathfrak{q}''=\mathfrak{d}$, whence $\mathfrak{q}'\subset\mathfrak{d}$. Proposition 1 is thereby proved.

LEMMA 1. *Let* $\mathfrak{p}$ *be a prime ideal in* $\mathfrak{o}$, *and let* $\mathfrak{a}$ *be an ideal contained in* $\mathfrak{p}$. *If* $\mathfrak{q}$ *is an ideal containing* $\mathfrak{a}$, *the statements* "$\mathfrak{q}$ *is primary for* $\mathfrak{p}$" *and* "$\mathfrak{q}/\mathfrak{a}$ *is primary for* $\mathfrak{p}/\mathfrak{a}$" *are equivalent.*

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[1] It was H. Grell who observed for the first time that, S being any multiplicatively closed set of nonzero divisors in a ring, it is possible to associate with S a ring of quotients, whose elements are the fractions whose denominators belong to S (Cf. H. Grell, *Beziehungen zwischen Ideale verschiedener Ringe*, Math. Ann. vol. 97 (1926) p. 510). For the properties of these rings of quotients, cf. Krull, *Idealtheorie* (Ergebnisse der Mathematik) or my paper *On the theory of local rings*, Ann. of Math. vol. 44 (1943) p. 690.

[2] Cf. van der Waerden, *Moderne Algebra*, vol. 2, chap. 12, p. 36.

Lemma 1 follows trivially from the definitions.

The zero ideal in $\mathfrak{o}/\mathfrak{s}$ is the intersection of the primary ideals $\mathfrak{q}_1/\mathfrak{s}, \cdots, \mathfrak{q}_g/\mathfrak{s}$ whose associated prime ideals are $\mathfrak{p}_1/\mathfrak{s}, \cdots, \mathfrak{p}_g/\mathfrak{s}$. Let S^* be the set of the residue classes modulo $\mathfrak{s}$ of the elements of S; then $S^* \cap (\mathfrak{p}_i/\mathfrak{s}) = \emptyset$ $(1 \leqq i \leqq g)$, which means that no element of S^* is a zero divisor in $\mathfrak{o}/\mathfrak{s}$.[3] We may therefore construct the ring of quotients $(\mathfrak{o}/\mathfrak{s})_{S^*}$ of S^* with respect to the ring $\mathfrak{o}/\mathfrak{s}$.

DEFINITION 1. *The ring* $(\mathfrak{o}/\mathfrak{s})_{S^*}$ *will be called the ring of quotients of S with respect to* $\mathfrak{o}$. *This ring will be denoted by* $\mathfrak{o}_S$.

This definition coincides with the usual one in the case where S does not contain any zero divisor. We shall now prove that the essential properties of rings of quotients in the usual sense still hold in our case.

If $\mathfrak{a}$ is an ideal in $\mathfrak{o}$, $(\mathfrak{a}+\mathfrak{s}/\mathfrak{s})\mathfrak{o}_S$ is an ideal in $\mathfrak{o}_S$ which we shall denote symbolically by $\mathfrak{a}\mathfrak{o}_S$ (in spite of the fact that $\mathfrak{o}$ is not in general a subring of $\mathfrak{o}_S$, so that we cannot multiply elements of $\mathfrak{o}$ by elements of $\mathfrak{o}_S$). If $\mathfrak{b}$ is any ideal in $\mathfrak{o}_S$, the set $\mathfrak{b} \cap (\mathfrak{o}/\mathfrak{s})$ may be written in the form $\mathfrak{a}/\mathfrak{s}$, where $\mathfrak{a}$ is an ideal in $\mathfrak{o}$ which contains $\mathfrak{s}$. We shall denote $\mathfrak{a}$ symbolically by $\mathfrak{b} \cap \mathfrak{o}$ (although $\mathfrak{b} \cap \mathfrak{o}$ is not a set theoretic intersection).

PROPOSITION 2. *If* $\mathfrak{b}$ *is any ideal in* $\mathfrak{o}_S$, *we have* $\mathfrak{b} = (\mathfrak{b} \cap \mathfrak{o})\mathfrak{o}_S$.

Since $\mathfrak{o}_S$ is a ring of quotients of $\mathfrak{o}/\mathfrak{s}$, we have $\mathfrak{b} = (\mathfrak{b} \cap (\mathfrak{o}/\mathfrak{s}))\mathfrak{o}_S$.[1] Proposition 2 follows immediately from this formula.

PROPOSITION 3. *Let* $\mathfrak{p}$ *be a prime ideal in* $\mathfrak{o}$, *and let* $\mathfrak{q}$ *be primary for* $\mathfrak{p}$. *If* $\mathfrak{q}$ *meets S, we have* $\mathfrak{p}\mathfrak{o}_S = \mathfrak{q}\mathfrak{o}_S = \mathfrak{o}_S$. *If* $\mathfrak{p}$ *does not meet S*, $\mathfrak{p}\mathfrak{o}_S$ *is prime*, $\mathfrak{q}\mathfrak{o}_S$ *is primary for* $\mathfrak{p}\mathfrak{o}_S$ *and* $\mathfrak{p}\mathfrak{o}_S \cap \mathfrak{o} = \mathfrak{p}$, $\mathfrak{q}\mathfrak{o}_S \cap \mathfrak{o} = \mathfrak{q}$.

If $\mathfrak{q}$ meets S, $\mathfrak{q}+\mathfrak{s}/\mathfrak{s}$ meets S^*, whence $\mathfrak{p}\mathfrak{o}_S = \mathfrak{q}\mathfrak{o}_S = \mathfrak{o}_S$. If $\mathfrak{p}$ does not meet S, the same holds for $\mathfrak{q}$, whence $\mathfrak{s} \subset \mathfrak{q} \subset \mathfrak{p}$. By Lemma 1, $\mathfrak{p}/\mathfrak{s}$ is prime and $\mathfrak{q}/\mathfrak{s}$ is primary for $\mathfrak{p}/\mathfrak{s}$. Furthermore, $\mathfrak{p}/\mathfrak{s}$ does not meet S^*. Proposition 3 follows therefore from the corresponding proposition which is known to hold for ordinary rings of quotients.[1] We see also that, if $\mathfrak{p}$ does not meet $\mathfrak{s}$, the formula $\mathfrak{q} \leftrightarrow \mathfrak{q}\mathfrak{o}_S$ establishes a one-to-one inclusion preserving correspondence between the primary ideals for $\mathfrak{p}$ in $\mathfrak{o}$ and the primary ideals for $\mathfrak{p}\mathfrak{o}_S$ in $\mathfrak{o}_S$.

PROPOSITION 4. *Let* $\mathfrak{a} = \mathfrak{v}_1 \cap \cdots \cap \mathfrak{v}_r$ *be an irredundant representation of an ideal* $\mathfrak{a}$ *in* $\mathfrak{o}$ *as an intersection of primary ideals. Let* $\mathfrak{u}_i$ *be the associated prime ideal of* $\mathfrak{v}_i$, *and assume that* $\mathfrak{u}_i$ *meets S for* $i > s$ *but not for* $i \leqq s$. *Then we have* $\mathfrak{a}\mathfrak{o}_S = \mathfrak{v}_1\mathfrak{o}_S \cap \cdots \cap \mathfrak{v}_s\mathfrak{o}_S$, *and this is an irredundant representation of* $\mathfrak{a}\mathfrak{o}_S$ *as intersection of primary ideals in* $\mathfrak{o}_S$.

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[3] If $s^* \in S^*$, we have $\{0\} : s^*(\mathfrak{o}/\mathfrak{s}) = \{0\}$ by the result quoted in footnote 2.

It is obvious that $\mathfrak{a}\mathfrak{o}_S \subset \mathfrak{d}_1\mathfrak{o}_S \cap \cdots \cap \mathfrak{d}_s\mathfrak{o}_S$. Let conversely a be any element of $\mathfrak{d}_1\mathfrak{o}_S \cap \cdots \cap \mathfrak{d}_s\mathfrak{o}_S$. We know[3] that $\mathfrak{d}_1\mathfrak{o}_S \cap \cdots \cap \mathfrak{d}_s\mathfrak{o}_S$ is equal to $(\mathfrak{d}_1/\mathfrak{s} \cap \cdots \cap \mathfrak{d}_s/\mathfrak{s})\mathfrak{o}_S$; it follows that a may be written in the form b^*/c^*, where $b^* \in \mathfrak{d}_1/\mathfrak{s} \cap \cdots \cap \mathfrak{d}_s/\mathfrak{s}$ and $c^* \in S^*$. If $i > s$, the ideal $\mathfrak{u}_i$ has an element u_i in common with S; if m is large enough, we have $u = (\prod_{i=s+1}^{r} u_i)^m \in \mathfrak{d}_{s+1} \cap \cdots \cap \mathfrak{d}_r$. Since $u_i \in S$ $(s+1 \leqq i \leqq r)$, we have $u \in S$, whence $u^* \in S^*$, if u^* is the residue class of u modulo $\mathfrak{s}$. We may write $a = (u^*b^*)/(u^*c^*)$, $u^*c^* \in S^*$. Let b be any element of the residue class b^* modulo $\mathfrak{s}$; since $b^* \in \mathfrak{d}_i/\mathfrak{s}$ $(1 \leqq i \leqq s)$, we have $b \in \mathfrak{d}_1 \cap \cdots \cap \mathfrak{d}_s$, whence $ub \in \mathfrak{a}$ and $u^*b^* \in \mathfrak{a}+\mathfrak{s}/\mathfrak{s}$, $a \in \mathfrak{a}\mathfrak{o}_S$. It is clear that none of the ideals $\mathfrak{d}_i/\mathfrak{s}$ contains the intersection of the others; making use of a known result[1] for ordinary rings of quotients, it follows that the representation $\mathfrak{a}\mathfrak{o}_S = \mathfrak{d}_1\mathfrak{o}_S \cap \cdots \cap \mathfrak{d}_s\mathfrak{o}_S$ is irredundant.

We shall now consider more specifically the case where S is the complement of a prime ideal $\mathfrak{u}$. The ring $\mathfrak{o}_S$ will then also be denoted by $\mathfrak{o}_{\mathfrak{u}}$. In that case, *the ideal* $\mathfrak{s}$ *coincides with the intersection of all primary ideals for* $\mathfrak{u}$. In fact, the set S^* is clearly the complement of $\mathfrak{u}/\mathfrak{s}$ with respect to $\mathfrak{o}/\mathfrak{s}$; the ring $\mathfrak{o}_S$ is the ring of quotients (in the ordinary sense) of the prime ideal $\mathfrak{u}/\mathfrak{s}$ with respect to $\mathfrak{o}/\mathfrak{s}$. It follows that $\mathfrak{u}\mathfrak{o}_S$ is the ideal of non-units in $\mathfrak{o}_S$, whence $\bigcap_{n=1}^{\infty}(\mathfrak{u}/\mathfrak{s})^n\mathfrak{o}_S = \{0\}$.[4] For every n, the ideal $(\mathfrak{u}/\mathfrak{s})^n\mathfrak{o}_S \cap \mathfrak{o}/\mathfrak{s}$ is a primary ideal for $\mathfrak{u}/\mathfrak{s}$ in $\mathfrak{o}/\mathfrak{s}$; it follows that the intersection of all primary ideals for $\mathfrak{u}/\mathfrak{s}$ is the zero ideal in $\mathfrak{o}/\mathfrak{s}$. Our assertion then follows from Lemma 1.[5] At the same time, we see that $\mathfrak{o}_{\mathfrak{u}}$ is a local ring in the sense of Krull.

Assume now that $\mathfrak{o}$ is a semi-local ring[6] and that $\mathfrak{u}$ is one of the maximal prime ideals in $\mathfrak{o}$. Let $\bar{\mathfrak{o}}$ be the completion of $\mathfrak{o}$; there corresponds to $\mathfrak{u}$ an idempotent ϵ in $\bar{\mathfrak{o}}$. We shall prove the following results:

PROPOSITION 5. *The ring* $\mathfrak{o}/\mathfrak{s}$ *is isomorphic with the subring* $\mathfrak{o}\epsilon$ *of the ring* $\bar{\mathfrak{o}}\epsilon$. *This isomorphism may be extended to an isomorphism of the completion of* $\mathfrak{o}_{\mathfrak{u}}$ *with* $\bar{\mathfrak{o}}\epsilon$.

The first statement will be proved if we show that $\mathfrak{s}$ coincides with the set of elements $x \in \mathfrak{o}$ which satisfy the condition $x\epsilon = 0$. If x is any element of $\mathfrak{o}$, we may write $x=x\epsilon+x(1-\epsilon)$, and we know that $1-\epsilon\in\mathfrak{u}^n\bar{\mathfrak{o}}$ for every n. If $x\epsilon=0$, we have $x\in\mathfrak{u}^n\bar{\mathfrak{o}}\cap\mathfrak{o}=\mathfrak{u}^n$ for every n; in particular, x belongs to every primary ideal for $\mathfrak{u}$, whence $x\in\mathfrak{s}$. If $x\in\mathfrak{s}$, we have $x\in\mathfrak{u}^n$ for every n ($\mathfrak{u}^n$ is primary because $\mathfrak{u}$ is a *maximal* prime ideal), whence $x\epsilon\in\mathfrak{u}^n\bar{\mathfrak{o}}\epsilon$. Since $\mathfrak{u}\bar{\mathfrak{o}}\epsilon$ is the ideal of non-units in $\bar{\mathfrak{o}}\epsilon$, it follows that $x\epsilon=0$.

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[4] Cf. Krull, *Dimensionstheorie in Stellenringen*, J. Reine Angew. Math. vol. 179 (1938) p. 204 or my paper quoted in footnote 1.

[5] This result, together with Proposition 1 above, yields a proof of a theorem of Krull; cf. Krull, *Primidealketten in allgemeinen Ringbereiche*, Sitzungsberichte der Heidelberger Akademie, 1928, p. 7.

[6] A semi-local ring is a Noetherian ring (that is, the maximal chain condition holds in the ring) in which there exist only a finite number of maximal prime ideals. For the proofs of the results on semi-local rings which are used in this paper, cf. my paper quoted in footnote 1.

If we identify $\mathfrak{o}/\mathfrak{s}$ with $\mathfrak{o}\epsilon$, every element of S^* is a unit in $\bar{\mathfrak{o}}\epsilon$. In fact, if $y\in S$, we have $y=y\epsilon+y(1-\epsilon)$, $y(1-\epsilon)\in\mathfrak{u}\bar{\mathfrak{o}}$. If we had $y\in\mathfrak{u}\bar{\mathfrak{o}}$, we would have $y\in\mathfrak{u}\bar{\mathfrak{o}}\cap\mathfrak{o}=\mathfrak{u}$, which is not the case. It follows that $\bar{\mathfrak{o}}\epsilon$ contains the ring $\mathfrak{o}_{\mathfrak{u}}$. The ring $\bar{\mathfrak{o}}\epsilon$ is a complete ring with $\mathfrak{u}\bar{\mathfrak{o}}\epsilon$ as unique maximal prime ideal; it is clear that $\mathfrak{o}\epsilon$ (and, a fortiori, $\mathfrak{o}_{\mathfrak{u}}$) is dense in $\bar{\mathfrak{o}}\epsilon$. In order to prove that $\bar{\mathfrak{o}}\epsilon$ is the completion of $\mathfrak{o}_{\mathfrak{u}}$, it is sufficient to prove that $\mathfrak{o}_{\mathfrak{u}}$ is topologically a subspace of $\bar{\mathfrak{o}}\epsilon$. We show first that $\mathfrak{u}^n\bar{\mathfrak{o}}\epsilon\cap\mathfrak{o}\epsilon=\mathfrak{u}^n\mathfrak{o}\epsilon$ for every n. Let $x\epsilon$ $(x\in\mathfrak{o})$ be an element of $\mathfrak{u}^n\bar{\mathfrak{o}}\epsilon\cap\mathfrak{o}\epsilon$; we have $x=x\epsilon+x(1-\epsilon)\in\mathfrak{u}^n\bar{\mathfrak{o}}$, whence $x\in\mathfrak{u}^n\bar{\mathfrak{o}}\cap\mathfrak{o}=\mathfrak{u}^n$, $x\epsilon\in\mathfrak{u}^n\epsilon$. The ideal $\mathfrak{u}^n\bar{\mathfrak{o}}\cap\mathfrak{o}_{\mathfrak{u}}$ is equal to $((\mathfrak{u}^n\bar{\mathfrak{o}}\cap\mathfrak{o}_{\mathfrak{u}}\mathfrak{o})\cap\epsilon)\mathfrak{o}_{\mathfrak{u}}=(\mathfrak{u}^n\mathfrak{o}\epsilon)\mathfrak{o}_{\mathfrak{u}}=\mathfrak{u}^n\mathfrak{o}_{\mathfrak{u}}$; Proposition 5 is thereby completely proved.

PRINCETON UNIVERSITY

UNION-PRESERVING TRANSFORMATIONS OF SPACE

EDWARD KASNER AND JOHN DECICCO

1. General statement. Sophus Lie studied transformations from lineal-elements into lineal-elements, and also transformations from surface-elements into surface-elements of space. The contact group is obtained by requiring all unions to be converted into unions. Lie's fundamental theorems may be stated as follows. All the contact lineal-element transformations form the group of extended point transformations. The contact surface-element transformations which are not merely extended point transformations are defined completely by either a single directrix equation, or a pair of directrix equations. In the first case, a point corresponds to a surface; and in the second case, a point corresponds to a curve.[1]

We extend the preceding results by studying transformations in space from differential curve-elements of order n: $(x, y, z, y', z', \cdots, y^{(n)}, z^{(n)})$, where n is 2 or more, into lineal-elements (X, Y, Z, Y', Z'). An example of such a transformation arises in the problem of finding the locus of the centers of spherical curvature for an arbitrary space curve. This problem leads to a transformation from curve-elements of third order into lineal-elements.[2]

We determine the general class of union-preserving transformations by means of a directrix equation. Lie has obtained directrix equations only for contact transformations of surface-elements since there are no contact transformations of lineal-elements besides the extended point transformations. For a point-to-surface transformation, Lie's standard directrix equation is of the form $\Omega(X, Y, Z, x, y, z)=0$. For a point-to-curve transformation, there are two standard directrix equations of the forms $\Omega_1(X, Y, Z, x, y, z)=0$, $\Omega_2(X, Y, Z, x, y, z)=0$. We find that any general union-preserving transformation from curve-elements of order n into lineal-elements is completely determined by our *new directrix equation*, involving derivatives, $\Omega(X, Y, Z, x, y, z, y', z', \cdots, y^{(n-2)}, z^{(n-2)})=0$.

In the final part of our paper, we shall prove that the only available union-preserving transformations (in the whole domain of curve-

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Presented to the Society, November 27, 1943; received by the editors August 17, 1943. We have studied the two-dimensional aspects of our new theory in Proc. Nat. Acad. Sci. (1943) and Revista des Matimaticas (1943). This leads to a large extension of the Huygens theory of evolutes and involutes and Lie theory.

[1] Lie-Scheffers, *Beruhrungstransformationen.*

[2] See Bull. Amer. Math. Soc. abstract 49-9-235 by Kasner and DeCicco, *A generalized theory of contact transformations.*

elements) are firstly, the group of point transformations, and secondly, the set of union-preserving transformations from curve-elements of order n into lineal-elements, together with the extensions of these two types.

2. **The union-preserving transformations.** Any transformation from curve-elements of order n, where n is 2 or more, into lineal-elements may be defined by equations of the form

$$
\begin{aligned}
X &= X(x, y, z, p_1, q_1, \cdots, p_n, q_n),\\
Y &= Y(x, y, z, p_1, q_1, \cdots, p_n, q_n),\\
Z &= Z(x, y, z, p_1, q_1, \cdots, p_n, q_n),\\
P &= P(x, y, z, p_1, q_1, \cdots, p_n, q_n),\\
Q &= Q(x, y, z, p_1, q_1, \cdots, p_n, q_n),
\end{aligned}
\tag{1}
$$

where $p_m = y^{(m)} = d^m y/dx^m$ and $q_m = z^{(m)} = d^m z/dx^m$ for $m = 1, 2, \cdots, n$.

A series of curve-elements of order n may be defined by $y = y(x)$, $z = z(x)$, $p_1 = p_1(x)$, $q_1 = q_1(x)$, $\cdots$, $p_n = p_n(x)$, $q_n = q_n(x)$. A series is a union if and only if the conditions $dy - p_1 dx = 0$, $dz - q_1 dx = 0$, $\cdots$, $dp_{n-1} - p_n dx = 0$, $dq_{n-1} - q_n dx = 0$ are satisfied.

A special type of union is the conical-union of order n. This consists of ∞^1 curve-elements of order n which have in common a fixed curve-element of order $(n-1)$. The equations of any conical-union are $x = x_0$, $y = y_0$, $z = z_0$, $p_1 = (p_1)_0$, $q_1 = (q_1)_0$, $\cdots$, $p_{n-1} = (p_{n-1})_0$, $q_{n-1} = (q_{n-1})_0$, $q_n = F(p_n)$.

Let us now consider what unions become unions under the transformation (1). These unions must satisfy the pair of Monge differential equations of order $(n+1)$

$$
\begin{aligned}
&(Y_{p_n} - PX_{p_n})p_{n+1} + (Y_{q_n} - PX_{q_n})q_{n+1}\\
&\quad + [(Y_x + p_1 Y_y + q_1 Y_z + \cdots + p_n Y_{p_{n-1}} + q_n Y_{q_{n-1}})\\
&\quad - P(X_x + p_1 X_y + q_1 X_z + \cdots + p_n X_{p_{n-1}} + q_n X_{q_{n-1}})] = 0,\\
&(Z_{p_n} - QX_{p_n})p_{n+1} + (Z_{q_n} - QX_{q_n})q_{n+1}\\
&\quad + [(Z_x + p_1 Z_y + q_1 Z_z + \cdots + p_n Z_{p_{n-1}} + q_n Z_{q_{n-1}})\\
&\quad - Q(X_x + p_1 X_y + q_1 X_z + \cdots + p_n X_{p_{n-1}} + q_n X_{q_{n-1}})] = 0.
\end{aligned}
\tag{2}
$$

THEOREM 1. *All transformations from curve-elements of order n into lineal-elements may be divided into three distinct types with respect to the number of unions which are converted into unions.*[3]

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[3] Kasner, *General transformation theory of differential elements*, Amer. J. Math. vol. 32 (1904) pp. 392–401.

Type 1°. *The union-preserving transformations. These carry every union into a union. The necessary and sufficient conditions for a union-preserving transformation are*

$$
(3)\qquad
\begin{aligned}
P &= \frac{Y_x + p_1Y_y + q_1Y_z + \cdots + p_nY_{p_{n-1}} + q_nY_{q_{n-1}}}{X_x + p_1X_y + q_1X_z + \cdots + p_nX_{p_{n-1}} + q_nX_{q_{n-1}}}\\
&= \frac{Y_{p_n}}{X_{p_n}} = \frac{Y_{q_n}}{X_{q_n}},\\
Q &= \frac{Z_x + p_1Z_y + q_1Z_z + \cdots + p_nZ_{p_{n-1}} + q_nZ_{q_{n-1}}}{X_x + p_1X_y + q_1X_z + \cdots + p_nX_{p_{n-1}} + q_nX_{q_{n-1}}}\\
&= \frac{Z_{p_n}}{X_{p_n}} = \frac{Z_{q_n}}{X_{q_n}}.
\end{aligned}
$$

Type 2°. *The intermediate transformations. These are not union-preserving transformations and satisfy the conditions*

$$
(4)\qquad
\begin{aligned}
&[(Y_x + p_1Y_y + q_1Y_z + \cdots + p_nY_{p_{n-1}} + q_nY_{q_{n-1}})\\
&\qquad - P(X_x + p_1X_y + q_1X_z + \cdots + p_nX_{p_{n-1}} + q_nX_{q_{n-1}})]\\
&\cdot [(Z_x + p_1Z_y + q_1Z_z + \cdots + p_nZ_{p_{n-1}} + q_nZ_{q_{n-1}})\\
&\qquad - Q(X_x + p_1X_y + q_1X_z + \cdots + p_nX_{p_{n-1}} + q_nX_{q_{n-1}})]^{-1}\\
&\qquad\qquad = \frac{Y_{p_n} - PX_{p_n}}{Z_{p_n} - QX_{p_n}} = \frac{Y_{q_n} - PX_{q_n}}{Z_{q_n} - QX_{q_n}}.
\end{aligned}
$$

Any intermediate transformation carries the ∞^∞ *unions of a Monge differential equation of third order into unions.*

Type 3°. *The general transformations. These do not satisfy the conditions* (4). *Any such transformation converts precisely* $\infty^{2(n+1)}$ *unions into unions.*

The general Type 3° is defined by *five* arbitrary functions of $(2n+3)$ independent variables. The intermediate Type 2° is defined by *three* arbitrary functions of $(2n+3)$ independent variables. For this intermediate type, it is seen that (X, Y, Z) may be taken arbitrarily and then the functions P and Q are completely determined by the equations (4). Later we shall show that the union-preserving Type 1° is defined by *one* arbitrary function of $(2n+1)$ variables or by *three* arbitrary functions of $(2n+1)$ variables.

We prove Theorem 1. For union-preserving transformations the equations (2) must be identities in p_{n+1} and q_{n+1}. Upon setting the various coefficients equal to zero and solving for P and Q, we obtain the conditions (3) for union-preserving transformations.

Next suppose we have the intermediate transformations so that the conditions (4) are satisfied. Then it is obvious that the equations (2) are equivalent. This shows that under any intermediate transformation, there are ∞^∞ unions preserved. These satisfy either one of the two equivalent Monge differential equations (2).

Finally suppose the conditions (4) are not satisfied. In the first place it is seen that if the last equality of (4) is not valid, we can solve the equations (2) for p_{n+1} and q_{n+1}. Therefore there are $\infty^{2(n+1)}$ unions which become unions. Finally if the last equality of (4) is valid, then it can be shown that at each curve-element of order $(n-1)$ there are ∞^1 conical-unions which become unions. Since there are $\infty^{(2n+1)}$ curve-elements of order $(n-1)$, it follows that altogether there are $\infty^{2(n+1)}$ conical-unions which are converted into unions. Therefore we have proved that our general Type 3° carries exactly $\infty^{2(n+1)}$ unions into unions under all conditions.

3. **The degenerate union-preserving transformations.** Henceforth we shall consider only union-preserving transformations. However, there are certain degenerate union-preserving transformations which we wish to exclude from consideration. These degenerate correspondences are those where P and Q assume identically the values 0 or ∞; and also those for which there exist two functional relations between the three functions definining (X, Y, Z).

It is found that these degenerate transformations may be divided into four cases: (a) those which convert every union into a union in a fixed plane parallel to one of the coordinate planes, (b) those which carry every union into a fixed straight line parallel to one of the coordinate axes, (c) those which carry every union into a conical-union of lineal-elements with fixed base-point, and (d) those which carry every union into a single fixed union.

In the rest of our article, whenever we speak of a union-preserving transformation we shall mean one which is not of the four preceding degenerate types.

4. **The special union-preserving transformations.** Any (nondegenerate) union-preserving transformation is said to be *special* if it carries every conical-union of order n into a conical-union of lineal-elements. Any such special transformation is given by equations of the form

$$
\begin{aligned}
X &= X(x, y, z, p_1, q_1, \cdots, p_{n-1}, q_{n-1}),\\
Y &= Y(x, y, z, p_1, q_1, \cdots, p_{n-1}, q_{n-1}),\\
Z &= Z(x, y, z, p_1, q_1, \cdots, p_{n-1}, q_{n-1}),
\end{aligned}
\tag{5}
$$

$$(5)\qquad \begin{aligned} P &= \frac{Y_x + p_1Y_y + q_1Y_z + \cdots + p_nY_{p_{n-1}} + q_nY_{q_{n-1}}}{X_x + p_1X_y + q_1X_z + \cdots + p_nX_{p_{n-1}} + q_nX_{q_{n-1}}}, \\ Q &= \frac{Z_x + p_1Z_y + q_1Z_z + \cdots + p_nZ_{p_{n-1}} + q_nZ_{q_{n-1}}}{X_x + p_1X_y + q_1X_z + \cdots + p_nX_{p_{n-1}} + q_nX_{q_{n-1}}}. \end{aligned}$$

These special union-preserving transformations depend on three arbitrary functions of $(2n+1)$ independent variables. Thus there are $\infty^{3f(2n+1)}$ special union-preserving transformations.

5. **The directrix equation of a general union-preserving transformation** T. We shall say that a union-preserving transformation T is *general* if it does not carry every conical-union of order n into a conical-union of lineal-elements. It follows that for such a general transformation T, we must have either $X_{p_n}\neq 0$, $Y_{p_n}\neq 0$, $Z_{p_n}\neq 0$, or $X_{q_n}\neq 0$, $Y_{q_n}\neq 0$, $Z_{q_n}\neq 0$, or both.

THEOREM 2: *For a general union-preserving transformation T, the four variables $(p_{n-1}, q_{n-1}, p_n, q_n)$ can be eliminated from the three functions defining (X, Y, Z), thus obtaining a single eliminant of the form*

$$(6)\qquad \Omega(X, Y, Z, x, y, z, p_1, q_1, \cdots, p_{n-2}, q_{n-2}) = 0.$$

We call this the *directrix equation* of our general union-preserving transformation T.

Let us proceed with the proof of the above result. In the first place, it is seen by (3) that p_n and q_n can be eliminated from the first and second, and also from the first and third of our equations (1) defining the general union-preserving transformation T. We therefore obtain the two relations

$$(7)\qquad \begin{aligned} Y &= F(X, x, y, z, p_1, q_1, \cdots, p_{n-1}, q_{n-1}), \\ Z &= G(X, x, y, z, p_1, q_1, \cdots, p_{n-1}, q_{n-1}). \end{aligned}$$

At this point, we wish to introduce the following notation. Let X be any function of $(X, Y, Z, x, y, z, p_1, q_1, \cdots, p_m, q_m)$. Then the operation *prime* on X denotes the following linear differential operator

$$(8)\qquad \mathrm{X}' = \mathrm{X}_x + p_1\mathrm{X}_y + q_1\mathrm{X}_z + \cdots + p_{m+1}\mathrm{X}_{p_m} + q_{m+1}\mathrm{X}_{q_m}.$$

That is, X' denotes the total derivative of X with respect to x while considering (X, Y, Z) as constants.

Next upon applying the conditions (3) for a union-preserving transformation, we discover that

$$(9)\qquad P = F_X, \qquad Q = G_X, \qquad F' = 0, \qquad G' = 0.$$

The eliminant with respect to X of the last two of the preceding equations must be identically zero. Differentiating the last two equations with respect to p_n, and also with respect to q_n, we find

$$(10)\qquad \begin{aligned} X_{p_n}F'_X + F_{p_{n-1}} &= 0, & X_{q_n}F'_X + F_{q_{n-1}} &= 0,\\ X_{p_n}G'_X + G_{p_{n-1}} &= 0, & X_{q_n}G'_X + G_{q_{n-1}} &= 0. \end{aligned}$$

From these four equations we can eliminate F_X' and G_X', thus finding the two eliminants

$$(11)\qquad X_{q_n}F_{p_{n-1}} - X_{p_n}F_{q_{n-1}} = 0, \qquad X_{q_n}G_{p_{n-1}} - X_{p_n}G_{q_{n-1}} = 0.$$

Since X_{p_n} and X_{q_n} can not both be zero, it follows from these equations that

$$(12)\qquad F_{p_{n-1}}G_{q_{n-1}} - F_{q_{n-1}}G_{p_{n-1}} = 0.$$

Observe that both F and G are not independent of p_{n-1} and q_{n-1}. For otherwise, if X is explicitly present in either F' or G', X would be independent of p_n and q_n as follows from equations (9). Hence X is not present in the last two of equations (9), and it results that both F and G are functions of X only. This is impossible since our transformation is not degenerate.

By the preceding remarks and equation (12), it follows that (7) has one and only one eliminant with respect to p_{n-1} and q_{n-1}, namely the equation (6). This completes the proof of Theorem 2.

THEOREM 3. *The directrix equation* (6) *completely determines the general union-preserving transformation* T, *and conversely.*

The general union-preserving transformations depend upon one arbitrary function of $(2n+1)$ independent variables. Thus there are $\infty^{f(2n+1)}$ general union-preserving transformations.

In the first place, it is seen that under the general union-preserving transformation T any point in the (X, Y, Z)-space corresponds to a family of ∞^∞ curves in the (x, y, z)-space, defined by a Monge equation of order $(n-2)$. These curves are defined by (6), where (X, Y, Z) are considered as constants. The derivatives of orders $(n-2)$, $(n-1)$, n of these curves are given by the equations

$$(13)\qquad \Omega = 0, \qquad \Omega' = 0, \qquad \Omega'' = 0.$$

Next any conical-union of order n in the (x, y, z)-space is converted into a union. This union is defined by the first two of the preceding equations where $(x, y, z, p_1, q_1, \cdots, p_{n-1}, q_{n-1})$ are regarded as constants. The first derivatives of this union are given by the equations

$$(14)\qquad \Omega_X + P\Omega_Y + Q\Omega_Z = 0, \qquad \Omega'_X + P\Omega'_Y + Q\Omega'_Z = 0.$$

We shall prove that the functions defining (X, Y, Z, P, Q) of our general union-preserving transformation T must satisfy the equations (13) and (14). In the first place (X, Y, Z) must satisfy the directrix equation (6), since it was obtained as a result of eliminating $(p_{n-1}, q_{n-1}, p_n, q_n)$ from the first three of equations (1) defining our general union-preserving transformation T.

Applying the conditions (3) for union-preserving transformations to our directrix equation (6), we obtain the first two of equations (13) and the first of equations (14).

Finally upon applying the conditions (3) for union-preserving transformations to the second of equations (13), we obtain the last of equations (13) and also the last of equations (14).

That we can actually solve equations (13) and (14) for (X, Y, Z, P, Q) follows from the fact that the directrix equation (6) represents a three-parameter family of Monge equations of order $(n-2)$ in the (x, y, z)-space (or a Monge equation of order $(n+1)$ obtained by eliminating the parameters (X, Y, Z), which may be integrated three times).

By Theorems 2 and 3, it follows that the transformation from curve-elements of third order into lineal-elements, which is obtained from the problem of finding the locus of the center of spherical curvature to an arbitrary curve, is general union-preserving. On the other hand, the problem of finding the locus of the center of circular curvature to an arbitrary curve leads to a transformation from curve-elements of third-order into lineal-elements which is special union-preserving.

We note the analogy to Lie's theory of contact transformations of surface-elements. In his work, either a transformation is point-to-point, or else it is point-to-surface or point-to-curve, in which cases it is defined by a single directrix equation or by a pair of directrix equations. In our theory, a union-preserving transformation is special or general, in which case it is defined by a single directrix equation.

6. The union-preserving transformations of space in the domain of curve-elements. We shall now discuss the following result.[4]

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[4] Kasner and DeCicco, *Transformation theory of integrable double-series of lineal-elements*, Bull. Amer. Math. Soc. vol. 46 (1940) pp. 93–100. Kasner, *Lineal element transformations of space for which normal congruences of curves are converted into normal congruences*, Duke Math. J. vol. 5 (1939) pp. 72–83. Kasner and DeCicco, *Curvature element transformations which preserve integrable fields*, Proc. Nat. Acad. Sci. U.S.A. vol. 25 (1939) pp. 104–111.

THEOREM 4. *If a union-preserving transformation T from curve-elements into lineal-elements carries any two unions which possess $n \geqq 2$ as the order of contact into two unions which have, at least, second order contact, then T must be an extended point-transformation.*

In the first place, it is obvious that our union-preserving transformation T is at most from curve-elements of order n into lineal-elements.

Let us first of all assume that T is a general union-preserving transformation. The first derivatives $P=dY/dX$ and $Q=dZ/dX$ of any transformed union under the general union-preserving transformation T defined by (13) and (14) are given by (14). We proceed to find the second derivatives $dP/dX=d^2Y/dX^2$ and $dQ/dX=d^2Z/dX^2$.

In the first place, by differentiating the last of equations (13) totally with respect to x, we find

$$(\Omega_X'' + P\Omega_Y'' + Q\Omega_Z'')dX/dx = -\Omega'''. \tag{15}$$

Next differentiating the equations (14) totally with respect to x and using the conditions (14) and (15), we find

$$\begin{aligned} &\Omega_Y dP/dX + \Omega_Z dQ/dX + \Omega_{XX} + P^2\Omega_{YY} + Q^2\Omega_{ZZ} \\ &\quad +2P\Omega_{XY} + 2Q\Omega_{XZ} + 2PQ\Omega_{YZ} = 0, \\ &\Omega_Y' dP/dX + \Omega_Z' dQ/dX + \Omega_{XX}' + P^2\Omega_{YY}' + Q^2\Omega_{ZZ}' \\ &\quad + 2P\Omega_{XY}' + 2Q\Omega_{XZ}' + 2PQ\Omega_{YZ}' = (1/\Omega'')(\Omega_X'' + P\Omega_Y'' + Q\Omega_Z'')^2. \end{aligned} \tag{16}$$

Now apply the conditions of our theorem. Then dP/dX and dQ/dX can not contain p_{n+1} and q_{n+1}. The only place where these appear are in Ω'''. Their coefficients are $\Omega_{p_{n-2}}$ and $\Omega_{q_{n-2}}$ which can not be zero. Therefore our transformation T is not a general union-preserving transformation from curve-elements of order n into lineal-elements.

Finally let us suppose that our transformation T is special union-preserving from curve-elements of order n into lineal-elements. By (5), it is found upon extending our transformation T once and imposing the condition that dP/dX and dQ/dX be independent of p_{n+1} and q_{n+1} that our special union-transformation T is a union-preserving transformation from curve-elements of order $(n-1)$, at most, into lineal-elements. This is impossible.

The preceding two contradictions lead us to the conclusion that there are no union-preserving transformations from curve-elements of order n into lineal-elements, such that it converts any two unions which possess $n \geqq 2$ as the order of contact into two unions which have at least second order contact.

Therefore the only possible union-preserving transformations which satisfy the conditions of our theorem are those from lineal-elements into lineal-elements. According to Lie's theorem, it follows that the only such transformations are the extended point transformations. This completes the proof of Theorem 4.

From this theorem, it follows that the only union-preserving transformations from curve-elements of order n into curve-elements of order m where $n \geqq m > 1$ are the extended point transformations. *Therefore the only successful union-preserving transformations (in the whole domain of curve-elements) are firstly, the point transformations, and secondly, the union-preserving transformations from curve-elements of order n, where n is 2 or more, into lineal-elements, or the extensions of these two types.*

Thus if T is a union-preserving transformation from curve-elements of order n into curve-elements of order m where $n \geqq m \geqq 1$, then, if $n = m$, T is the extension of order m of a point-transformation; and if $n > m$, T is the extension of order $(m-1)$ of a union-preserving transformation from curve-elements of order $(n-m+1)$ into lineal-elements.

7. Alternate characterization of union-preserving transformations.

Under a union-preserving transformation T, a single union in the (x, y, z)-space corresponds to a single union in the (X, Y, Z)-space, but to a single union in the (X, Y, Z)-space there correspond $\infty^{2(n-1)}$ unions in the (x, y, z)-space.

(It is noted that a point (X_0, Y_0, Z_0) is not considered to be a union of lineal-elements in the (X, Y, Z)-space, but consists of $\infty^{f(1)}$ conical-unions of lineal-elements, each such conical-union corresponding to $\infty^{2(n-1)}$ unions of the Monge equation of order $(n-2)$, defined by our directrix equation (6)).

THEOREM 5. *Any transformation from curve-elements of order n of the (x, y, z)-space into lineal-elements of the (X, Y, Z)-space, by which there correspond exactly $\infty^{2(n-1)}$ unions in the (x, y, z)-space to any single union of the (X, Y, Z)-space, is a union-preserving transformation.*

Let $Y = F(x)$, $Z = G(x)$, $P = F_X$, $Q = G_X$ be any union in the (X, Y, Z)-space. Upon applying our transformation (1) to this union, we find that the corresponding unions in the (x, y, z)-space must satisfy the differential equations

$$
\begin{aligned}
&Y(x, y, z, p_1, q_1, \cdots, p_n, q_n) = F[X(x, y, z, p_1, q_1, \cdots, p_n, q_n)],\\
(17)\quad &Z(x, y, z, p_1, q_1, \cdots, p_n, q_n) = G[X(x, y, z, p_1, q_1, \cdots, p_n, q_n)],\\
&P(x, y, z, p_1, q_1, \cdots, p_n, q_n) = F_X, Q(x, y, z, p_1, q_1, \cdots, p_n, q_n) = G_X.
\end{aligned}
$$

In general, we can solve these equations for $(p_{n-1}, q_{n-1}, p_n, q_n)$ in terms of $(x, y, z, p_1, q_1, \cdots, p_{n-2}, q_{n-2})$. Thus we find that there are at most $\infty^{2(n-1)}$ unions in the (x, y, z)-space which correspond to a given union in the (X, Y, Z)-space.

We now impose the conditions that these represent exactly $\infty^{2(n-1)}$ unions. In the preceding equations, we can think of $(p_{n-1}, q_{n-1}, p_n, q_n)$ as being functions of $(x, y, z, p_1, q_1, \cdots, p_{n-2}, q_{n-2})$. We can differentiate the first equation partially with respect to $(x, y, z, p_1, q_1, \cdots, p_{n-2}, q_{n-2})$, obtaining a set of $(2n-1)$ partial equations. Multiplying these equations by $(1, p_1, q_1, \cdots, p_{n-1}, q_{n-1})$ respectively, adding the results, and using the conditions that p_n is the derivative of p_{n-1} and q_n is the derivative of q_{n-1}, we obtain an equation of the same form as the first of equations (2). Similarly operating upon the second of the preceding equations, we are led to an equation of the same form as the second of equations (2). From these equations of the form (2), we are led to the conclusion that our transformation is union-preserving since the equations must be independent of p_{n+1} and q_{n+1}. This completes the proof of Theorem 5.

As an application, consider the osculating sphere to any arbitrary curve. To a single curve there corresponds a single spherical evolute, that is, the locus of the centers of spherical curvature. But to a given spherical evolute there correspond ∞^4 spherical involutes. Thus our work may be considered to be a generalized theory of spherical evolutes and involutes in space.

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REMARKS ON TRANSITIVITIES OF BETWEENNESS

W. R. TRANSUE

This note provides lattice theoretic interpretations of the transitivities

T_8. $abc\cdot dab\cdot xcd\cdot a\neq b\rightarrow acx$,
T_9. $abc\cdot dab\cdot xcd\cdot a\neq b\rightarrow bcx$,
T_{10}. $abc\cdot abd\cdot xbc\cdot a\neq b\cdot b\neq c\rightarrow xbd$,

introduced by Pitcher and Smiley.[1] It may be recalled that in a lattice the relation abc (b is between a and c) is said to hold if and only if

$$(a\cup b)\cap(b\cup c)=b=(a\cap b)\cup(b\cap c).$$

THEOREM 1. *If L is a lattice then its betweenness relation has one of the transitivities T_8 or T_9 if and only if L is linearly ordered.*

PROOF. It is obvious that T_8 and T_9 are satisfied if L is linearly ordered. To show that T_8 implies linear order, consider two elements a, $c\in L$. Suppose that a and c are not comparable, that is, that none of the relations $a=c$, $a<c$, $a>c$ holds. Then $a\neq a\cup c$, $c\neq a\cup c$. Moreover, we have

$$a\ a\cup c\ c\cdot a\cap c\ a\ a\cup c\cdot a\cup c\ c\ a\cap c\cdot a\neq a\cup c$$

and by T_8 this implies $a\ c\ a\cup c$ which, with $a\ a\cup c\ c$, implies $c=a\cup c$, contrary to our assumption that $c\neq a\cup c$. In the same way T_9 can be shown to imply linear order.

THEOREM 2. *If L is a lattice then its betweenness relation has the transitivity T_{10} if and only if L is linearly ordered or is composed of two linearly ordered systems with a common greatest element, I, and a common least element,* 0.

PROOF. It is easy to see that lattice betweenness in such a lattice has the transitivity T_{10}. Denote the two linearly ordered systems by L_1 and L_2. Then if, in the hypotheses of T_{10}, $b\neq 0$, $b\neq I$, $b\in L_1$, all the elements a, c, d, and x must belong to L_1 and the conclusion follows from the fact that T_{10} holds for linear order. If $b=0$ or $b=I$ in the hypotheses of T_{10} and if $a\in L_1$, then we must have $c\in L_2$, $d\in L_2$, $x\in L_1$ and the conclusion again follows.

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[1] Everett Pitcher and M. F. Smiley, *Transitivities of betweenness*, Trans. Amer. Math. Soc. vol. 52 (1942) pp. 95–114. We shall use the notations and terminology of Pitcher and Smiley.

To show that T_{10} implies that the lattice must have one of these forms, we show first that the lattice consists of a number of chains connecting 0 and I unless it is linearly ordered and then show that the number of these chains cannot be greater than two. To prove the first of these statements let us show that if a and b are two elements which are not comparable then there is a greatest element, I, and a least element, 0, and $a \cup b = I$, $a \cap b = 0$. Suppose $a \cup b \neq I$, that is, that $a \cup b \geqq x$ for every $x \in L$ fails. Then there exists $u \in L$ such that $u > a \cup b$ and

$$a\ a \cup b\ u \cdot a\ a \cup b\ b \cdot b\ a \cup b\ u \cdot a \neq a \cup b \cdot a \cup b \neq u \rightarrow b\ a \cup b\ b$$

which implies $a \cup b = b$ and $b \geqq a$, contrary to the assumption that a and b are not comparable. This shows that $a \cup b = I$, and $a \cap b = 0$ may be shown in the same way.

To show that the number of chains is not greater than two we again suppose the contrary. Then we have three elements a, b, and c, no two of which are comparable. Then from T_{10}, since $a \cup b = a \cup c = b \cup c = I$,

$$a\ I\ b \cdot a\ I\ c \cdot c\ I\ b \cdot a \neq I \cdot I \neq b \rightarrow c\ I\ c$$

which implies $c = I$, contrary to our assumption that it was not comparable with a and b. This completes the proof of Theorem 2.

Finally we include a correction to the previously quoted paper of Pitcher and Smiley.[2] It is stated (p. 113) that "Examples of semi metric spaces are easily given in which τ_2 fails."

$$\tau_2. \quad abc \cdot adb \cdot dbc \rightarrow adc.$$

That this transitivity is, in fact, present in all semi metric spaces may be seen as follows. The hypotheses of τ_2 for betweenness in a semi metric space are

$$\delta(a, b) + \delta(b, c) = \delta(a, c), \tag{1}$$

$$\delta(a, d) + \delta(d, b) = \delta(a, b), \tag{2}$$

$$\delta(d, b) + \delta(b, c) = \delta(d, c). \tag{3}$$

Substituting from (2) in (1) and using (3) we have $\delta(a, d) + \delta(d, c) = \delta(a, c)$. A consequence of this fact is that no non-modular lattice can be supplied with a semi metric so that metric and lattice betweenness are the same.

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[2] This error was noticed by the authors and is included here at their request.

CONTINUED FRACTIONS AND BOUNDED ANALYTIC FUNCTIONS

H. S. WALL

1. **Introduction.** In this paper we use the characterization given by Schur [6][1] for analytic functions bounded in the unit circle together with the Stieltjes integral representation of F. Riesz [5] for analytic functions with positive real parts, to obtain a new proof of a theorem [8] characterizing totally monotone sequences in terms of Stieltjes continued fractions. In the first place, Schur used an algorithm which he called a "continued fraction-like" algorithm. We begin by constructing from this an *actual* continued fraction algorithm, and we then characterize the class of analytic functions bounded in the unit circle in terms of this continued fraction. Next, we obtain by a simple transformation a continued fraction for functions with positive real parts.[2] This along with the above mentioned theorem of F. Riesz leads to the theorem [8, pp. 165–166] that the sequence $\{c_p\}$ is totally monotone if and only if the power series $c_0-c_1z+c_2z^2-\cdots$ is the expansion of a continued fraction of the form

$$\frac{g_0}{1}+\frac{g_1z}{1}+\frac{(1-g_1)g_2z}{1}+\frac{(1-g_2)g_3z}{1}+\cdots,$$

where $g_0\geqq 0$, $0\leqq g_p\leqq 1$, $p=1, 2, 3, \cdots$.

2. **An actual continued fraction algorithm derived from the "continued fraction-like" algorithm of Schur.** The continued fraction which we shall consider is as follows:[3]

$$\alpha_0+\frac{(1-\alpha_0\bar{\alpha}_0)z}{\bar{\alpha}_0z}\;\frac{1}{-\alpha_1}+\frac{(1-\alpha_1\bar{\alpha}_1)z}{\bar{\alpha}_1z}\;\frac{1}{-\alpha_2}+\cdots, \tag{2.1}$$

in which the α_p are complex constants with moduli not exceeding unity, and z is a complex variable. It will be convenient to suppose that if for some p, $|\alpha_p|=1$, then the continued fraction *terminates* with the first identically vanishing partial numerator.

The pth approximant of (2.1) will be denoted by $A_p(z)/B_p(z)$, where $A_0(z)=\alpha_0$, $B_0(z)=1$, $A_1(z)=z$, $B_1(z)=\bar{\alpha}_0z$, and the other nu-

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[1] Numbers in brackets refer to the bibliography at the end of the paper.

[2] A special case of this was given in [9, p. 415].

[3] Hamel [2] used a somewhat different continued fraction for the purpose of characterizing analytic functions bounded in the unit circle. He was obliged to use an unconventional definition of convergence.

merators and denominators are to be computed by means of the recursion formulas

$$(2.2)\quad \begin{aligned} A_{2p}(z) &= \alpha_p A_{2p-1}(z) - A_{2p-2}(z),\\ B_{2p}(z) &= \alpha_p B_{2p-1}(z) - B_{2p-2}(z),\\ A_{2p+1}(z) &= \bar{\alpha}_p z A_{2p}(z) + (1-\alpha_p\bar{\alpha}_p) z A_{2p-1}(z),\\ B_{2p+1}(z) &= \bar{\alpha}_p z B_{2p}(z) + (1-\alpha_p\bar{\alpha}_p) z B_{2p-1}(z), \end{aligned} \qquad p = 1, 2, 3, \cdots.$$

If we write

$$\pi_p = \prod_{q=0}^{p} (1 - \alpha_q\bar{\alpha}_q),$$

then we have the "determinant formulas"

$$(2.3)\qquad A_{2p+1}(z)B_{2p}(z) - A_{2p}(z)B_{2p+1}(z) = (-1)^p z^{p+1}\pi_p,$$

$$(2.4)\qquad A_{2p+2}(z)B_{2p}(z) - A_{2p}(z)B_{2p+2}(z) = (-1)^p z^{p+1}\pi_p\alpha_{p+1}.$$

These may be readily derived from the recursion formulas.

We shall now establish the following theorem:

THEOREM A. *A function $f(z)$ is analytic and has modulus not greater than unity for $|z| < 1$ if and only if it is equal to a terminating continued fraction of the form* (2.1), *or is the limit for $|z| < 1$ of the sequence of even approximants of a nonterminating continued fraction of the form* (2.1).

PROOF. Except in the case where $|\alpha_0| = 1$, so that the continued fraction is equal to the constant α_0, the moduli of the even approximants are all *less* than 1 for $|z| < 1$. In fact, consider the linear fractional transformation

$$t = t(w) = \frac{\alpha - zw}{1 - \bar{\alpha}zw} = \alpha + \frac{(1 - \alpha\bar{\alpha})z}{\bar{\alpha}z - (1/w)}, \qquad |\alpha| < 1,$$

of the w-plane into the t-plane, the transformation depending upon the parameter z. If $|z| < 1$, this transformation has the property that $|t| < 1$ for $|w| \leqq 1$. The same is true of the product of two or more such transformations, and inasmuch as $A_{2p}(z)/B_{2p}(z)$ is equal to the product of p such transformations applied to the point $w = \alpha_p$, we conclude that

$$(2.5)\qquad \left| \frac{A_{2p}(z)}{B_{2p}(z)} \right| < 1 \quad \text{for} \quad |z| < 1.$$

In the case where (2.1) does not terminate, the sequence of even approximants converges uniformly for $|z| \leqq r$ for every positive constant r less than 1, and represents an analytic function with modulus not greater than 1 for $|z| < 1$. In fact, from the determinant formula (2.4) we have the relation

$$\frac{A_{2p+2}(z)}{B_{2p+2}(z)} - \frac{A_{2p}(z)}{B_{2p}(z)} = \frac{(-1)^p \pi_p \alpha_{p+1} z^{p+1}}{B_{2p+2}(z) B_{2p}(z)}. \tag{2.6}$$

It is easy to see from the recursion formulas and (2.3), (2.4) that the denominators $B_{2p}(z)$ are different from zero for $|z| \leqq 1$. Hence, the expansion in ascending powers of z of the right member of (2.6) begins with the $(p+1)$th power, or a higher power, of z. Consequently, there exists a power series $P(z) = c_0 - c_1 z + c_2 z^2 - \cdots$ which agrees term by term with the series for $A_{2p}(z)/B_{2p}(z)$ for more and more terms as p is increased. Inasmuch as the coefficients in the expansion of this rational function do not exceed 1 in numerical value by virtue of (2.5), we conclude that $|c_p| \leqq 1$, $p = 0, 1, 2, \cdots$, so that $P(z)$ converges for $|z| < 1$. Moreover, if we put $A_{2p}(z)/B_{2p}(z) = a_0 - a_1 z + a_2 z^2 - \cdots$, then if $|z| \leqq r < 1$,

$$\left| P(z) - \frac{A_{2p}(z)}{B_{2p}(z)} \right| = \left| \sum_{q=0}^{\infty} (-1)^q (c_q - a_q) z^q \right| \leqq \frac{2r^{p+1}}{1 - r}, \tag{2.7}$$

from which we conclude that the sequence $\{A_{2p}(z)/B_{2p}(z)\}$ converges uniformly to $P(z)$ for $|z| \leqq r < 1$; and from (2.5) it follows that $|P(z)| \leqq 1$ for $|z| < 1$.

We must now show, conversely, that if $P(z) = c_0 - c_1 z + \cdots$ is any function which is analytic and has modulus not greater than unity for $|z| < 1$, then there exists a continued fraction of the form (2.1) such that (2.7) holds. To do this, we define the function $P_1(z)$ by

$$P_1(z) = \frac{1}{z} \frac{c_0 - P(z)}{1 - \bar{c}_0 P(z)} = \frac{c_1 - c_2 z + c_3 z^2 - \cdots}{1 - \bar{c}_0 (c_0 - c_1 z + c_2 z^2 - \cdots)}.$$

It is clear that $|c_0| \leqq 1$, being the value of $P(0)$, and that if $|c_0| = 1$, then $P(z) \equiv c_0$. In either case, we put $\alpha_0 = c_0$. If we suppose that $|c_0| < 1$, then, from the character of the above transformation and from Schwarz's lemma, it follows that $P_1(z)$ is analytic and has modulus not exceeding unity for $|z| < 1$. Take $\alpha_1 = P_1(0)$. If $|\alpha_1| = 1$, then $P_1(z) \equiv \alpha_1$. If, however, $|\alpha_1| < 1$, we write

$$P_2(z) = \frac{1}{z} \frac{\alpha_1 - P_1(z)}{1 - \bar{\alpha}_1 P_1(z)}, \qquad \alpha_2 = P_2(0).$$

The function $P_2(z)$ is analytic and has modulus not greater than unity for $|z|<1$. If $|\alpha_2|=1$, then $P_2(z)\equiv\alpha_2$, while if $|\alpha_2|<1$, the process may be continued. In this way we obtain a finite or infinite sequence of functions

$$P_0(z) = P(z),\ P_1(z),\ P_2(z),\ \cdots, \tag{2.8}$$

satisfying the relations

$$P_{k+1}(z) = \frac{1}{z}\,\frac{\alpha_k - P_k}{1-\bar{\alpha}_k P_k}, \qquad P_k = \frac{\alpha_k - zP_{k+1}}{1-\bar{\alpha}_k zP_{k+1}}, \qquad \alpha_k = P_k(0). \tag{2.9}$$

From these we derive immediately a formal terminating or nonterminating continued fraction expansion (2.1) for $P(z)$. In the terminating case, the expansion is obviously valid, being in fact an identity. In the nonterminating case we have, for arbitrary k, the identity

$$P(z) = \frac{P_k(z)A_{2k-1}(z) - A_{2k-2}(z)}{P_k(z)B_{2k-1}(z) - B_{2k-2}(z)},$$

so that, by (2.3),

$$P(z) - \frac{A_{2k-2}(z)}{B_{2k-2}(z)} = \frac{P_k(z)(-1)^{k-1}\pi_{k-1}z^k}{B_{2k}(z)\left[P_k(z)B_{2k-1}(z) - B_{2k-2}(z)\right]}.$$

From this it readily follows that the power series for $A_{2k-2}(z)/B_{2k-2}(z)$ agrees term by term with $P(z)$ for more and more terms as k is increased, and consequently (2.7) holds.

This completes the proof of Theorem A. This proof is the same as that of Schur [6], except that we have used the formulas and notation of continued fractions. It should be observed that the constants α_p are uniquely determined by means of the given function $P(z)$, either as an infinite sequence in the one case or as a finite sequence in the other.

3. A continued fraction expansion for functions with positive real parts. A function $k(z)$ is analytic and has a nonnegative real part for $|z|<1$ and has the value 1 for $z=0$ if and only if there exists a function $P(z)$ which is analytic and has modulus not greater than 1 for $|z|<1$, such that

$$k(z) = \frac{1+zP(z)}{1-zP(z)}. \tag{3.1}$$

This follows from Schwarz's lemma and the relation

$$R(k(z)) = \frac{1 - |zP(z)|^2}{|1 - zP(z)|^2}.$$

We shall now obtain a continued fraction expansion for $k(z)$.

Let $\beta_0 = 1$, and determine $\beta_1, \beta_2, \cdots$ by the relations

$$\beta_{p+1} = -\frac{\bar{\alpha}_p - \beta_p}{1 - \alpha_p \beta_p}, \qquad p = 0, 1, 2, \cdots, \tag{3.2}$$

where the α_p are the numbers appearing in the continued fraction (2.1) for the function $P(z)$ determined by (3.1). The numbers β_p have moduli equal to 1, and form an infinite sequence or a finite sequence according as the sequence $\{\alpha_p\}$ is infinite or finite, respectively. Instead of the functions (2.9) we now introduce functions $h_p(z)$ by

$$h_p(z) = \frac{1 - \beta_p P_p(z)}{1 + z\beta_p P_p(z)}, \qquad p = 0, 1, 2, \cdots. \tag{3.3}$$

By means of (2.9) and (3.2) we then find that

$$h_p(z) = \frac{\beta_p - \bar{\alpha}_p}{\beta_{p+1} - \beta_p z + (\beta_{p+1} + \bar{\alpha}_p) z h_{p+1}(z)}, \qquad p = 0, 1, 2, \cdots. \tag{3.4}$$

Remembering that $\beta_0 = 1$, we therefore have the formal continued fraction expansion:

$$\begin{aligned} \frac{1 - P(z)}{1 + zP(z)} = \frac{\beta_0 - \bar{\alpha}_0}{\beta_1 - \beta_0 z} + \frac{(\beta_1 + \bar{\alpha}_0)(\beta_1 - \bar{\alpha}_1)z}{\beta_2 - \beta_1 z} \\ + \frac{(\beta_2 + \bar{\alpha}_1)(\beta_2 - \bar{\alpha}_2)z}{\beta_3 - \beta_2 z} + \cdots. \end{aligned} \tag{3.5}$$

On multiplying both members of (3.5) by $2z/(1-z)$, adding 1 to both members, and then taking reciprocals, we have, using (3.1),

$$\begin{aligned} k(z) = \frac{1 + z}{1 - z} + \frac{2(\beta_0 - \bar{\alpha}_0)z}{\beta_1 - \beta_0 z} + \frac{(\beta_1 + \bar{\alpha}_0)(\beta_1 - \bar{\alpha}_1)z}{\beta_2 - \beta_1 z} \\ + \frac{(\beta_2 + \bar{\alpha}_1)(\beta_2 - \bar{\alpha}_2)z}{\beta_3 - \beta_2 z} + \cdots. \end{aligned} \tag{3.6}$$

In case $|\alpha_p| < 1$, $p = 0, 1, 2, \cdots, n-1$, $|\alpha_n| = 1$, this continued fraction terminates, the last partial quotient being equal to

$$\frac{(\beta_n + \bar{\alpha}_{n-1})(1 - \alpha_n \beta_n)z}{1 + \alpha_n \beta_n z};$$

while if $|\alpha_p|<1$, $p=0, 1, 2, \cdots$, the continued fraction does not terminate. In the first case, $k(z)$ is of course equal to the continued fraction. In the second case, the continued fraction converges uniformly in the neighborhood of the origin by a well known theorem [4, p. 259]. An easy argument (cf. [9, pp. 415–416]) then shows that its value is $k(z)$.

If, in particular, $k(z)$ is real when z is real, then $\beta_p=1$, $p=0, 1, 2, \cdots$, the α_p are real, and the continued fraction for $k(z)$ can be thrown by means of an equivalence transformation into the form

$$(3.7) \qquad k(z) = \frac{1+z}{1-z}\cdot\frac{1}{1\;+}\;\frac{g_1w}{1\;+}\;\frac{(1-g_1)g_2w}{1}\;+\;\frac{(1-g_2)g_3w}{1}\;+\cdots$$

where $w=4z/(1-z)^2$ and $g_p=(1-\alpha_{p-1})/2$, $p=1, 2, 3, \cdots$. Thus $0\leqq g_p\leqq 1$, the continued fraction terminating in case equality holds for some value of p.

It is readily seen that, conversely, if the α_p are given, then the function $k(z)$ given by (3.6) or (3.7) has the stated properties. We therefore have the following theorem:

THEOREM B. *A function $k(z)$ is analytic and has a nonnegative real part for $|z|<1$, and is equal to* 1 *for $z=0$, if and only if it has a continued fraction expansion of the form* (3.6), *where the α_p are constants with moduli not greater than* 1, *and the β_p are given in terms of the α_p by* (3.2), *β_0 being equal to* 1. *If $k(z)$ is real when z is real, the continued fraction can be thrown into the form* (3.7).

4. A characterization of totally monotone sequences in terms of continued fractions.

A sequence $\{c_p\}$ of real numbers is called *totally monotone* if all the differences $\Delta^m c_n = c_n - C_{m,1}c_{n+1} + C_{m,2}c_{n+2} - \cdots + (-1)^m C_{m,m}c_{n+m}$, $m, n=0, 1, 2, \cdots$, are nonnegative. If $c_0=0$, then $c_p=0$ for $p=1, 2, 3, \cdots$. Excepting in this trivial case, we may normalize by dividing every member of the sequence by c_0, and may thus assume that $c_0=1$.

Hausdorff [3] showed that $\{c_p\}$ is totally monotone if and only if there exists a bounded nondecreasing function $\phi(u)$ such that

$$(4.1) \qquad c_p = \int_0^1 u^p d\phi(u), \qquad p = 0, 1, 2, \cdots.$$

This is equivalent to saying that

$$(4.2) \qquad c_0 - c_1w + c_2w^2 - \cdots = \int_0^1 \frac{d\phi(u)}{1+wu}\cdot$$

We now turn our attention to a theorem of F. Riesz [5] which, along with Theorem B, will furnish a characterization of totally monotone sequences in terms of continued fractions.

THEOREM C. *A function $k(z)$ is analytic and has a nonnegative real part for $|z|<1$, and is equal to 1 for $z=0$, if and only if it has an integral representation of the form*

$$k(z) = \int_0^{2\pi} \frac{e^{it}+z}{e^{it}-z}\,d\alpha(t), \tag{4.3}$$

where $\alpha(t)$ is a nondecreasing function such that $\alpha(0)=0$, $\alpha(2\pi)=1$. The function $\alpha(t)$ is determined uniquely to an additive constant at all its points of continuity by $k(z)$.

PROOF. Put

$$\begin{aligned} k(z) &= 1 + a_1z + a_2z^2 + \cdots \\ &= 1 + \sum_{p=1}^{\infty} (b_p + ic_p)r^p(\cos p\theta + i \sin p\theta), \end{aligned} \tag{4.4}$$

where $z=re^{i\theta}$, $0<r<1$. Then, if

$$u(r, \theta) = \Re(k(z)) = 1 + \sum_{p=1}^{\infty} r^p(b_p \cos p\theta - c_p \sin p\theta) \geqq 0,$$

we conclude that the function

$$\alpha_r(t) = \frac{1}{2\pi}\int_0^t u(r, \theta)d\theta, \qquad 0 \leqq t \leqq 2\pi,$$

is a nondecreasing function of t; and $\alpha_r(0)=0$, $\alpha_r(2\pi)=1$. Moreover,

$$\int_0^{2\pi} d\alpha_r(t) = 1, \qquad \int_0^{2\pi} 2\cos qt d\alpha_r(t) = r^q b_q,$$
$$\int_0^{2\pi} -2\sin qt d\alpha_r(t) = r^q c_q.$$

Using a well known theorem, we may now determine a nondecreasing function $\alpha(t)$ such that $\alpha(0)=0$, $\alpha(2\pi)=1$, and a sequence of values of r approaching 1, such that these equations go over into

$$\int_0^{2\pi} d\alpha(t) = 1, \qquad \int_0^{2\pi} 2\cos qt d\alpha(t) = b_q,$$
$$\int_0^{2\pi} -2\sin qt d\alpha(t) = c_q.$$

When these values are substituted in (4.4), that series may be written as the integral of a geometric series. On summing the latter, we obtain (4.3). The essential uniqueness of the function $\alpha(t)$ follows from the fact that an arbitrary continuous function can be approximated uniformly by a trigonometric polynomial [5, pp. 38–39].

Conversely, if $\alpha(t)$ is any bounded nondecreasing function such that $\alpha(0)=0$, $\alpha(2\pi)=1$, then one may verify at once that the function $k(z)$ defined by (4.3) has the required properties.

We now make in (4.3) the change of variable $u=\sin^2(t/2)$, and that formula becomes

$$k(z) = \frac{1+z}{1-z}\int_0^1 \frac{d\phi(u)}{1+wu} - iw\int_0^1 \frac{[u(1-u)]^{1/2}\cdot d\psi(u)}{1+wu}, \tag{4.5}$$

where $\alpha(t)+\alpha(2\pi-t)=\psi(u)$, $1-\alpha(2\pi-t)=\phi(u)$, $w=4z/(1-z)^2$. The function $\psi(u)$ is of bounded variation, and $\phi(u)$ is bounded and nondecreasing: $\phi(0)=0$, $\phi(1)=1$. The second integral in (4.5) vanishes identically if and only if $k(z)$ is real when z is real. On the other hand, $k(z)$ is of this character if and only if it has an expansion of the form (3.7). On putting these two facts together we obtain the relation

$$\int_0^1 \frac{d\phi(u)}{1+wu} = \frac{1}{1} \, \underset{+}{} \, \frac{g_1w}{1} \, \underset{+}{} \, \frac{(1-g_1)g_2w}{1} \, \underset{+}{} \, \frac{(1-g_2)g_3w}{1} \, \underset{+}{} \cdots \tag{4.6}$$

Recalling now the definition of a totally monotone sequence, and the statement made concerning (4.2), we have the following theorem:

THEOREM D. *The sequence* $\{c_p\}$ *of real numbers, of which* $c_0=1$, *is totally monotone if and only if the power series* $c_0-c_1w+c_2w^2-\cdots$ *has a continued fraction expansion of the form*

$$\frac{1}{1} \, \underset{+}{} \, \frac{g_1w}{1} \, \underset{+}{} \, \frac{(1-g_1)g_2w}{1} \, \underset{+}{} \, \frac{(1-g_2)g_3w}{1} \, \underset{+}{} \cdots \tag{4.7}$$

where $0\leqq g_p\leqq 1$, $p=1, 2, 3, \cdots$, *it being agreed that the continued fraction shall terminate in case some partial numerator vanishes identically.*

An interesting consequence of this theorem is the fact that there exists a sequence of polynomials $G_1(g_1)$, $G_2(g_1, g_2)$, $G_3(g_1, g_2, g_3)$, $\cdots$, where G_p depends upon p variables, such that every totally monotone sequence with $c_0=1$ can be represented parametrically in the form

$$c_p = G_p(g_1, g_2, \cdots, g_p), \qquad p = 0, 1, 2, \cdots \ (G_0 = 1),$$

where $0\leqq g_p\leqq 1$, $p=1, 2, 3, \cdots$. Conversely, every sequence of this form is totally monotone. To obtain the polynomials G_p it is but necessary to observe that the power series expansion of (4.7) is the series $c_0 - c_1 w + c_2 w^2 + \cdots$ where the c_p are polynomials in the g_p. These may be most conveniently calculated by means of formulas given by Stieltjes [7, pp. 419–420]. We mention only that

$$(4.8) \qquad G_1(g_1) = g_1, \qquad G_2(g_1, g_2) = g_1^2(1 - g_2) + g_1 g_2.$$

F. Riesz [5, p. 56] showed that a sequence $\{c_p\}$, where $c_0 = 1$, is totally monotone if and only if for every n the point $(c_0, c_1, \cdots, c_{n-1})$ lies in the convex extension of the continuous curve given in parametric form by $(1, u, u^2, \cdots, u^{n-1})$, $0 \leqq u \leqq 1$. The polynomials G_p therefore furnish a parametric representation for the points of this convex set. This can be verified for $n = 3$ by means of (4.8).

5. **Continued fraction transformations.** If the α_p are real in (3.5), that relation may be written in the form

$$(5.1) \qquad \frac{(1 - z)}{2} \frac{1 - P(z)}{1 + zP(z)} = \frac{g_1}{1} + \frac{(1 - g_1)g_2 w}{1} + \frac{(1 - g_2)g_3 w}{1} + \cdots$$

where $g_p = (1 - \alpha_{p-1})/2$, $p = 1, 2, 3, \cdots$, and $w = 4z/(1 - z)^2$. To indicate the dependence of $P(z)$ upon the α_p we shall now write $P(z) = (z; \alpha_0, \alpha_1, \alpha_2, \cdots)$. The following relations may be readily verified [6]:

$$(5.2) \qquad -P(z) = (z; -\alpha_0, -\alpha_1, -\alpha_2, \cdots),$$

$$(5.3) \qquad P(-z) = (z; \alpha_0, -\alpha_1, \alpha_2, -\alpha_3, \cdots),$$

$$(5.4) \qquad P(z^n) = (z; \alpha_0, 0, \cdots, 0, \alpha_1, 0, \cdots, 0, \alpha_2, 0, \cdots),$$

where 0 occurs $n - 1$ times between α_p and α_{p+1}. We observe that the effect of replacing α_{p-1} by $-\alpha_{p-1}$ is to replace g_p by $1 - g_p$; and that the effect of replacing α_{p-1} by 0 is to replace g_p by 1/2. These facts together with the preceding relations enable us to obtain a number of transformations of the continued fraction in (5.1).

Let us denote the right member of (5.1) by $F(w)$, and write $F(w) = [w; g_1, g_2, g_3, \cdots]$. On replacing $P(z)$ by $-P(z)$ and α_{p-1} by $-\alpha_{p-1}$, that is, g_p by $1 - g_p$, $p = 1, 2, 3, \cdots$, we obtain at once the relation [8, p. 166; 9, p. 416]

$$(5.5) \qquad \frac{1 - F(w)}{1 + wF(w)} = [w; 1 - g_1, 1 - g_2, 1 - g_3, \cdots].$$

Similarly, on replacing $P(z)$ by $P(-z)$ and α_p by $(-1)^p \alpha_p$ we get [1, p. 191]

$$(5.6)\qquad 1 - F(-w/(1+w)) \doteq [w; 1-g_1, g_2, 1-g_3, g_4, \cdots].$$

Using (5.4), replacing $f(z)$ by $f(z^n)$ in (5.1), we obtain a relation of the form

$$(5.7)\qquad \frac{F(U_n(w))}{V_n(w) + W_n(w)F(U_n(w))} = [w; g_1, 1/2, \cdots, 1/2, g_2, 1/2, \cdots, 1/2, g_3, 1/2, \cdots],$$

where $1/2$ appears $n-1$ times in each place, and where $U_n(w)$, $V_n(w)$, $W_n(w)$ are rational functions of w given by

$$U_n(w) = \frac{4w^n}{\{((1+w)^{1/2}+1)^n - ((1+w)^{1/2}-1)^n\}^2},$$

$$V_n(w) = (1+w)^{1/2}\frac{\{((1+w)^{1/2}+1)^n - ((1+w)^{1/2}-1)^n\}^2}{((1+w)^{1/2}+1)^{2n} - ((1+w)^{1/2}-1)^{2n}},$$

$$W_n(w) = \frac{w\{((1+w)^{1/2}+1)^{n-1} - ((1+w)^{1/2}-1)^{n-1}\}^2}{2w^{n-1} - \{((1+w)^{1/2}+1)^{2n-1} - ((1+w)^{1/2}-1)^{2n-1}\}}.$$

Here, that branch of $(1+w)^{1/2}$ is to be taken which reduces to 1 for $w=0$.

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ON MINIMUM CIRCUMSCRIBED POLYGONS

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This paper contains the proofs of two theorems on the n-gon M_n of minimum area circumscribed about a convex region R in the plane. Theorem 1 shows that the area of M_n is a convex function of n and Theorem 3 shows that if R is symmetric about a point there exists an M_{2n} which is also symmetric. The corresponding theorems on inscribed polygons are also given. These theorems were conjectured by R. B. Kershner.

The symbols a and b with subscripts will be used to represent the sides of circumscribed polygons or the vertices of inscribed polygons. It will be convenient to replace the circular order of the sides (vertices) $a_0, a_1, \cdots, a_{n-1}$ of a polygon by an artificial linear ordering $a_0 < a_1 < \cdots < a_{n-1} < a_n$, where a_n represents the same side (vertex) as a_0. The order of the sides (vertices) of circumscribed (inscribed) polygons is established by the order of the contact points (vertices) on the boundary of the convex region.

LEMMA 1. *Let* $A_n = a_0a_1 \cdots a_{n-1}$ *and* $B_m = b_0b_1 \cdots b_{m-1}$ *be two polygons circumscribed about the convex region* R *and let either* (1) $a_0 \leqq b_0 < b_1 < a_1 \leqq a_{r-1} \leqq b_{s-1} < b_s < a_r$ *or* (2) $a_0 \leqq b_0 < b_1 < a_1 \leqq b_{i-1} \leqq a_{r-1} < a_r < b_s$. *Let* $C_{n-r+s} = a_0b_1b_2 \cdots b_{s-1}a_r \cdots a_{n-1}$ *and* $D_{m-s+r} = b_0a_1a_2 \cdots a_{r-1}b_s \cdots b_{m-1}$. *Then the areas satisfy the inequality* $A + B \geqq C + D$ *and there is equality if and only if* $a_0 = b_0$ *and* $a_{r-1} = b_{s-1}$.

PROOF. The common part of A and B is the common part of C and D. The remaining part of $A+B$ is the remaining part of $C+D$ together with the areas of the quadrilaterals $a_0a_1b_0b_1$ and $a_{r-1}a_rb_{s-1}b_s$. Equality holds if and only if both these areas are zero, that is, if $a_0 = b_0$ and $a_{r-1} = b_{s-1}$.

LEMMA 2. *If* M_n *is an* n*-gon of maximum area inscribed in the convex region* R, *the vertices of* M_n *are contact points of the sides of a circumscribed polygon.*

PROOF. If $M_n = a_0a_1 \cdots a_{n-1}$, the line through a_1 parallel to the line a_0a_2 is a supporting line of R, for otherwise M_n would not have maximum area. Supporting lines determined similarly at all vertices of M_n are seen to form a circumscribed polygon.

LEMMA 3. *Let* $A_n = a_0a_1 \cdots a_n$ *and* $B_m = b_0b_1 \cdots b_m$ *be two polygons*

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inscribed in the convex region R, each having as vertices points of contact of the sides of some circumscribed polygon, and let either (1) $a_0 \leqq b_0 < b_1 < a_1 \leqq a_{r-1} \leqq b_{s-1} < b_s < a_r$ *or* (2) $a_0 \leqq b_0 < b_1 < a_1 \leqq b_{s-1} \leqq a_{r-1} < a_r < b_s$. *Let* $C_{n-r+s} = a_0 b_1 b_2 \cdots b_{s-1} a_r \cdots a_{n-1}$ *and* $D_{m-s+r} = b_0 a_1 a_2 \cdots a_{r-1} b_s \cdots b_{m-1}$. *Then the areas satisfy the inequality* $A+B \leqq C+D$ *and there is equality if and only if* $a_0 = b_0$ *and* $a_{r-1} = b_{s-1}$.

PROOF. The vertices b_0 and b_1 are in or on the triangle consisting of a_0a_1 and the adjacent sides of the circumscribed polygon corresponding to a_0 and a_1. Hence (if $a_0 \neq b_0$) a_0b_0 produced meets a_1b_1 produced. Hence $\Delta a_0b_0b_1 \leqq \Delta a_0b_0a_1$, with equality only if $a_0 = b_0$. Let a_0b_1 intersect a_1b_0 at p. Then $\Delta a_0pa_1 \geqq \Delta b_0pb_1$, with equality only if $a_0 = b_0 = p$.

Let $a_{r-1}b_s$ intersect a_rb_{s-1} at q. Then similarly, if $a_{r-1} < b_{s-1} < b_s < a_r$, $\Delta a_{r-1}qa_r > \Delta b_{s-1}qb_s$, and if $b_{s-1} < a_{r-1} < a_r < b_s$, $\Delta b_{s-1}qb_s > \Delta a_{r-1}qa_r$.

If $a_{r-1} \leqq b_{s-1} < b_s < a_r$, $C+D-A-B = \Delta a_0pa_1 - \Delta b_0pb_1 + \Delta a_{r-1}qa_r - \Delta b_{s-1}qb_s \geqq 0$ and equality holds only if $a_0 = p = b_0$ and $a_{r-1} = q = b_{s-1}$. If $b_{s-1} \leqq a_{r-1} < a_r < b_s$, $C+D-A-B = \Delta a_0pa_1 - \Delta b_0pb_1 + \Delta b_{s-1}qb_s - \Delta a_{r-1}qa_r \geqq 0$ and equality holds only if $a_0 = p = b_0$ and $a_{r-1} = q = b_{s-1}$. Thus in either case $A+B \leqq C+D$ with equality if and only if $a_0 = b_0$ and $a_{r-1} = b_{s-1}$.

THEOREM 1. *If* M_n *is an n-gon of minimum area circumscribed about the convex region R, the areas satisfy the inequality* $M_n + M_{n+2} \geqq 2M_{n+1}$.

PROOF. Let $M_n = a_0a_1 \cdots a_{n-1}$, $M_{n+2} = b_0b_1 \cdots b_{n+1}$.

Case 1. Let there be a sequence of sides $a_0 \leqq b_0 < b_1 < b_2 < a_1$. In this case, if $C_{n+1} = a_0b_1a_1 \cdots a_{n-1}$, $D_{n+1} = b_0b_2 \cdots b_{n+1}$, $M_n + M_{n+2} - C - D = a_0a_1b_0b_1 + b_0a_1b_1b_2 > 0$. Hence $M_n + M_{n+2} > C+D \geqq 2M_{n+1}$.

Case 2. Let there be no such sequence of sides. Then there must be a sequence of the form $a_0 \leqq b_0 < b_1 < a_1 \leqq a_{r-1} \leqq b_{s-1} < b_s < a_r$ since there are n intervals $[a_i, a_{i+1})$ and $n+2$ b's in these intervals. We may assume $s \leqq r+1$ since otherwise we could renumber the sides starting from a_{r-1} which we would call a_0. We may even assume that $s = r+1$, for otherwise $b_{r+1} > a_r$ and $b_{n+1} < a_n$ and hence there is a first subscript $t > r$ for which $b_{t+1} < a_t$. Then $b_t \geqq a_{t-1}$ and we may replace t by r and $t+1$ by s so that $s = r+1$. Let $C_{n+1} = a_0b_1 \cdots b_{s-1}a_r \cdots a_{n-1}$ and $D_{n+1} = b_0a_1 \cdots a_{r-1}b_s \cdots b_{n+1}$. Then by Lemma 1, $M_n + M_{n+2} \geqq C+D \geqq 2M_{n+1}$.

THEOREM 2. *If* M_n *is an n-gon of maximum area inscribed in the convex region R, the areas satisfy the inequality* $M_n + M_{n+2} \leqq 2M_{n+1}$.

PROOF. Let $M_n = a_0a_1 \cdots a_n$, and $M_{n+2} = b_0b_1 \cdots b_{n+1}$.

Case 1. Let there be a sequence of sides $a_0 \leqq b_0 < b_1 < b_2 < a_1$. In this case, if $C_{n+1} = a_0 b_1 a_1 \cdots a_{n-1}$ and $D_{n+1} = b_0 b_2 \cdots b_{n-1}$, $C + D - M_n - M_{n+2} = a_0 b_1 a_1 - b_0 b_1 b_2$ which is greater than 0 as can be shown using Lemma 2 and elementary geometry. Hence $M_n + M_{n+2} < C + D \leqq 2M_{n+1}$.

Case 2. Let there be no such sequence of sides. The proof is similar to that of Case 2 of Theorem 1 except that Lemma 3 is used instead of Lemma 1.

THEOREM 3. *If R is a convex region symmetric about a point* 0, *among the $2n$-gons of minimum area circumscribed about R there is one which is symmetric about* 0.

PROOF. Let $M_{2n} = a_0 a_1 \cdots a_{2n-1}$ be a minimum circumscribed $2n$-gon and let $M' = b_0 b_1 \cdots b_{2n-1}$ where b_i is the image in 0 of $a_{i \pm n}$. If M_{2n} is not symmetric there is a side $a_i \neq b_i$. We may assume $a_0 \neq b_0$. By reordering the sides if necessary we may assume $a_0 < b_0$ and hence $b_n < a_n$. Let t be the largest number, $0 \leqq t < n$, such that $a_t \leqq b_t$. Then $b_{t+1} < a_{t+1}$ and if we renumber starting from a_t, which we shall call a_0, we have $a_0 \leqq b_0 < b_1 < a_1$. Since a_0 and a_1 are successive sides of M_{2n}, $a_1 < b_n$. Hence $a_0 \leqq b_0 < b_1 < a_1 < b_n \leqq a_n < a_{n+1} < b_{n+1}$. Let $C_{2n} = a_0 b_1 b_2 \cdots b_n a_{n+1} \cdots a_{2n-1}$ and $D_{2n} = b_0 a_1 a_2 \cdots a_n b_{n+1} \cdots b_{2n-1}$. Then, by Lemma 1, $M + M' \geqq C + D$. But C and D are both symmetric and M and M' are equal. Therefore either C or D is a symmetric circumscribed $2n$-gon of area not greater than M_{2n}.

THEOREM 4. *If R is a convex region symmetric about a point* 0, *among the $2n$-gons of maximum area inscribed in R there is one which is symmetric about* 0.

PROOF. The proof is similar to that of Theorem 3 except that Lemma 3 is used instead of Lemma 1.

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ON r-REGULAR CONVERGENCE

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In his paper *On sequences and limiting sets* [1],[1] G. T. Whyburn introduced the notion of regular convergence. He showed that in the cases of 0 and 1 regular convergence (see definition below) that the limit of sequences of many simple topological sets is of the same type as the members of the sequence. It is the purpose of this paper to extend some of these results to higher dimensions. The lack of simple characterizations of the higher dimension sets (such as the n-sphere) makes the results much weaker than in the 0 and 1 dimensional cases.

It is assumed throughout the paper that all sets lie in a compact metric space. All our complexes and cycles will be non-oriented, and the Vietoris cycles and chains (V-cycles and V-chains) will have these as coordinates. The set of all points x whose distance from a set A is less than ϵ will be denoted by $U_\epsilon(A)$. Finally we shall denote the boundary of an r-dimensional complex (or V-chain) z^r by $\dot{z}^r$.

DEFINITION: *A sequence of closed sets (A_i) converging to a limit set A is said to converge r-regularly ($\rightarrow r$) if for every $\epsilon > 0$ there exist numbers $\delta > 0$ and $N > 0$ such that, if $n > N$, any r-dimensional V-cycle in A_n of diameter less than δ is ~ 0 in a subset of A_n of diameter less than ϵ. If $A_i \rightarrow sA$ for all $s \leqq r$, we write $A_i \rightarrow \leqq rA$* [1].

DEFINITION. *A Vietoris cycle $\xi^r = (x_i^r)$ is called a projection cycle if* $\lim_{i\to\infty}$ *(point set x_i^r) $= X$ and each $x_i^r \subset X$. Clearly X is the smallest carrier* [2] *of ξ^r.*

Note. Corresponding to any cycle $\xi^r = (x_i^r)$ of a compact set F, there always exists a projection cycle $\xi_1^r \sim \xi^r$ in F. In fact if a convergent subsequence of (x_i^r) is chosen, this set can be used as the set X of the definition.

THEOREM 1. *If $A_i \rightarrow rA$, then for any $\epsilon > 0$ there exist positive numbers δ and N such that if x^{r+1} is a simplex of A_i $(i > N)$ whose boundary has a V-chain realization* [3] *of diameter less than δ, then x^{r+1} has a V-chain realization of diameter less than ϵ.*

PROOF. Let δ and N be the numbers corresponding to ϵ in the definition of r-regular convergence and consider a simplex x^{r+1} of A_i $(i > N)$

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[1] Numbers in brackets refer to the bibliography at the end of the paper.

whose boundary has a V-chain realization $\eta^r=(y_i^r)$ of diameter less than δ. By the choice of δ and N, $\eta^r\sim 0$ in a subset of A_i of diameter less than ϵ. Thus there is a V-chain $\xi^{r+1}=(x_i^{r+1})$ of diameter less than ϵ such that $\dot{x}_i^{r+1}=y_i^r$. Clearly ξ^{r+1} is the V-chain realization of x^{r+1} of diameter less than ϵ.

THEOREM 2. *If* $A_i\rightarrow\leqq rA$ *then for any* $\epsilon>0$ *there exist positive numbers* $\delta^r(\epsilon)$ *and* $N^r(\epsilon)$ *such that if* x^s $(s\leqq r+1)$ *is a* $\delta^r(\epsilon)$*-simplex of* A_i $(i\geqq N^r(\epsilon))$, *then* x^s *has a V-chain realization in a subset of* A_i *of diameter less than* ϵ.

PROOF. The proof shall be by induction.

The case $r=0$ is clearly a direct consequence of the definition of 0-regular convergence where the numbers $\delta^0(\epsilon)$ and $N^0(\epsilon)$ are the numbers δ and N of the definition.

Suppose the theorem to be true for $r=k-1$ and consider the case $r=k$. Let ϵ be an arbitrary positive number and $\delta<\epsilon$, N be the numbers corresponding to it in Theorem 1. Now by hypothesis there exist positive numbers $\delta^{k-1}(\delta/3)$ and $N^{k-1}(\delta/3)$, and we define $\delta^k(\epsilon)=\min\ (\delta^{k-1}(\delta/3),\ \delta/3)$ and $N^k(\epsilon)=N^{k-1}(\delta/3)+N$. Let x^s $(s\leqq k+1)$ be a $\delta^k(\epsilon)$-simplex of A_i $(i>N^k(\epsilon))$. If $s\leqq k$, we know that x^s has a V-chain realization of diameter less than $\delta/3<\epsilon$ since $\delta^k(\epsilon)\leqq\delta^{k-1}(\delta/3)$ and $N^k(\epsilon)>N^{k-1}(\delta/3)$. If $s=k+1$, the boundary of x^{k+1} is a k-dimensional $\delta^{k-1}(\delta/3)$-cycle of A_i $(i>N^{k-1}(\delta/3))$. Thus each simplex of this cycle has a V-chain realization in a subset of A_i of diameter less than $\delta/3$. Furthermore we may suppose these realizations to be chosen so that common sides of two simplices have the same realizations. Now adding these V-chains for the simplices of the cycle we obtain a sequence of k-dimensional cycles, a subsequence of which yields a k-dimensional V-cycle of diameter less than $\delta^k(\epsilon)+2(\delta/3)<\delta/3+2(\delta/3)=\delta$ which is clearly a realization of the boundary of x^{k+1}. Now by the choice of δ, we know that x^{k+1} has a V-chain realization in a subset of A_i $(i>N^k(\epsilon)>N)$ of diameter less than ϵ. Thus the theorem is true for $r=k$ and hence for all cases.

THEOREM 3. *If* $A_i\rightarrow\leqq(r-1)A$ *and C is the smallest carrier of an essential projection cycle* $\xi^r=(x_j^r)$ *of A, then C can be expressed as* $\lim_{i\to\infty} C_i$ *where* C_i *is the smallest carrier of a cycle* ξ_i^r *of* A_i *which will be essential for all sufficiently large i.*

PROOF. Since ξ^r is essential, there is a positive number η such that ξ^r is not $\sim_\eta 0$ in C. Let $\epsilon_k\rightarrow 0$ be a sequence of positive numbers and let $3\delta_k=\min\ \delta^{r-1}(\epsilon_k/3),\ \delta^{r-1}(\eta/3)$ (from Theorem 2). Choose N_k' such that for $i>N_k'$ we have $U_{\delta_k}(A_i)\supset A$, $U_{\delta_k}(A)\supset A_i$ and let $N_k''=N_k'$

$+N^{r-1}(\delta_k/3)+N^{r-1}(\eta/3)$. This defines a number N_k'' for each k, and by letting $N_k=\sum_{j=1}^{k}N_j''$ we obtain a monotone increasing sequence of numbers. Consider a δ_k-cycle $x^r_{j(k)}$ of ξ^r that is not $\sim_\eta 0$ in C and such that $U_{\delta_k}(x^r_{j(k)})\supset C$. Now pick a number $n>N_k$ and let $a_0, a_1, \cdots, a_g$ be the vertices of $x^r_{j(k)}$, then for each $s\leqq g$, let b_s be a point of A_n such that $\rho(a_s, b_s)<\delta_k$. For each simplex $(a_{i_0}, a_{i_1}, \cdots, a_{i_r})$ in $x^r_{j(k)}$ let $(b_{i_0}, b_{i_1}, \cdots, b_{i_r})$ be a simplex and let x^r be the cycle consisting of these simplices. Clearly x^r will be a $3\delta_k$-cycle of A_n, and we shall call any cycle obtained in this manner a δ_k-projection of $x^r_{j(k)}$ [1]. Our choice of $3\delta_k$ allows us to realize each simplex of x^r in a subset of A_n of diameter less than $\epsilon_k/3$, $\eta/3$, from which we obtain a V-cycle realization $\xi^r_n=(x^r_{jn})$ of x^r in $U_{\epsilon_k/3}(x^r)$ as well as in $U_{\eta/3}(x^r)$. Now C_n, the smallest carrier of ξ^r_n, will satisfy the conditions of our theorem, for $C_n\subset\overline{U}_{\epsilon_k/3}(x^r)$, $x^r\subset U_{\delta_k}(x^r_{j(k)})$, $x^r_{j(k)}\subset U_{\delta_k}(x^r)$, $C\subset U_{\delta_k}(x^r_{j(k)})$. Therefore, $C_n\subset U_{\epsilon_k}(C)$, $C\subset U_{\epsilon_k}(C_n)$ as $(\epsilon_k/3)+\delta_k+\delta_k<\epsilon_k/3+\epsilon_k/3+\epsilon_k/3=\epsilon_k$. Now for all n such that $N_k\leqq n<N_{k+1}$, choose C_n corresponding to ϵ_k; then $\lim_{n\to\infty} C_n=C$.

Finally, since $\xi^r_n\subset U_{\eta/3}(x^r)$, we obtain by an application of the prism construction [4] the homology $x^r_{jn}\sim_{\eta/3}x^r$ for all j. Thus ξ^r_n is not $\sim_{\eta/3}0$ in C_n, for if it were then $x^r\sim_{\eta/3}0$; but by a δ_k-projection into C of the $\eta/3$-complex bounded by x^r we could obtain an η-complex in C bounded by $x^r_{j(k)}$, contrary to our hypothesis. This shows that ξ^r_n is essential for sufficiently large n and concludes the proof.

The necessity of the regular convergence in the preceding theorem is shown by the following example. Let A_i be the arc of the circle $\rho=1$ where θ varies from $1/i$ to $2\pi-(1/i)$; then A will be the circle $\rho=1$, and the 0-regular convergence is clearly violated. Now A is the smallest carrier of an essential 1-cycle, but the theorem cannot be satisfied, as A_i contains no essential 1-cycles.

COROLLARY 3.1. *If* $A_i\to\leqq(r-1)A$, *where* A_i *is a* T_r*-set* [5] *for each* i, *then* A *is a* T_r*-set.*

PROOF. If the theorem were not true, then T would contain an essential r-dimensional cycle; but by the theorem there would exist essential r-dimensional cycles in some of the A_i, which contradicts their property of being T_r-sets.

THEOREM 4. *If* $M_i\to\leqq rM$ *and* B *is an irreducible membrane* [2] *for the homology* $\xi^r=(x^r_j)\sim 0$ *in* M, *then* B *can be expressed as* $\lim_{i\to\infty}B_i=B$, *where* $B_i\subset M_i$ *is an irreducible membrane for a homology* $\xi^r_i\sim 0$ *in* M_i.

PROOF. Let $\epsilon_k\to 0$ be a sequence of positive numbers and let $3\delta_k=\delta^r(\epsilon_k)$ and $N^r(\epsilon_k)$ be the numbers corresponding to ϵ_k in Theorem

2. Choose N_k' such that for $i>N_k'$, $U_{\delta_k}(M_i)\supset M$, $U_{\delta_k}(M)\supset M_i$, and let $N_k''=N^r(\epsilon_k)+N_k'$. This defines a number N_k'' for each k, and by letting $N_k=\sum_{j=1}^{k}N_j''$ we obtain a monotone increasing sequence of numbers. Since B is an irreducible membrane of $\xi^r\sim 0$ in M, we can find a V-chain $\eta^{r+1}=(y_j^{r+1})$ in M such that $\dot{y}_j^{r+1}=x_j^r$ for each j, and such that the point sets y_j^{r+1} converge to B. Let $j(k)$ be chosen such that $y_{j(k)}^{r+1}$ is a δ_k-complex and $U_{\epsilon_k}(y_{j(k)}^{r+1})\supset B$, $U_{\epsilon_k}(B)\supset y_{j(k)}^{r+1}$. Let n be any fixed number greater than N_k, and project $y_{j(k)}^{r+1}$ by means of a δ_k-projection into a $(3\delta_k)$-complex y^{r+1} of M_n. Since $3\delta_k=\delta^r(\epsilon_k)$, each simplex of y^{r+1} has a V-chain realization in a subset of M_n of diameter less than ϵ_k. Combining these realizations for all simplices of y^{r+1}, we obtain a V-chain realization $\eta_n^{r+1}=(y_{jn}^{r+1})$ of y^{r+1} in $U_{\epsilon_k}(y^{r+1})$. Now a V-cycle can be formed from a subsequence of $(\dot{y}_{jn}^{r+1})$, and a further subsequence can be chosen so that the remaining (y_{jn}^{r+1}) converge to a set B_n'. We shall denote this subsequence by the same notation $\eta_n^{r+1}=(y_{jn}^{r+1})$ and we shall let $\xi_n^r=(\dot{y}_{jn}^{r+1})$. Let $B_n\subset B_n'$ be an irreducible membrane of the homology $\xi_n^r\sim 0$ in M_n. Now $U_\epsilon(B)\supset y_{j(k)}^{r+1}$, $U_{\delta_k}(y_{j(k)}^{r+1})^r\supset y^{r+1}$, $\overline{U}_{\epsilon_k}(y^{r+1})\supset B_n'\supset B_n$, $B_n\supset y^{r+1}$ $U_{\delta_k}(y^{r+1})\supset y_{j(k)}^{r+1}$, $U_{\delta_k}(y_{j(k)}^{r+1})\supset B$; therefore $U_{3\epsilon_k}(B)\supset B_n$ and $U_{3\epsilon_k}(B_n)\supset B$. Thus if we choose ξ_n^r and B_n in M_n corresponding to ϵ_k for all n such that $N_k\leqq n<N_{k+1}$, we shall have $\lim_{n\to\infty} B_n=B$, and the conclusion of the theorem.

The necessity of the regular convergence in the preceding theorem is shown by allowing M_i to be a totally disconnected set for each i such that $\lim_{i\to\infty} M_i=M$ is a unit interval, and hence the irreducible membrane of the homology of the 0-cycle consisting of its end points. Now clearly the convergence is not 0-regular, and the conclusion of the theorem is violated since no 0-cycle of any M_i is ~ 0.

COROLLARY 4.1. *If $M_i\to\leqq rM$ and $A_i\to A$ where A_i is an A_r-set* [5] *of M_i for each i, then A is an A_r-set of M.*

PROOF. Consider any irreducible membrane B of the homology $\xi^r\sim 0$ in M where $\xi^r\subset A$. By Theorem 4 there exist cycles ξ_i^r in M_i for each i and irreducible membranes (B_i) of the homologies $\xi_i^r\sim 0$ in M_i, such that $B_i\to B$. Also since $A_i\to A$ and $\xi^r\subset A$, we can choose a ξ_i^r in A_i for each i. By the definition of an A_r-set we have $B_i\subset A_i$ for each i; therefore $B=\lim_{i\to\infty} B_i\subset\lim_{i\to\infty} A_i=A$. Thus A is an A_r-set of M.

THEOREM 5. *If $C_i\to\leqq rC$, where C_i is the irreducible carrier of an r-dimensional projection cycle ξ_i^r for each i, then C is the irreducible carrier of a projection cycle ξ^r. Finally ξ^r will be essential if and only if all but a finite number of the ξ_i^r are essential.*

PROOF. Corresponding to $\delta_i \rightarrow 0$, we can pick a subsequence of (C_i), which we shall suppose to be the whole sequence, such that $U_{\delta_i/3}(C_i) \supset C$, $U_{\delta_i/3}(C) \supset C_i$. Let $\xi_i^r = (x_{ij}^r)$; then there exists a $\delta_i/3$-cycle $x_{in_i}^r$ such that $U_{\delta_i/3}(x_{in_i}^r) \supset C_i$, $U_{\delta_i/3}(C_i) \supset x_{in_i}^r$. Project $x_{in_i}^r$ into a δ_i-cycle x_i^r of C. Clearly $U_{\delta_i}(x_i^r) \supset C$, $U_{\delta_i}(C) \supset x_i^r$. Pick a subsequence of (x_i^r), which we shall suppose to be the whole sequence, forming a V-cycle ξ^r. Now the point sets $x_i^r \rightarrow C$ and $\xi^r = (x_i^r) \subset C$; therefore C is the irreducible carrier of ξ^r.

Now suppose that ξ^r is not essential, then $\xi^r \sim 0$ in C. Since $C_i \rightarrow \leqq rC$, we know by a theorem of G. T. Whyburn [7] that C is an lc^r; hence by a theorem of R. L. Wilder [3] the r-dimensional Betti number of C is a finite number n. A result of H. A. Arnold [8] implies that all but a finite number of the C_i have this same finite Betti number. Let $\xi_{10}^r, \cdots, \xi_{n0}^r$ be a basis for r-dimensional cycles in C, which we can choose to be projection cycles with smallest carriers $C_{01}, \cdots, C_{0n}$. By Theorem 3 there exist cycles $\xi_{1i}^r, \cdots, \xi_{ni}^r$ with smallest carriers $C_{1i}, \cdots, C_{ni}$ in C_i such that $\lim_{i\to\infty} C_{ji} = C_j$ $(j=1, \cdots, n)$. The (ξ_{ji}^r) will be linearly independent for i greater than some integer N_1, for if not we can establish by a projection a linear dependence of the (ξ_{j0}^r). Since the Betti number of each C_i for i greater than some number N_2 is n, it follows that (ξ_{ji}^r) is a basis for cycles in C_i for $i > N = N_1 + N_2$. Thus $\xi_i^r \sim \sum_{j=1}^n \alpha_j \xi_{ji}^r$ $(i > N)$ $(\alpha_j = 0$ or $1)$, where we can suppose that N was chosen large enough so that the same linear combination holds for each i. Now by projecting the complexes bounded by $\xi_i^r + \sum_{j=1}^n \alpha_j \xi_{ji}^r$, we can establish a homology $\xi^r \sim \sum_{j=1}^n \alpha_j \xi_j^r$. But $\xi^r \sim 0$; therefore $\alpha_j = 0$ for all j. Thus $\xi_i^r \sim 0$ in C_i for $i > N$, which implies that ξ_i^r is inessential.

Conversely, if ξ^r is essential then exactly the same procedure as was used in Theorem 3 can be used to show that all but a finite number of the ξ_i^r are essential.

The necessity of the regular convergence in the preceding theorem is shown by the following example. Let C_i be the collection of points $(x = j/3^i, y = 0)$ for $j = 0, 1, \cdots, 3^i$, then $\lim_{i\to\infty} C_i = C =$ the unit interval from 0 to 1, and clearly the 0-regular convergence is violated. Now each C_i is an essential 0-dimensional V-cycle and hence is its own irreducible carrier, but C clearly cannot be the irreducible carrier of an essential 0-cycle as all 0-cycles are ~ 0 in C.

THEOREM 6. *If $B_i \rightarrow \leqq rB$, where B_i is an irreducible membrane for an homology of a projection cycle $\xi_i^r \sim 0$ in B_i for each i, then B is an irreducible membrane of an homology of a projection cycle $\xi^r \sim 0$ in B.*

PROOF. In the proof of Theorem 5 we have seen how to establish a projection cycle ξ^r in B. Furthermore by projections into B of the chains bounded by the ξ_i^r, we can establish an homology $\xi^r \sim 0$ in B. It remains to show that ξ^r is not ~ 0 in a proper subset B' of B. To this end suppose $\xi^r \sim 0$ in B'. By Theorem 4, $B' = \lim_{i\to\infty} B_i'$ where B_i' is an irreducible membrane of the homology $\xi_i^r \sim 0$ in B_i. (Since the carrier of ξ^r was chosen as the limit of the carriers of the ξ_i^r, we can choose the B_i' corresponding to the homologies of our original ξ_i^r.) Thus $B' = \lim_{i\to\infty} B_i' = \lim_{i\to\infty} B_i = B$ and B is an irreducible membrane of $\xi^r \sim 0$ in B.

THEOREM 7. *If* $M_i^r \rightarrow \leqq r M$, *where* M_i^r *is an* r*-dimensional closed Cantorian manifold* [6] *for each* i, *then if* dim $M \leqq r$, M *is also a closed* r*-dimensional Cantorian manifold.*

PROOF. Since $p^r(M_i^r) \neq 0$ for each i ($p^r(M_i^r)$ = the rth dimensional Betti number of M_i^r) and the convergence is regular, it follows that $p^r(M) \neq 0$. Next suppose M' is a proper closed subset of M with $p^r(M') \neq 0$. Then there exists an essential (projection) cycle ξ^r with irreducible carrier C in M'. By Theorem 3, $C = \lim_{k\to\infty} C_k$, where C_k is the smallest carrier of an essential cycle ξ_k^r in M_k^r. Now for some integer n, C_n will be a proper closed subset of M_n^r, but by the definition of M_n^r, $p^r(C_n) = 0$ contrary to the fact that ξ_n^r is essential. Thus $p^r(M') = 0$ for every proper closed subset M' of M. Finally dim $M \geqq r$, for $p^r(M) \neq 0$, implies the existence of an essential ξ^r in M. Since by hypothesis dim $M \leqq r$, we have dim $M = r$, and M is an r-dimensional closed Cantorian manifold.

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TOPICS IN THE THEORY OF ABELIAN GROUPS. I
DIVISIBILITY OF HOMOMORPHISMS

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1. **Introduction.** The theory of character groups of Abelian groups has in recent years become of great importance, especially through applications to topology and algebra. The character group of G is the group H of homomorphisms of G into the real numbers mod 1. Extending this, we may consider the group $H = \text{Hom}\ (G, Z)$ of homomorphisms of G into a third group Z, or more generally, a "pairing" of groups H and G into Z: a multiplication $h \cdot g = z$, satisfying both distributive laws.

Of course the duality theorems for character groups will not hold in the more general cases; but, under certain conditions, substitutes may hold. We expect in later notes to give various facts about pairings, and the closely associated problem of divisibility by integers. In the present note, we answer the question of when a homomorphism of G into Z is divisible by an integer m; this has an immediate application to a theorem in combinatorial topology.

2. **Divisibility theorems, discrete groups.** We use the following notations for a group X and an integer m.

$$mX = \text{all } mx, \qquad x \in X,$$

$$_mX = \text{all } x \in X \text{ such that } mx = 0.$$

Note that $x \in mX$ means that x is divisible by m.

In the theorems below, we use

$$H = \text{Hom}\ (G, Z). \tag{1}$$

Let $H'(G')$ denote all $h(g)$, $h \in H'$, $g \in G'$.

LEMMA 1. *Suppose G' is a subgroup of G,*

$$h \in H, \qquad h' \in \text{Hom}\ (G', Z).$$

Then h' can be extended over G so that $h = mh'$ in G' if and only if:

(α) *$h(G) \subset mZ$.*

(β) *$h'(mg) = h(g)$ if $mg \in G'$.*

(γ) *If m' is a divisor of m, then $h'(m'g) \in m'Z$ if $m'g \in G'$.*

The necessity of the conditions is clear. We prove the sufficiency first for the case

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$$m = p^n, \qquad p \text{ prime}.$$

To begin with, extend h' over $mG+G'$ by setting

$$h'(mg + g') = h(g) + h'(g') \qquad (g' \in G'). \tag{2}$$

This definition is unique. For if $mg+g' = mg_1+g_1'$, then $m(g_1-g) \in G'$, and (β) gives

$$0 = h'(0) = h'[m(g_1 - g) + (g_1' - g')] = h(g_1 - g) + h'(g_1' - g'),$$

so that

$$h(g) + h'(g') = h(g_1) + h'(g_1').$$

Clearly (β) holds in $mG+G'$. To prove (γ), suppose $m = m'm''$. Take any

$$m'g = mg_1 + g' \in mG + G'.$$

Then $m'(g - m''g_1) = g' \in G'$, and hence, using (2), (α), and (γ) in G', we find

$$h'(m'g) = h(g_1) + h'[m'(g - m''g_1)] \in m'Z.$$

Let S be the set of all pairs (ϕ, G''), where

$$mG + G' \subset G'' \subset G,$$

$$\phi \in \operatorname{Hom}(G'', Z), \qquad \phi = h' \text{ in } mG + G',$$

and such that (β) and (γ) hold with ϕ and G'' in place of h' and G'. Partially order these by setting

$$(\phi_1, G_1) < (\phi_2, G_2) \quad \text{if} \quad G_1 \subset G_2 \quad \text{and} \quad \phi_2 = \phi_1 \text{ in } G_1,$$

that is, ϕ_2 is an extension of ϕ_1. By[1] Zorn's lemma, there is a maximal simply ordered set S' of elements of S. Let G_0 be the set of all g in some G'' with $(\phi, G'') \in S'$, and set $\phi_0(g) = \phi(g)$ for such a (ϕ, G''). Clearly G_0 is a subgroup of G, $\phi_0(g)$ is independent of the (ϕ, G'') chosen, and $(\phi_0, G_0) \in S'$; hence it is the last element in S', and there is no later element in S. Therefore ϕ_0 cannot be further extended in G. To prove the lemma for $m = p^n$, it is sufficient to show that $G_0 = G$; for $\phi_0 = h'$ in G' by hypothesis, and (β) gives

$$m\phi_0(g) = \phi_0(mg) = h(g), \qquad m\phi_0 = h.$$

Suppose $G_0 \neq G$. Then choose an element $g_1 \notin G_0$ such that for a largest possible divisor m_1 of m, $g_1 \in m_1G$. Since $mG \subset G_0$, it follows that $m_1 < m$, and $m_1 = p^\mu$, $\mu < n$. Choose g_3 and g_2 so that

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[1] Compare Remark 2 in §3.

$$g_1 = m_1 g_3, \qquad g_2 = p g_1 = p m_1 g_3; \tag{3}$$

then $g_2 \in G_0$. Set $m_2 = p^{n-\mu-1}$; then $p m_1 m_2 = m$. Since $\phi_0 = h'$ in $mG + G'$, (2) gives

$$m_2 \phi_0(g_2) = \phi_0(p m_1 m_2 g_3) = h(g_3). \tag{4}$$

Since $g_2 = p m_1 g_3 \in G_0$, we may choose z_2 by (γ) so that

$$\phi_0(g_2) = p m_1 z_2. \tag{5}$$

Let G_1 be the subgroup of G generated by G_0 and g_1. Since p is prime, and pg_1 is in G_0 while g_1 is not, $ag_1 \in G_0$ if and only if $a \equiv 0 \pmod{p}$. Extend ϕ_0 through G_1 by setting

$$\phi_0(ag_1 + g) = a m_1 z_2 + \phi_0(g) \qquad (a \text{ integral}; \ g \in G_0). \tag{6}$$

To show that this is unique, suppose

$$a g_1 + g = b g_1 + g', \qquad g, g' \in G_0.$$

Then

$$(a - b) g_1 = g' - g \in G_0, \qquad a - b = kp,$$

and

$$\begin{aligned}
[a m_1 z_2 + \phi_0(g)] - [b m_1 z_2 + \phi_0(g')] &= k p m_1 z_2 + \phi_0(g - g') \\
&= k \phi_0(g_2) - \phi_0(k p g_1) \\
&= \phi_0(k g_2 - k g_2) = 0,
\end{aligned}$$

as required. Clearly ϕ_0 is now a homomorphism in G_1.

Since $mG \subset G_0$, (β) holds in G_1. To prove (γ), take any

$$m' g = a g_1 + g_0 \in G_1 \qquad (g_0 \in G_0).$$

If $m'g \in G_0$, then $\phi_0(m'g) \in m'Z$, since (γ) holds in G_0. Suppose it is not. By the choice of g_1 and m_1, $m' \leqq m_1$; hence for some m'', $m'm'' = m_1$. Now by (3),

$$g_0 = m' g - a m_1 g_3 = m'(g - a m'' g_3),$$

and by (γ) in G_0, $\phi_0(g_0) = m' z_0$ for some z_0. Now by (6),

$$\phi_0(m' g) = a m_1 z_2 + \phi_0(g_0) = m'(a m'' z_2 + z_0),$$

as required.

We have now shown that $(\phi_0, G_1) \in S$. Since ϕ_0 has been extended from G_0, $(\phi_0, G_0) \prec (\phi_0, G_1)$. But this contradicts the fact that S' was maximal. It follows that $G_0 = G$, and the lemma is proved for the case $m = p^n$.

For a general m, we shall use induction on the number $\rho(m)$ of distinct prime divisors of m. Say, $m=m_1m_2$, $(m_1, m_2)=1$, $\rho(m_1)$ and $\rho(m_2)$ being less than $\rho(m)$. Set

$$h_1 = m_2h', \qquad h_2 = m_1h', \qquad \text{in } G'.$$

Then the conditions of the lemma hold for both h_1 and h_2, with m_1 and m_2 respectively. Property (α) for m_1 and m_2 is clear. To prove (β), we have

$$h_1(m_1g) = m_2h'(m_1g) = h'(mg) = h(g).$$

To prove (γ), suppose m_1' is a divisor of m_1 and $m_1'g\in G'$; then m_1' is a divisor of m, and (γ) for h' gives

$$h_1(m_1'g) = m_2h'(m_1'g) \in m_2(m_1'Z) \subset m_1'Z.$$

Therefore h_1 and h_2 may be extended over G, so that

$$h = m_1h_1 = m_2h_2.$$

Now choose integers r, s so that $rm_1+sm_2=1$. Then

$$rh_2 + sh_1 = (rm_1 + sm_2)h' = h' \text{ in } G',$$

$$m(rh_2 + sh_1) = rm_1m_2h_2 + sm_2m_1h_1 = h \text{ in } G,$$

so that $h'=rh_2+sh_1$ is the required homomorphism.

THEOREM 1. *For any integer m, $h\in mH$ if and only if*

(a) $h(G)\subset mZ$,

(b) $h({}_mG)=0$.

Let G' be the identity alone, and set $h'(0)=0$. We need merely prove (β) and (γ) of Lemma 1. Clearly (β) reduces to (b); (γ) is trivial.

3. Divisibility properties, topological groups. We shall consider only topological groups with the following property:

(P) For each integer $m>0$, the mapping mg of G into itself is interior, that is, maps open sets into open sets.

An equivalent statement is the following: For every neighborhood U of 0 in G and every $m>0$ there is a neighborhood U' of 0 such that $U'\subset mU$; that is, all sufficiently small elements of G are divisible by m, a solution lying in U. The property holds in the commonly used groups.

REMARK 1. It follows that mG is open in $G(m>0)$; hence, for any subgroup G', $mG+G'$ is open in G.

EXAMPLE. If G is the group of dyadic rational numbers, using the topology of the real numbers, and $m=3$, then $3G$ contains no element of the form $1/2^n$. Thus G does not satisfy (P).

LEMMA 2. *Let G and Z be topological groups, and let* (P) *hold in G. Let h' be a homomorphism of G into Z, and set $h=mh'$. Then h' is continuous if and only if h is.*

That continuity of h follows from that of h' is simple (and does not use (P)). Suppose h is continuous. Given any neighborhood V of 0 in Z, choose a neighborhood U_1 of 0 in G so that $h(U_1)\subset V$, and choose $U\subset mU_1$. Now take any $g\in U$. Then $g=mg_1$, $g_1\in U_1$, and $h'(g)=h(g_1)\in V$, proving that h' is continuous.

By Hom (G, Z) we shall now mean the group of *continuous homomorphisms* of G into Z. We do not need to consider a topology in H here.

THEOREM 2. *If G and Z are topological groups, and* (P) *holds in G, then Lemma 2 and Theorem 1 hold.*

Using the former proofs, we need merely prove the continuity of h'. But this follows from the continuity of h and the last lemma.

REMARK 2. Since mG is open in G and h' is first extended over mG, if the proof in Lemma 1 is given in the equivalent form of extending h' over larger and larger subgroups of G (as from G_0 to G_1 in the proof given), only a finite number of steps will be required in the case of most groups G.

4. **On the complete resolution of H by G.** We use the following definitions and facts.[2] If H, G are paired into Z, (H, G') (the *nullifier* or *annihilator* of G' in H) is the set of all h with $h\cdot G'=0$. If $(H, G')=0$, then G' *resolves* H. (Then if $h_1\neq h_2$, there is an element $g\in G'$ with $h_1\cdot g\neq h_2\cdot g$.) If $(H, {}_mG')=mH$, G' *m-resolves* H. (Then if h is not divisible by m, there is an element $g\in{}_mG'$ such that $h\cdot g\neq 0$.) "Resolves" and "0-resolves" are equivalent. If G' m-resolves H for all integers $m\geqq 0$, G' *resolves H completely. Z is completely divisible* if $mZ=Z$ for each integer $m\neq 0$. The main application of the Pontrjagin duality theory to topology lies in the fact[2] that any group resolves completely and is resolved completely by its character group.

THEOREM 3. *If Z is completely divisible, then G resolves H $=$Hom (G, Z) completely.*

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[2] Compare H. Whitney, *On matrices of integers and combinatorial topology*, Duke Math. J. vol. 3 (1937) pp. 35–45. The main statement in footnote 8 of this paper is not true.

If the groups are topological, we assume G satisfies (P).

If $h\in mH$, say $h=mh'$, then $mg=0$ implies

$$h(g) = h'(mg) = h'(0) = 0,$$

and $h\in(H, {}_mG)$. Conversely, if $h\in(H, {}_mG)$, that is, $h({}_mG)=0$, Theorem 1(or Theorem 2) gives $h\in mH$.

5. **Application to combinatorial topology.** Let $H^r(K, X)$ and $H_r(K, X)$ denote the rth homology and cohomology groups, respectively, of K, with the coefficient group X.

THEOREM 4. *Let Z be completely divisible. Then for any G, and $H=\mathrm{Hom}\ (G, Z)$, we have the isomorphisms*

$$H_r(K, H) \approx \mathrm{Hom}\ [H^r(K, G), Z],$$

$$H^r(K, H) \approx \mathrm{Hom}\ [H_r(K, G), Z].$$

If G and Z are topological, we assume G satisfies (P).

This is Theorem 8 of Whitney, loc. cit., with the hypothesis weakened; it is essentially the Pontrjagin Duality Theorem, slightly generalized. For the proof, we need merely apply Theorem 3 to obtain the remainder of the hypothesis.

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MODULARITY IN BIRKHOFF LATTICES

L. R. WILCOX

The purpose of this note is to identify upper semi-modular lattices originally defined by G. Birkhoff[1] and subsequently studied by Dilworth[2] with those M-symmetric lattices[3] (introduced independently by the author without assumption of chain conditions) which satisfy a condition of finite dimensionality.

The definitions and notations are these. In a lattice L, $a \succ b (b \prec a)$ means that a "covers" b, that is, $a > b$, together with $a \geqq x \geqq b$ implies $x = a$ or $x = b$; $(b, c)M$ means $(a+b)c = a + bc$ for every $a \leqq c$ (where $a+b$, ab are the "join" and "meet" respectively of a, b). We say that L is *M-symmetric* if the binary relation M is symmetric; L is a *Birkhoff lattice* if

$$(1) \qquad a, b \succ ab \textit{ implies } a + b \succ a, b;$$

L is *of finite-dimensional type*[4] if for every $a < b$ there exists a finite "principal chain"

$$a_1 \prec a_2 \prec \cdots \prec a_n,$$

with $a_1 = a$, $a_n = b$. When a, b satisfy this condition for a specific n, we say that *b is $n-1$ steps over a.*

The properties of the relation M are given in part in a previous paper.[5] Additional properties needed here are contained in the following lemma.

LEMMA 1. *Suppose $b, c \in L$. Then*
(a) *$(b, c)M$ if and only if $bc \leqq a \leqq c$ implies $(a+b)c = a$;*
(b) *if $(b, c)M$, then $(b', c')M$ for $bc \leqq b' \leqq b$, $bc \leqq c' \leqq c$.*

PROOF. The forward implication in (a) is obvious. To prove the

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[1] G. Birkhoff, *Lattice theory*, Amer. Math. Soc. Colloquium Publications vol. 25, New York, 1940, p. 62.

[2] R. P. Dilworth, *Ideals in Birkhoff lattices*, Trans. Amer. Math. Soc. vol. 49 pp. 325–353; also *The arithmetical theory of Birkhoff lattices*, Duke Math. J. vol. 8 (1941) pp. 286–299.

[3] L. R. Wilcox, *Modularity in the theory of lattices*, Ann. of Math. vol. 40 (1939) pp. 490–505; see also *A note on complementation in lattices*, Bull. Amer. Math. Soc. vol. 48 (1942) pp. 453–457.

[4] This property is weaker than finite dimensionality as used by Birkhoff (loc. cit. p. 11), even if 0 and 1 exist.

[5] L. R. Wilcox, *Modularity in the theory of lattices*, pp. 491–495.

converse, let $a \leqq c$. Then $a' = a+bc$ has the property $bc \leqq a' \leqq c$, whence

$$(a+b)c = (a+bc+b)c = (a'+b)c$$
$$= a' = a+bc.$$

To prove (b) we use the condition in (a). Let $b'c' \leqq a \leqq c'$. Then

$$(a+b')c' \leqq (a+b)cc' = ac' = a \leqq (a+b')c',$$

whence (b) follows.

THEOREM 1. *Every M-symmetric lattice is a Birkhoff lattice.*

PROOF. Suppose $a, b \succ ab$. Then it is immediate that $a+b > a, b$. To prove $a+b \succ a$, let $a \leqq c \leqq a+b$. Since $b \geqq cb \geqq ab$, we have $cb = b$ or $cb = ab$ from the hypothesis $b \succ ab$. If $cb = b$, then $a+b \leqq c$, whence $c = a+b$. Suppose $cb = ab$. We shall prove $(c, b)M$. Let $ab = cb \leqq x \leqq b$. Then $x = ab$ or $x = b$, whence either

$$(x+c)b = (ab+c)b = (cb+c)b = cb = x,$$

or

$$(x+c)b = (b+c)b = b = x,$$

and it follows by Lemma 1 (a) that $(c, b)M$. Now the symmetry of M yields $(b, c)M$, and thus, since $bc \leqq a \leqq c$,

$$c = (a+b)c = a.$$

In all cases $c = a+b$ or $c = a$, and consequently $a+b \succ a$. Similarly $a+b \succ b$.

REMARK. The theorem just proved generalizes the known result[6] that every modular lattice is a Birkhoff lattice, since modular lattices are M-symmetric.

In order to consider the converse of Theorem 1, let, for the purposes of the following lemmas, L be a fixed Birkhoff lattice of finite-dimensional type.

LEMMA 2. *If $b, c \in L$ and $c \succ bc$, then $b+c \succ b$.*

PROOF.[7] Observe that $b \geqq bc$; if $b = bc$, $b \leqq c$, and $b+c = c \succ bc = b$. If $b > bc$, then there exists $n = 1, 2, \cdots$ such that b is n steps over bc. If $n = 1$, the result is obvious from condition (1) defining a Birkhoff lattice. Suppose the result has been proved for all b, c for which b is

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[6] Birkhoff, loc. cit. p. 34.

[7] This is MacLane's second "exchange axiom" in the convex lattice of all $x \geqq bc$; as such it follows for finite-dimensional lattices from remarks on p. 63 of Birkhoff. Since L need not be finite-dimensional, we give the proof in full.

k steps over bc, and let b be $k+1$ steps over bc. Clearly there exists $b' \prec b$ such that b' is k steps over bc. Since $b'c \leqq bc \leqq b'c$, we have $c \succ b'c$, and by the induction hypothesis applied to b', c it follows that $b'+c \succ b'$. But $b' \leqq (b'+c)b \leqq b$, whence $(b'+c)b = b'$ or $(b'+c)b = b$. In the latter case $b' \prec b \leqq b'+c$, and thus $b = b'+c$, whence $c \leqq b$, contrary to $c \succ bc$. Consequently $(b'+c)b = b'$. Since $b'+c$, $b \succ (b'+c)b$, (1) yields

$$b + c = b + (b' + c) \succ b.$$

LEMMA 3. *If $b, c \in L$, $c \succ bc$, then $(c, b)M$, $(b, c)M$.*

PROOF. If $bc \leqq a \leqq c$, then $a = bc$ or $a = c$, so that either

$$(a + b)c = (bc + b)c = bc = a,$$

or

$$(a + b)c = (c + b)c = c = a,$$

and $(b, c)M$. Now suppose $bc \leqq a \leqq b$. Then $bc \leqq ac \leqq bc$ yields $ac = bc$. Hence $c \succ ac$, and $a+c \succ a$ by Lemma 2. But $a \leqq (a+c)b \leqq a+c$, whence $(a+c)b = a$ or $(a+c)b = a+c$. In the latter case $a+c \leqq b$, and $c \leqq b$, which is impossible. Hence $(a+c)b = a$, and $(c, b)M$.

LEMMA 4. *Suppose $b, c \in L$, $(b, c)M$. Then $bc \leqq a \leqq b$, $a+c = b+c$ implies $a = b$.*

PROOF. If $c = bc$, that is, $c \leqq b$, or if c is one step over bc then $(c, b)M$ either by direct verification or by Lemma 3; hence

$$a = a + cb = (a + c)b = (b + c)b = b.$$

Suppose the result holds for all b, c with c n steps over bc, and let b, c satisfy the hypotheses, c being $n+1$ steps over bc. Then there exists c' with $bc \leqq c' \prec c$, where c' is n steps over bc. Since $(b, c')M$ by Lemma 1 (b), and since $bc' = bc \leqq a \leqq b$, we need only verify $a+c' = b+c'$ in order to show $a = b$. Since $(a, c)M$ by Lemma 1 (b), $(c'+a)c = c'$. Thus

$$c \succ c' = (c' + a)c,$$

and by Lemma 2,

$$c + b = c + a = c + (c' + a) \succ c' + a.$$

But

$$c' + a \leqq c' + b \leqq c + b,$$

whence

$$c' + b = c' + a \quad \text{or} \quad c' + b = c + b.$$

In the second case, since $(b, c)M$,

$$c' = (c' + b)c = (c + b)c = c,$$

which is impossible. This completes the proof.

THEOREM 2. *Every Birkhoff lattice L of finite-dimensional type is M-symmetric.*

PROOF. Suppose $(b, c)M$, and in proof of $(c, b)M$ let $bc \leqq a \leqq b$. Define

$$b_1 = (a + c)b \geqq a;$$

we shall prove that $b_1 = a$ by applying Lemma 4 to a, b_1, c in place of a, b, c. First, $(b_1, c)M$ by Lemma 1 (b), since $bc \leqq b_1 \leqq b$, and $(b, c)M$. Moreover,

$$b_1c = (a + c)cb = bc \leqq a \leqq b_1.$$

Finally, $a + c \geqq b_1$, c, whence

$$a + c \geqq b_1 + c \geqq a + c,$$

and $a + c = b_1 + c$. The hypotheses of Lemma 4 have been verified, and thus $a = b_1$, as was to be proved.

The effect of Theorems 1 and 2 is to show that not necessarily finite-dimensional M-symmetric lattices are a true generalization of the Birkhoff lattices. Moreover, the condition defining M-symmetry does not lose its strength in infinite-dimensional cases as does condition (1). For example, an interval of real numbers ordered as usual satisfies (1) vacuously; it is modular, hence M-symmetric. However, define a lattice L as consisting of the closed real interval $I = [0, 1]$, ordered naturally, together with an element ϵ, with $0 < \epsilon < 1$, but $x \nless \epsilon$, $\epsilon \nless x$, $\epsilon \neq x$ for $x \in I$. This is a lattice in which the only covering relations are $\epsilon \succ 0$, $1 \succ \epsilon$. Hence (1) is vacuously true, but M-symmetry fails violently, since $(x, \epsilon)M$ for every $x \in L$, but $(\epsilon, x)M$ is false except for $x = 0$, 1 or ϵ.

Interesting questions are these. What infinite-dimensional generalization of the Jordan chain condition holds in M-symmetric lattices? Moreover, in finite-dimensional lattices, (1) together with its dual implies modularity; what can be said generally of lattices which together with their duals are M-symmetric?

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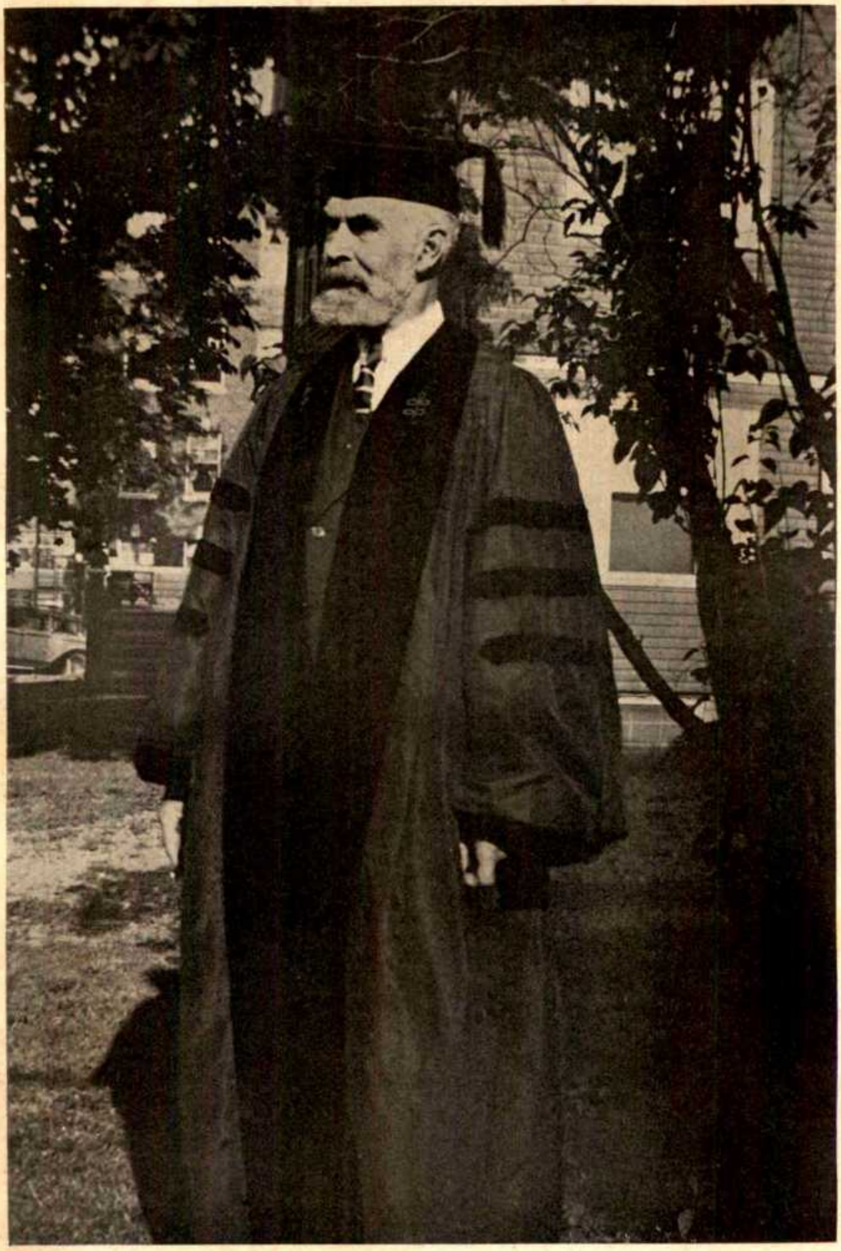

William Fogg Osgood
1864–1943

WILLIAM FOGG OSGOOD—IN MEMORIAM

With the death on July 22, 1943, of William Fogg Osgood, the American Mathematical Society has lost a beloved and distinguished member, and The Fraternity of Science a most faithful comrade.

William Fogg Osgood was born in Boston on March 10, 1864, the only child surviving beyond infancy of Dr. William Osgood, a physician and fourth in an unbroken line of that profession, and of Mary Rogers (Gannett) Osgood of Milton, Massachusetts. Both Osgoods and Gannetts were old New England families, the earliest American Osgood in the mathematician's line being John Osgood (1595–1651) who came to Ipswich, Massachusetts, in 1638 from Wherwell, near Andover (Hampshire), England, where the family's freehold seems to have antedated the Conquest. On the same side of the family, Osgood was a great great grandson of General Joseph Otis, the brother of James Otis the patriot.

Osgood attended the Boston Latin School, and entered Harvard in 1882, whence he graduated in 1886 summa cum laude with highest honors in mathematics and the second in a class numbering 286. In school his education had been centered in the classics, and at Harvard the study of the Greek authors formed a large part of his program; indeed he received second year honors in classics. But before long his interest in the experimental and mathematical sciences was aroused by his teachers at Harvard, most of all by his distant kinsman Benjamin Osgood Peirce; and his intention to devote his life to mathematics became established. After obtaining the A.M. in 1887, he received a travelling fellowship, and spent two years in Göttingen and a year at Erlangen where he obtained his Ph.D. (1890). He was awarded an L.L.D. by Clark University in 1909.

Osgood was married in 1890 to Therese Ruprecht (deceased) of Göttingen, and returned to become an instructor in the department of mathematics at Harvard (1890–93). He served there as assistant professor (1893–1903) and as professor (1903–33), and became professor emeritus in 1933. He was the father of three children: Dr. William Ruprecht Osgood of the Bureau of Standards, Mrs. Freda Osgood Silz (deceased), and Dr. Rudolph Osgood, a physician of Fall River, Massachusetts. In 1932 Osgood was married to Mrs. Céleste Phelps Morse. During 1934–36 he was visiting professor of mathematics at the National University of Peking. From 1936 until his death he lived in Belmont, Massachusetts.

Osgood was elected to the National Academy of Sciences in 1904. He was the eighth president of the American Mathematical Society

(1905–06). He was a member of the American Philosophical Society, the Deutsche Mathematiker-Vereinigung, the Leopoldinische-Carolinische Deutsche Akademie der Naturforscher, the Circolo Mathematico de Palermo, corresponding member of the Mathematical Society of Charkow, the Göttinger Gesellschaft der Wissenschaften, and honorary member of the Calcutta Mathematical Society. He also belonged to Phi Beta Kappa and Sigma Xi.

We are fortunate in having an exhaustive bibliography of Osgood's works in the first volume of the Semicentennial Publications of this Society.[1] Moreover the original papers, numbering upward of four score, are too widely known and too technical to make more than their briefest mention desirable in this place. His study of term by term integration of non-uniformly convergent series (1896) contained in germ ideas of Borel measure. His example of a Jordan curve of positive area settled an important controversy. His work on conformal mapping is definitive. His treatment of the uniformization of algebraic functions formed an important contribution. Osgood initiated a branch of the calculus of variations. His work on the gyroscope showed how his mind could illuminate simple and important new aspects of a classical problem hitherto regarded as quite completed.

At the invitation of Felix Klein, Osgood wrote the article on the theory of functions of a complex variable in the Encyklopädie der Mathematischen Wissenschaften. Out of this grew his best known book, the *Lehrbuch der Funktionentheorie.* This book is a classic; it formed the first thorough and systematic treatment of the subject; it combines the deep impress of an individuality with the effective expression of a great mathematical tradition. The second volume deals with several complex variables and is today the only deep and broad systematization of the subject. It is a mine from which much material for future research will be drawn. So much of it is original that it has largely the import of a series of memoirs. During his last years, Osgood was working on the hypergeometric theory, which was intended to find a place in a later part of this book.[2]

In forming a picture of the mode and temper of Osgood's thought, three major influences should be regarded as having had a decisive effect in the formation of his mind: his predilection for physics, the early classical training, and the sojourn in Germany.

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[1] R. C. Archibald, *A semicentennial history of the American Mathematical Society* 1888–1938, Amer. Math. Soc. Semicentennial Publications, vol. 1, New York, 1936, pp. 153–158.

[2] Circumstances have made it impossible as yet to ascertain how much has been left in a form suitable for publication.

Osgood once told me that he was at heart a physicist, and that if during his student days the career of physics had offered a more deeply mathematical and less disproportionately experimental form, he might well have made it his. And there was ever present in his mind the notion of mathematics finding one of its deepest justifications in serving as an instrument whereby the mind of man can penetrate into the mysteries of nature.

This is not to say that Osgood was any the less alive to that inner order which makes mathematics a fit subject of study in its own right. Not only did there appear to be nothing inconsistent between the two evaluations of the subject, but on the contrary he always saw the two—the extrinsic and the intrinsic justification of mathematics—as mutually reënforcing one another.

And this brings us to the second point. I have always felt that one of Osgood's most distinguishing characteristics was this sense of balance between values: applications and theory; traditional mathematics and innovations; teaching and research; and with it all the constant awareness that a mathematician is a human being who is using his human mind for his enlightenment, rather than a theorem-producing machine. And at the risk of being involved in a well known controversy, I hazard the opinion that this was due in no small degree to the fact that Osgood received his earliest intellectual impressions in the study of the classics. That sense of the dignity of reason, of tradition in excellence, and the instinct of historical continuity are qualities which are less to be derived from modern studies than from the writings of antiquity.

The third current of influence was the academic Germany of the eighteen-eighties, and especially the power of Felix Klein. The young Bostonian of four and twenty with his turn of mind at once scientific and classical came in Göttingen under the force of the great teacher whose ambition it was to unify mathematics and to set it forth in all its coördination and clarity. It is easy to realize the magnitude of the impression which was produced: the old scholar of the classics became the classical mathematician, and a flame was kindled which was to light a long career of science.

When Osgood and Bôcher returned from Germany to take positions on the Harvard faculty, they found the Department of Mathematics rather like that of a provincial college. It had contained individuals of eminence, but it could scarcely have supplied a real training in advanced modern mathematics. Then in a few decades it became one of the leading departments in the country, with a respectable place among the departments of mathematics in the world. This was largely the natural result of the general movement of American education, and every member of the department at Harvard had his share in it. But if the credit of this rise to eminence can be given to any one man, Osgood was undoubtedly that one. His influence was exerted upon teaching at all levels, on the increased valuation of research, and perhaps most important of all, in making the many eminent appointments to the teaching staff—the only way in the long run in which any individual can assure the greatness of his department. Osgood was departmental chairman in 1918–22, and acting dean of the Graduate School in 1922 (February–July). He also belonged to various educational committees.

Osgood was one of the most notable teachers in American mathematics, and this at every grade from freshman to specialized graduate course. He was too true a democrat ever to allow himself to lower mathematics to the level of the student: it was his ambition on the contrary to raise the latter to the height of the subject. Ever speaking the language of the student, constantly mindful of his difficulties, Osgood spared no patient effort in bringing him to see what mathematics truly is: the powerful agent of illumination of the physical world, and the gem of human reason.

In his personal contacts with the students Osgood was a happy combination of the wise master and the cordial friend. This gentleman of the old school, whose charm and dignity made his acquaintance not only a pleasure but an honor, was so inclined to see the best in the men that he brought out their very best. Nor could it ever be ascertained whether a student's academic promise, or his being in trouble and in need of a friend constituted the greater claim to Osgood's kindness.

There are many with whom the memory of Osgood will dwell: With all those who have fallen heir to his intellectual heritage; and what state in the union fails to contain them? With his old students, to whom he brought the undistorted spirit of science; they must form by now a goodly host. With his contemporaries, whose ranks time has thinned, those companions in the task of bringing the Great Tradition to our shores. And also with some others, into whose lives he entered as a genial light and inspiration, beyond the power of language to convey.

BERNARD OSGOOD KOOPMAN

THE HIGHER COMMUTATOR SUBGROUPS OF A GROUP

REINHOLD BAER

It is not the object of this address to introduce you to new theories or to tell of great discoveries. Quite on the contrary; I intend to speak of unsolved problems and of conjectures. In order to describe these, certain concepts will have to be discussed; and for obtaining a proper perspective it will be necessary to mention a number of theorems, some of them new. The proofs of the latter will be relegated to appendices so that the hurried reader may skip them easily. The bibliography is in no sense supposed to be complete. We just selected convenient references for facts mentioned and beyond that just enough to be a basis for further reading.

1. **The hierarchy of invariant subgroups.** The subgroups of a group may be classified according to the operations which leave them invariant. There are first the *normal* subgroups of the group G, characterized by the fact that they are transformed into themselves by the inner automorphisms of G; and for this reason they had at one time appropriated the term "invariant subgroup." There are next the *characteristic* subgroups of G which are left invariant by every automorphism of G. Clearly not every subgroup of G is normal, unless G belongs to a comparatively special class of groups, the so-called abelian and hamiltonian groups; and neither is in general every normal subgroup characteristic, though this may happen too (for example, in cyclic groups and in simple groups).

These two classes of subgroups are well known, but for our purposes they are too big. There is next the class of subgroups which we shall term for lack of a better name *strictly characteristic.* A subgroup S of G belongs to this class if $S^f \leqq S$ whenever f is an endomorphism[1] of G and $G^f = G$. The distinction between characteristic and strictly characteristic subgroups does not cut very deep, since there exists a very big class of groups with the following property:

(Q) *If f is an endomorphism of G such that $G^f = G$, then f is an automorphism of G.*

This postulate (Q) is satisfied by every finite group and more generally by every group satisfying the ascending chain condition for normal subgroups. But the Q-groups are not exhausted by the groups

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An address delivered before the Annual meeting of the Society in Chicago on November 27, 1943; received by the editors January 5, 1944.

[1] An endomorphism of the group G is a single-valued and multiplicative G to G function.

just mentioned, witness the free groups on a finite number of generators.[2] However, there exist groups that are not Q-groups, for instance the rational numbers modulo 1 and the free groups on an infinity of generators; and an example of a characteristic subgroup which is not strictly characteristic will be constructed in Appendix I.

Finally we have the *fully invariant* subgroups.[3] A subgroup S of the group G is termed fully invariant if $S^f \leq S$ for every endomorphism f of G. Examples of groups whose centers are not fully invariant are easily constructed.[4] On the other hand it is readily verified that the center is always a strictly characteristic subgroup. For if z is an element in the center of the group G, and if $G^f = G$ is satisfied by the endomorphism f of G, then there exists to every element x in G an element y in G satisfying $x = y^f$; and we find

$$z^f x = z^f y^f = (zy)^f = (yz)^f = y^f z^f = x z^f$$

so that z^f is in the center too. Examples of fully invariant subgroups are obtained by forming the subgroup G^n generated by all the nth powers of elements in G, or the commutator subgroup (G, G) of G which is generated by all the commutators $(x, y) = x^{-1}y^{-1}xy$. More generally it may be said that the "word subgroups" in the sense of B. Neumann and P. Hall[5] are fully invariant whereas the "marginal subgroups" may only be shown to be characteristic.

The importance of characteristic subgroups stems from the fact that they themselves and their quotient groups are group invariants. But if we compare different groups, we have to have a method for deciding which are "corresponding" invariants of the different groups.[6] For it would not do to compare the structure of G^3 with the structure of H^5 or the order of G with the maximum order of the elements in H. We indicate one such possibility of comparing the fully invariant subgroups of different groups. If S is a subgroup of the group G, then we denote by $(G \rightarrow H; S)$ the subgroup of the group H which is generated by all the elements s^f for s an element in S and f a homomorphism of G into H. It is readily seen that $(G \rightarrow H; S)$

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[2] For proofs of this theorem see F. Levi [1, p. 95] and W. Magnus [1, p. 276, VIII].

Criteria for Q-groups may be found in R. Baer [2], D. I. Fouxe-Rabinowitsch [1], and A. Malcew [1]. Numbers in brackets refer to the Bibliography at the end of the paper.

[3] This concept is due to F. Levi [1].

[4] See, for example, Appendix I below.

[5] P. Hall [2], B. Neumann [1].

[6] See in this context the functors discussed by Eilenberg and MacLane [2].

is always a fully invariant subgroup of H; and if S is a fully invariant subgroup of G, then we have

$$(H\rightarrow G;(G\rightarrow H;S))\leqq S.$$

Equality we cannot always expect. But if G and H are both free groups on an infinity of generators, then equality holds, so that the fully characteristic subgroups of G and H form isomorphic partially ordered sets.[7]

APPENDIX I: EXISTENCE OF CHARACTERISTIC, BUT NOT STRICTLY CHARACTERISTIC SUBGROUPS

Let F be a finite group which is not abelian. Then F contains a cyclic subgroup T which is not part of the center of F. Denote by Z a cyclic group whose order equals the order of T and denote by $G(0)$ the direct product of F and Z. It is clear that Z is part of the center of $G(0)$ and that there exists an endomorphism $f(0)$ of $G(0)$ which maps F upon 1 and which effects an isomorphism of Z upon T.

Since $G(0)$ is a finite group, there exists a free group $G(1)$ on a finite number of generators and a homomorphism $f(1)$ of $G(1)$ upon the (full) group $G(0)$. Denote by $G(i)$ for $1<i$ groups isomorphic to $G(1)$ and by $f(i)$ an isomorphism of $G(i)$ upon $G(i-1)$ for $1<i$. Finally let G be the free product[8] of all the groups $G(i)$ for $0\leqq i$. Then there exists[9] one and only one endomorphism f of G which induces $f(i)$ in $G(i)$ for $0\leqq i$; and it is clear that $G=G^f$.

Suppose now that B is a finite subgroup of G. Then it follows from a theorem on the subgroups of free products[10] that B is the free product of a free group V and of groups of the form[11] $B\cap x^{-1}G(i)x$. But B is a finite group and therefore it cannot contain a free group different from 1, nor can it be the free product of two groups different from 1. On the other hand free groups do not possess finite subgroups different from 1, since subgroups of free groups are free.[12] Thus we have shown the following fact.

If B is a finite subgroup of G, then $B\leqq x^{-1}G(0)x$ or $xBx^{-1}\leqq G(0)$ for suitable x in G.

Denote now by C the subgroup of G which is generated by the center D of $G(0)$ and by all its conjugates in G. If g is an automorphism of G, then $G(0)^g$ is a finite subgroup of G. Hence it follows from the lemma just proved that $G(0)^g = y^{-1}G(0)y$ for some y in G. Consequently $D = yD^g y^{-1}$, since D is a characteristic subgroup of $G(0)$, and since the succession of g and of the inner automorphism $s \to ysy^{-1}$ of G induces an automorphism of $G(0)$. But from this fact one readily deduces that *C is a characteristic subgroup of G.*

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[7] Baer [3].

[8] For the theory of free products see, for example, Baer and Levi [1].

[9] If we had formed the direct product of the groups $G(i)$ instead of forming their free product, then it would have been impossible to construct this endomorphism f.

[10] Baer and Levi [1, p. 392].

[11] By $S\cap T$ we designate the crosscut of the sets S and T.

[12] By Schreier's theorem.

From the theorems[13] on subgroups of free products one deduces now that C is the free product of D and of some conjugates to D in G. Thus D is a free factor of C and is consequently the crosscut of C and $G(0)$. This shows in particular that T is not part of C. But T is part of C^f, since $D^f = D^{f(0)} = T$. Hence

C is not a strictly characteristic subgroup of G.

2. Commutator subgroups. If S and T are subsets of the group G, then we denote by (S, T) the subgroup generated by all the commutators $(s, t) = s^{-1}t^{-1}st$ for s in S and t in T. If S and T are fully invariant subgroups of G, then so is (S, T). Thus it is possible to use this operator for the inductive definition of fully invariant subgroups. We mention two important instances.

The derived series: $G^{(0)} = G$, $G^{(i+1)} = (G^{(i)}, G^{(i)})$.

The lower central series: ${}^{0}G = G$, ${}^{i+1}G = (G, {}^{i}G)$.

These are not the only possibilities. For instance, one could define a series $G(i)$ of fully invariant subgroups by the following recursion formulas:

$$G(0) = G, G(1) = (G, G), G(i + 1) = (G(i), G(i - 1)) \text{ for } 0 < i.$$

But so far only the derived and the lower central series seem to have proved interesting. A systematic method of constructing higher commutator subgroups has been evolved by P. Hall.[14] But his constructions are too involved to be sketched in a few words.

The calculus of commutator subgroups is based on the following simple formulas, which are easily verified by direct computation.[15]

$$(1) \qquad (x, y) = (y, x)^{-1}.$$

$$(2) \qquad (xy, z) = y^{-1}(x, z)y(y, z) = (x, z)((x, z), y)(y, z).$$

$$(3) \qquad (x, yz) = (x, z)z^{-1}(x, y)z = (x, z)(z, (y, x))(x, y).$$

If X, Y, Z are subgroups of the group G, then one deduces from these formulas the following results.

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[13] Baer and Levi [1, p. 396, Folgerung 3].

[14] Hall [1], in particular p. 43–48.

[15] Hall [1, p. 43].

(4) $$(X, Y) = (Y, X).$$

(5)[16] $$(X, (Y, Z)) \leqq (Y, (Z, X))(Z, (X, Y)).$$

An induction argument using the definitions of derived and lower central series as well as formulas (4) and (5) leads now to the inequalities

(6)[17] $$({}^{i}G, {}^{j}G) \leqq {}^{i+j+1}G, \quad G^{(i)} \leqq {}^{2^i-1}G.$$

Thus the derived series decreases much more rapidly than the lower central series. Another significant distinction between these two series is the fact that $G^{(i+1)}$ is a fully invariant subgroup of $G^{(i)}$ whereas ${}^{i+1}G$ need not be a fully invariant subgroup of ${}^{i}G$. A property common to both these series is the fact that the quotient groups of successive members are abelian.

Special properties of such series of fully invariant subgroups are invariant properties of the underlying groups and may therefore be used for characterizing special classes of groups. We give two examples of such properties.

(S) 1 *is the crosscut of the groups* $G^{(i)}$.

(N) 1 *is the crosscut of the groups* ${}^{i}G$.

It is an immediate consequence of formula (6) that every N-group is an S-group, though the converse is not true. In fact, finite S-groups are just the soluble groups whereas finite N-groups are exactly the nilpotent groups.[18] Finite nilpotent groups are known to be direct products of p-groups; and this theorem may be generalized as follows.

Every N-group without elements of order[19] 0 *is the direct product of (finite or infinite) p-groups.*[20]

In Appendix II we prove this theorem and we construct an example of a p-group P satisfying $P^{(2)}=1$, though it is not an N-group, showing that the converse of the above theorem does not hold.

N-groups generated by a finite number of elements may be shown to be Q-groups (as defined in §1),[21] though there exist S-groups generated by two elements which are not Q-groups.[22]

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[16] Hall [1, p. 47, Theorem 2.3].

[17] Hall [1, p. 53, Theorem 2.51 and p. 54, Theorem 2.54].

[18] See, for example, Baer [1], where further references may be found.

[19] An element in a group is said to be of order 0 if it generates an infinite cyclic group.

[20] A p-group is a group all of whose elements are of order a power of p.

[21] See, for example, Fouxe-Rabinowitsch [1].

[22] For proofs of these facts, cp. Baer [2].

Finally we indicate a method[23] for proving that a given group G is an N-group. Suppose that the ring R contains an identity element 1 (satisfying $r1=1r=r$ for every r in R), that 0 is the crosscut of the powers P^i of the two-sided ideal P in R, and that G is a multiplicative group of elements in R meeting the requirement:

$g\equiv 1$ *modulo* P *for* g *in* G.

If g is an element in G, then there exist uniquely determined elements g', g'' in P such that $g=1+g'$ and $g^{-1}=1+g''$; and g'' belongs to P^i if g' is an element in P^i. If g, h are elements in G, and if h', h'' belong to P^i, then

$$\begin{aligned}(g, h) &= (1+g'')(1+h'')(1+g')(1+h')\\ &\equiv 1+g''+h''+g'+h'+g''g'+h''h' \text{ modulo } P^{i+1}\\ &\equiv 1 \text{ modulo } P^{i+1}.\end{aligned}$$

Thus one is able to prove by complete induction the following fact: If g belongs to ${}^iG$, then $g\equiv 1$ modulo P^{i+1}. Consequently G itself is an N-group, since the crosscut of the ideals P^i is 0.

If T and $S\leqq T$ are normal subgroups of the group G, then we denote by $S\div T$ the set of all the elements x in G satisfying $(x, T)\leqq S$. It is readily seen[24] that the set $S\div T$ is a normal subgroup of G; and if S and T are characteristic subgroups of G, so is their *commutator quotient* $S\div T$. The best known example is the center $1\div G$ of the group G or more generally:

The upper central series: $Z_0(G)=1$, $Z_i(G)=Z_{i-1}(G)\div G$ for $0<i$.

Interesting relations between upper and lower central series of a finite p-group have been discovered by P. Hall.[25] There are, however, fundamental differences, since the one series is descending, the other ascending, the one consists of fully invariant subgroups, the other of strictly characteristic subgroups which need not be fully characteristic.

In analogy to the definition of N- and S-groups one is led to the Z-groups, meeting the following requirement.[26]

(Z) *Every element in* G *is contained in at least one* $Z_i(G)$.

A finite group is known to be a Z-group if, and only if, it is an N-group. But in general no such relation holds. The non-abelian free groups are examples of N-groups with centers equal to 1, and there exist p-groups with this property.[27] An example of a Z-group, not an N-group, will be constructed in Appendix II.[28]

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[23] This method has been introduced by W. Magnus [1] who used it to prove that free groups are N-groups.

[24] Baer [3], Zassenhaus [1].

[25] Hall [1, p. 53, Theorem 2.51].

[26] Baer [1].

APPENDIX II: NILPOTENCY OF N-GROUPS

If x and y are elements of order a power of p in the N-group G without elements of order 0, then the order of xy is of the form ip^j where i is an integer prime to p. We denote by W the subgroup generated by x and y; and we put $(xy)^{p^j}=z$ so that z is an element of the order i prime to p. From ${}^kW\leqq {}^kG$ we infer that 1 is the crosscut of the subgroups ${}^kW$.

We are going to prove by complete induction with regard to k that $W/{}^kW$ is a finite group for every k. This is certainly true for $k=0$; and thus we assume its validity for $k-1$. Since W is generated by a finite number of elements, and since $W/{}^{k-1}W$ is finite, we deduce from the Reidemeister-Schreier method[29] that ${}^{k-1}W$ is generated by a finite number of elements. Consequently ${}^{k-1}W/{}^kW$ is an abelian group, generated by a finite number of elements whose orders are different from 0. This implies the finiteness of ${}^{k-1}W/{}^kW$, proving that $W/{}^kW$ is finite too.

Since $W/{}^kW$ is a finite group whose lower central series ends with 1, and since $W/{}^kW$ is generated by two elements of order a power of p, it follows from well known theorems[30] that $W/{}^kW$ is a p-group. Hence $z\equiv 1$ modulo ${}^kW$ for every positive k. Thus z belongs to the crosscut of the subgroups ${}^kW$. But this crosscut is 1, showing that $z=1$. Therefore we have proved the following fact.

If x and y are elements of order a power of p in the N-group G without elements of order 0, *then xy is an element of order a power of p.*

If we denote by $G(p)$ the set of all the elements of order a power of p in G, then it follows from the fact just established that $G(p)$ is a subgroup of G. Hence $G(p)$ is a fully invariant and therefore a normal subgroup of G. If p and q are different primes, then $G(p)$ and $G(q)$ are normal subgroups of G which have only the identity in common. Consequently $rs=sr$ for r in $G(p)$ and s in $G(q)$. Now it is readily seen that G is the direct product of its primary components $G(p)$, as we desired to prove.

The condition that the group G be an N-group, though indispensable for the preceding proof, is not a necessary condition, as will be seen from the following *example of a p-group which is not an N-group.*

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[27] Baer [1, pp. 412, 413, Example 3.4].

[28] For another example of a Z-group, not an N-group, see Baer [1, p. 406, Example 2.5].

[29] See, for example, Baer [1, p. 396 (1.3)] or Zassenhaus [1, p. 108].

[30] See, for example, Zassenhaus [1, p. 107].

Denote by B the abelian group generated by elements b, $b(1)$, $b(2), \cdots, b(i), \cdots$, subject to the relations:

$$1 = b^p,\ b = b(i)^{p^i} \text{ for every positive } i.$$

It is readily seen that the crosscut B^{p^ω} of all the groups B^{p^i} is the cyclic group of order p, generated by b, and that B/B^{p^ω} is the direct product of a cyclic group of order p by a cyclic group of order $p^2 \cdots$ by a cyclic group of order $p^i \cdots$ and that the elements $b(i)$ represent a basis of B/B^{p^ω}.

Denote by $g(i)$ the automorphism of B which maps $b(i)$ upon $b(i)^{1+p}$ (and thus leaves b invariant) and which leaves all the $b(j)$ for $j \neq i$ invariant. It is obvious that the automorphism $g(i)$ is of order p^i and that the automorphisms $g(i)$, for positive i, generate an abelian group.

Let finally G be the group obtained by adjoining to B elements $t(i)$ subject to the relations:

$$1 = t(i)^{p^i},\ t(i)^{-1}xt(i) = x^{g(i)} \text{ for } x \text{ in } B \text{ and positive } i.$$

This group G is an extension of the group B by the group of automorphisms, generated by the $g(i)$'s.

Since G/B and B are both abelian p-groups, G itself is a p-group. Furthermore $G^{(2)} = 1$. One verifies inductively that ${}^iG = B^{p^i}$ and that the crosscut of the subgroups ${}^iG$ is therefore $B^{p^\omega} \neq 1$. Thus G has been shown to be a p-group, an S-group, but not an N-group.

Furthermore one verifies by complete induction that $Z_i(G)$, the ith term in the upper central series of G, is generated by the elements $t(j)$ for $0 < j < i$ and by the elements x in B which satisfy $x^{p^i} = 1$. Thus G is a Z-group.

3. Burnside's problem.

Before the impact of topology on group theory made itself felt, group theory was concerned almost exclusively with the study of finite groups. Practically the first fact noticed by the student of finite groups is that the orders of the elements in a finite group are bounded and different from 0. This latter property defines a larger class of groups, which we are going to discuss now.

If the orders of the elements in the group G are bounded, then there exists the l.c.m. of the orders of all the elements in G. If m is this number, then $x^m = 1$ for every x in G. Every group G is the homomorphic image of a free group F. If N is the kernel[11] of the homomorphism mapping F upon G, then $x^m=1$ for x in G is equivalent to[32] $F^m \leqq N$; and thus G is not only the homomorphic image of the free group F, but even of F/F^m. These *free groups reduced modulo m* may therefore be considered as universal groups for the class of groups under consideration. There is little known about their nature[33] and Burnside's problem might be stated somewhat loosely as the problem of determining the structure of F/F^m.

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[11] The kernel of the homomorphism f consists of those elements which are mapped upon 1 by f.

Let us start with a simple observation. The commutator (x, y) may be seen readily to equal

$$(x, y) = x^{-1}y^{-1}xy = (x^{-1}y^{-1}x)^2x^{-2}(xy)^2$$

so that we have in every group G the following relation:

$$(G, G) \leqq G^2.$$

This shows that F/F^2 is abelian and this makes it possible to answer completely any question we might have concerning the structure of these groups F/F^2.

It may easily be derived from a not at all trivial result of F. Levi and B. L. van der Waerden[34] that

$${}^2G \leqq G^3;$$

and thus one is led to the question whether there exists to every positive integer n an integer n' such that

$${}^{n'}G \leqq G^n.$$

The answer to this question is clearly in the negative, since we have noted before that ${}^{n'}G \leqq G^n$ implies that G/G^n is the direct product of its p-components whereas not every group satisfying $G^n=1$, for some n, is the direct product of its p-components. As a matter of fact a still stronger result may be proved.

If the free group F is not abelian, and if F/F^n is an N-group, then n is a power of a prime.

For a proof of this theorem see Appendix III. For prime powers n we prove in Appendix III a still stronger result; to wit:

If there exists an integer n' such that ${}^{n'}G \leqq G^n$ for every group G, then n is a prime.

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[32] F^m is the subgroup generated by all the mth powers of elements in the group F.

[33] This is the reason why so much more emphasis is laid upon the commutator subgroups than upon these subgroups F^m.

[34] Levi and van der Waerden [1, p. 155].

Whether or not the converse of either of these theorems holds true is not known at present; and a decision would constitute an important step towards a solution of Burnside's problem.

The problem as Burnside himself conceived it was stated in a more limited fashion than we have considered it, since he imposed the additional restriction upon the groups under consideration that they should be generated by a finite number of elements. For these groups he has stated two famous conjectures.[35]

(I) *F/F^n is finite for every free group F of finite rank.*[36]

(II) *F/F^n is soluble for n odd and F a free group of finite rank.*

These conjectures are certainly true for $n=2$ and 3.

The conjecture (II) may be restated in a somewhat more precise form, since $(F/F^n)^{(i)} = (F^nF^{(i)})/F^n$.

(II′) *If n is an odd integer and if m is a positive integer, then there exists an integer $k=k(n, m)$ such that*

$$F^{(k)} \leqq F^n \text{ for } F \text{ a free group of rank } m.$$

A similar restatement of (I) is impossible because of the existence of simple groups. But for prime powers n this is possible, if one recalls that the lower central series of a finite p-group ends with the identity and that ${}^{i}(F/F^n) = ({}^{i}F\,F^n)/F^n$; and thus we obtain the following conjecture:

(I′) *If n is a power of a prime, and if m is a positive integer, then there exists an integer $h=h(n, m)$ such that*

$${}^{h}F \leqq F^n \text{ for } F \text{ a free group of rank } m.$$

That (I′) and (II′) imply the finiteness of F/F^n is a consequence of the following easily verified fact. If the group G is generated by a finite number of elements, and if every element in G is of finite order, then $G/{}^{i}G$ and $G/G^{(i)}$ are both finite groups. But (I′) and (II′) for prime powers are essentially equivalent conjectures, as may be seen from the following theorem, a proof of which may be found in Appendix III.

If G is a p-group, generated by a finite number of elements, then there exist to every positive integer i integers i', i'' such that

$${}^{i'}G \leqq G^{(i)}, \quad G^{(i'')} \leqq {}^{i}G.$$

But in case these conjectures should not be verified, then this theorem still asserts that *a p-group which is generated by a finite number of elements is an N-group if, and only if, it is an S-group.*

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[35] Burnside [1].

[36] The rank $r(F)$ of the free group F is the number of elements in any free set of generators of F. It is at the same time the rank of the free abelian group $F/(F, F)$.

The importance of these statements may be seen from the following almost obvious equivalence.

If n is a power of a prime, and if F is a free group of finite rank, then the following two conditions are necessary and sufficient for finiteness of F/F^n.

(a) *F/F^n is an N-group.*

(b) *The lower central series of F/F^n is finite.*

But by the preceding remarks we may substitute for (a) and (b) the following conditions.

(a′) *F/F^n is an S-group.*

(b′) *The derived series of F/F^n is finite.*

If n is a prime power such that F/F^n is an N- (or S-)group for every free group F of finite rank, then one may verify the corresponding statement for every F/F^n with free F. Furthermore it is immediately seen that F/F^n is an N-group if, and only if, F^n is the crosscut of the subgroups ${}^{i}F\,F^n$ for all the integers i. The fact that free groups are N-groups may be considered an encouraging sign. But more is known. For Zassenhaus [2] proved a theorem from which it is easy to derive the fact that 1 is the crosscut of the groups ${}^{i}F\,F^{p^i}$ for p a prime and $i=1, 2, \cdots$. All this, however, does not suffice to prove (a). Thus it may be a consolation to know that it is possible to substitute for (a) and (a′) the following weaker conditions without impairing the validity of the above statements.

(a*) *If $(F/F^n)/{}^{i}(F/F^n)\neq 1$, then ${}^{i+1}(F/F^n)<{}^{i}(F/F^n)$.*

(a′*) *If $(F/F^n)/(F/F^n)^{(i)}\neq 1$, then $(F/F^n)^{(i+1)}<(F/F^n)^{(i)}$.*

Let us turn now to explaining the significance of condition (b) and its equivalent (b′). Denote by N the crosscut of all the subgroups ${}^{i}(F/F^n)$. For n a power of a prime, N is at the same time the crosscut of the subgroups $(F/F^n)^{(i)}$. The following fact is now readily verified.

If condition (b) *is satisfied by the prime power n, if the free group F is of finite rank, then the following conditions are necessary and sufficient for the group G to be a homomorphic image of the group $(F/F^n)/N$:*

(i) *G is finite.*

(ii) $G^n=1$.

(iii) *G may be generated by $r(F)$ elements.*

Condition (b) has been verified for $n=5$ and $r(F)=2$ by P. Hall.[37]

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[37] For a proof see Grün [2].

Both conditions (a*) and (b) have been investigated recently with rather powerful tools.[38] So far, however, the results obtained have not been of that striking nature that lends itself to easy narration.

APPENDIX III: THE LOWER CENTRAL SERIES OF FREE GROUPS MODULO n

THEOREM 1. *If F is a non-abelian free group and n a positive integer such that F/F^n is an N-group, then n is a power of a prime.*

PROOF. Suppose that $n=n'n''$ is the product of the two relatively prime integers n' and n'' both of which are different from 1. Denote by S the direct product of n' cyclic groups each of which is of order n'', and let $s(1), \cdots, s(n')$ be a basis of S. There exists one and only one automorphism k of S which maps $s(i)$ for $0<i<n'$ upon $s(i+1)$ and $s(n')$ upon $s(1)$. This automorphism k is clearly of order n'. Consequently there exists a group T which is obtained by adjoining to S an element t subject to the relations:

$$t^{n'} = 1, \quad t^{-1}st = s^k \text{ for } s \text{ in } S.$$

This group T is an extension of the abelian group S of order $n''^{n'}$ by the cyclic group T/S of order n'. If x is an element in T, then $x^{n'}$ is in S and $x^n=(x^{n'})^{n''}=1$, since $S^{n''}=1$. Furthermore it is readily seen that T may be generated by the two elements t and $s(1)$. But T is not the direct product of its primary components, since the two elements t and $s(1)$ of the relatively prime orders n' and n'' do not permute.

Since F is a non-abelian free group, its rank is at least 2 and consequently there exists a homomorphism h of F upon T. If H is the kernel of the homomorphism h, then we deduce $F^n \leqq H$ from $T^n=1$; and thus h induces a homomorphism of F/F^n upon T. Since T is not the direct product of its primary components, neither is F/F^n, as homomorphisms map primary components upon primary components. We proved in Appendix II that N-groups without elements of order 0 are the direct products of their primary components. Hence F/F^n is not an N-group if n is not a power of a prime.

THEOREM 2. *If there exists an integer n' such that ${}^{n'}G \leqq G^n$ for every group G, then n is a prime.*

PROOF. Suppose that n, n' are positive integers such that ${}^{n'}F \leqq F^n$ for every free group F of finite rank (this hypothesis which is weaker than the one enunciated in the theorem will be shown to be sufficient). Then F/F^n satisfies ${}^{n'}(F/F^n)=1$. Hence F/F^n is an N-group and it follows from Theorem 1 that n is a power of a prime.

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[38] Grün [1], Magnus [1, 2], Witt [1], Zassenhaus [2].

It has been shown elsewhere[19] that there exists a group G with the following properties:

(i) Both $G/(G, G)$ and (G, G) are direct products of cyclic groups of order p.

(ii) The center of G is equal to 1.

From (i) one infers in particular that $G^{(2)}=1$ and that $G^{p^2}=1$. Thus there exists a representation of G as a quotient group of a free group: $G=F/N$ satisfying $F^{p^2}\leqq N$.

Suppose now that the prime power $n=p^m$ is not a prime, so that $1<m$. Then it follows from a preceding remark that $F^n\leqq N$ and that G is a homomorphic map of F/F^n. Suppose now that the element x in F belongs to ${}^{n'}F$. Then there exists a free factor V of F which is a free group of finite rank such that x belongs to ${}^{n'}V$. Now we may apply the hypothesis stated at the beginning of the proof, namely that ${}^{n'}V\leqq V^n$. Thus x belongs to V^n and therefore to F^n, and we have shown: ${}^{n'}F\leqq F^n\leqq N$. But this shows that the lower central series of $G=F/N$ is finite and ends with 1, a fact that is readily seen to contradict (ii). This contradiction proves that n is a prime.

Let us term the positive integer n *Burnsidean* if F/F^n is finite for every free group F of finite rank. Then we may prove the following converse of Theorem 1.

THEOREM 1'. *If n is a Burnsidean prime power, and if F is a free group, then F/F^n is an N-group.*

PROOF. If x is an element not 1 in F, then there exists a homomorphism h of F upon a free group V of finite rank such that $x^h\neq 1$, since F is the free product of two free groups V and W, the first one of which contains x and is of finite rank. This homomorphism h maps F^i upon V^i and ${}^{i}F$ upon ${}^{i}V$ for every positive i. If x were contained in the crosscut ${}^{\omega}F$ of the subgroups ${}^{i}F$, then x^h would belong to the crosscut ${}^{\omega}V$ of the subgroups ${}^{i}V$. But n is a Burnsidean prime power. Thus V/V^n is finite and of order a power of a prime. Hence there exists an integer w such that ${}^{w}V\leqq V^n$. Since x^h belongs to ${}^{\omega}V$, it belongs to ${}^{w}V$ and therefore to V^n. But this implies that x belongs to F^n, and hence we have shown ${}^{\omega}F\leqq F^n$, a fact that is equivalent to the assertion: F/F^n is an N-group.

We note that it would have been sufficient for our proof to make the hypothesis: F/F^n is an N-group for every free group F of finite rank, a hypothesis that is considerably weaker than the assumption that n is Burnsidean.

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[19] Baer [1, pp. 412, 413, Example 3.4].

As an application of Theorem 1′ we prove a converse of the theorem proved in Appendix II.

THEOREM 3. *Suppose that the positive integer n is the product of Burnsidean prime powers, and that the group G satisfies $G^n=1$. Then G is the direct product of its primary components if, and only if, G is the homomorphic image of an N-group H satisfying $H^n=1$.*

PROOF. It is a consequence of a theorem established in Appendix II that N-groups H satisfying $H^n=1$ are the direct products of their primary components. Homomorphisms map direct products of p-groups upon direct products of p-groups, proving the sufficiency of our condition.

Suppose conversely that G is the direct product of its primary components. If $n=p(1)^{m(1)}\cdots p(k)^{m(k)}$ with $0<m(i)$, $0<k$, then G is the direct product of groups $G(i)$ satisfying $G(i)^{n(i)}=1$ for $n(i)=p(i)^{m(i)}$. The group $G(i)$ may be represented as a quotient group $F(i)/M(i)$ of a free group $F(i)$; and clearly $F(i)^{n(i)}\leqq M(i)$. Thus $G(i)$ is the homomorphic image of $F(i)/F(i)^{n(i)}$. By hypothesis $n(i)$ is a Burnsidean prime power and thus it follows from Theorem 1′ that $R(i)=F(i)/F(i)^{n(i)}$ is an N-group. If R is the direct product of the N-groups $R(i)$, then $R^n=1$, R is an N-group and G is the homomorphic image of R, proving our contention.

THEOREM 4. *If G is a p-group, generated by a finite number of elements, then there exist to every positive integer i integers i', i'' such that*

$$ {}^{i'}G \leqq G^{(i)}, \quad G^{(i'')} \leqq {}^{i}G. $$

PROOF. It is a consequence of a theorem quoted in §1 that we may select as i'' any integer satisfying $i<2^{i''}$.

Since ${}^{j}G/{}^{j+1}G$ and $G^{(j)}/G^{(j+1)}$ are abelian groups the orders of whose elements are powers of the prime p, since such abelian groups are finite if they are generated by a finite number of elements, one proves inductively by the usual application of the Reidemeister-Schreier method[40] that $G/{}^{j}G$ and $G/G^{(j)}$ are finite groups. Thus $G/G^{(i)}$ is a finite p-group, and such groups are known to have the property that their lower central series is finite and ends with 1. Hence there exists an integer i' such that ${}^{i'}(G/G^{(i)})=1$ and this is equivalent to ${}^{i'}G\leqq G^{(i)}$.

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[40] See footnote 29.

4. Invariants of the Hopf type. The group invariants discussed so far were subgroups and quotient groups of subgroups of the group under discussion. But there exist other possibilities of connecting groups with groups in an invariant fashion. An instance is the automorphism group of a group.

A group may be given either by inner properties or by some sort of representation. An example of the first kind of definition is the following one: the abelian group of order p^n the orders of whose elements are 1 and p. An example of the second kind is the customary definition of the symmetric group of degree n. Though it is certainly more desirable to characterize a group by inner properties, it is more common to define it by some sort of representation, and very often a suitable representation makes a group better accessible to treatment.

H. Hopf[41] has recently discovered a group invariant derived from a representation of a group as a quotient group. He showed that for normal subgroups N of free groups F the quotient group

$$(F,F)/(F,N)$$

depends only on the structure of F/N, and is an invariant of F/N. Now the groups F/F^n, discussed in §3, are defined by their representations only. Thus these Hopfian invariants and their generalizations may be a tool in their investigation.

We indicate a new proof of Hopf's invariance theorem which has the advantage of being a convenient basis for many important generalizations. To simplify notation we restrict ourselves to a discussion of the lower central series, which is quite typical.[42]

Suppose that M is a normal subgroup of the group H. Then H/M is a representation of a certain group G. We define inductively subgroups ${}_iM$ by the formulas: ${}_0M = M$, ${}_iM = (H, {}_{i-1}M)$. Then every ${}_iM$ is a normal subgroup of H, and we note the following inequalities:

$${}_iM \leqq {}^iH,\ {}_iM \leqq {}_{i-1}M \text{ and } M \leqq {}_iM + {}^iH.$$

1. Let us consider an endomorphism h of the group H, satisfying $x^h \equiv x$ modulo M. Of h we say then that it induces the identity in H/M. Assume now that we have shown already that h induces the identity in ${}^iH/{}_iM$. Then a simple application of formulas (2), (3) of §2 shows that h induces the identity in ${}^{i+1}H/{}_{i+1}M$.

2. Consider now two representations H/M and K/N of the same group G and homomorphisms h, k of H into K and K into H respectively with the following property: h induces an isomorphism a of

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[41] Hopf [1].

[42] Baer [3].

H/M upon K/N and k induces the isomorphism a^{-1} of K/N upon H/M.

This hypothesis implies in particular $M^h \leqq N$ and $N^k \leqq M$; and one verifies readily that ${}^{i}H^h \leqq {}^{i}K$, ${}^{i}K^k \leqq {}^{i}H$, ${}_{i}M^h \leqq {}_{i}N$, ${}_{i}N^k \leqq {}_{i}M$. But hk induces the identity in H/M; and thus it follows from what we have shown under 1 that hk induces the identity in every ${}^{i}H/{}_{i}M$. From this fact and the previously stated inequalities one deduces immediately that h and k induce reciprocal isomorphisms between ${}^{i}H/{}_{i}M$ and ${}^{i}K/{}_{i}N$.

This implies in particular that the isomorphism a maps $({}_{i}M \div {}^{i}H)/M$ upon $({}_{i}N \div {}^{i}K)/N$.

3. We say now that two representations H/M and K/N of the group G are *similar* if every isomorphism between H/M and K/N may be induced by homomorphisms. Using this terminology the result obtained under 2 implies the following generalization of Hopf's invariance theorem: ${}^{i}H/{}_{i}M$ and $({}_{i}M \div {}^{i}H)/M$ are invariants of the class of similar representations of G to which H/M belongs.

It should be mentioned that not every representation belongs to a class of similar representations.

That this theorem is really a generalization of Hopf's theorem is a consequence of the fairly obvious fact that all representations of the group G as a quotient group of a free group (or of a free abelian group,[43] or of a free group reduced modulo n, and so on) are similar.

The range of applicability of these invariants seems to be fairly wide. So far they have been used mainly to show that homomorphisms meeting certain requirements are isomorphisms.[42]

As these invariants have not been defined in an invariant fashion, one may desire an invariant definition. This may be obtained either by using a normal form of the representation which is trivial as far as it can be done and not very interesting, or by a properly invariant definition which does not refer to any representation. This latter problem seems to be fairly deep and has been solved so far apparently in only one instance.[44]

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[43] Eilenberg and MacLane [1, p. 765, Lemma 4.3].

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University of Illinois

THE NOVEMBER MEETING IN PASADENA

The four hundred first meeting of the American Mathematical Society was held at the California Institute of Technology on Saturday, November 27, 1943. The attendance was about one hundred thirty, including the following thirty-five members of the Society:

E. T. Bell, W. Z. Chien, P. H. Daus, R. P. Dilworth, H. J. Hamilton, F. C. Hinds, P. G. Hoel, D. G. Humm, D. H. Hyers, C. G. Jaeger, Glenn James, R. C. James, C. C. Lin, W. E. Mason, A. D. Michal, M. M. Mills, K. T. Millsaps, G. H. Peebles, Sam Perlis, W. C. Randels, William Shand, G. E. F. Sherwood, S. C. Snowdon, D. V. Steed, Alfred Tarski, A. E. Taylor, B. P. Taylor, T. Y. Thomas, S. E. Urner, F. A. Valentine, Morgan Ward, L. E. Wear, W. M. Whyburn, E. R. Worthington, Max Zorn.

After a brief opening session for contributed papers at which Professor A. E. Taylor presided, the meeting in the morning was devoted to the invited one hour addresses at which Professor T. Y. Thomas presided. By invitation of the Program Committee, Dr. Alfred Tarski of the University of California spoke on *Paradoxes of geometrical congruence* and Professor Max Zorn of the University of California at Los Angeles on *Mathematical induction.*

The Symposium on Applied Mathematics was held in the afternoon with Professor A. D. Michal presiding. Professor Theodore von Kármán of the California Institute of Technology and Northrop Aircraft Corporation spoke on *Generalized airplane wing theory*, Professor P. Y. Chou of National Tsing Hua University and the California Institute of Technology on *Velocity correlations and the solutions of the equations of turbulent fluctuation*, and Mr. Paul Lieber of the Douglas Aircraft Corporation on *Some applications of matrices to flutter and related phenomena.* In the absence of Mr. Lieber, the last address was read by Mr. K. Latzer of the Douglas Aircraft Corporation.

Titles and cross references to the abstracts of the papers read follow below. Papers whose abstract numbers are followed by the letter *t* were read by title. Papers 1 to 3 were read in the morning session while papers 4 to 6 were read by title. Paper 3 was read by Dr. Perlis.

1. R. P. Dilworth: *A decomposition theorem for partially ordered sets.* (Abstract 50-1-7.)

2. R. C. James: *Orthogonality and differentiability in normed linear spaces.* (Abstract 50-1-26.)

3. Roy Dubisch and Sam Perlis: *On the radical of a non-associative algebra.* Preliminary report. (Abstract 50-1-8.)

4. František Wolf: *On a problem of L. M. K. Boelter. Viscous fluid forced through a heated vertical pipe.* (Abstract 50-1-43-*t*.)

5. E. J. Purcell: *Variety congruences of order one in n-dimensional space.* (Abstract 50-1-47-*t*.)

6. K. T. Millsaps: *A note on generalized Hilbert space.* (Abstract 50-1-48-*t*.)

A. D. MICHAL,
Associate Secretary

BOOK REVIEWS

On growth and form. By D'Arcy W. Thompson. Cambridge University Press; New York, Macmillan, 1942. 1116 pp. $12.50.

It is a rare privilege to write a review of a new edition of a book which has already taken its place as one of the great classics of science. *On growth and form* by D'Arcy Thompson in this new edition is much as it was in the first. Its fundamental ideas have been very little revised, but a wealth of new material has been added, expanding it from 793 to 1116 pages.

To summarize its contents would be a very difficult matter. It is almost an encyclopedia of all the relations that have ever been discussed between mathematics and organic form. Among the subjects treated are: the form of the cell, tissues, concretions produced by living things, shells, horns, and teeth; from the dynamic point of view, growth and the relation between form and mechanical efficiency; and such perennial favorites of the geometrician as the form of the bee's cell and the arrangement of leaves.

For the most part the mathematics used in the book is elementary. Thompson makes no pretentions, but says he is using the tools he has, leaving it to better equipped workmen to carry on the work. Professor Archibald[1] in a review of the first edition deals more competently than I could with the mathematics. The general point of view seems more important than the mathematics itself. In fact the author frequently talks more about mathematics than in its language. He states his purpose in the introductory chapter (p. 14): "to correlate with mathematical statement and physical law certain of the simpler outward phenomena of organic growth and structure or form, while all the while regarding the organism, *ex hypothesi*, as a material and mechanical configuration." And in the Epilogue he concludes (p. 1096) "My task is finished if I have been able to show that a certain mathematical aspect of morphology, to which as yet the morphologist gives little heed, is interwoven with his problems, complementary to his descriptive task, and helpful, nay essential, to his proper study and comprehension of Growth and Form."

Various biologists have placed very different estimates on the value of the book. Professor Sinnott,[2] in his review, has been exceedingly generous and attributes to it a wide and important influence. Professor McClung[3] on the other hand has stated that its influence is slight. In spite of the fact that the book has been very much admired, I am afraid that Professor McClung's estimate is nearer the truth. The ideas of *On growth and form* have played little part in the spectacular advances of biology since the book was first written. For this I think there is a very good reason: the point of view which the book represents is out of fashion, and is indeed the antithesis of the one now in vogue, to which these advances have been due.

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[1] R. C. Archibald, Bull. Amer. Math. Soc. vol. 24 (1918) p. 403.

[2] E. W. Sinnott, Quarterly Review of Biology vol. 18 (1943) p. 64.

[3] C. E. McClung, Science N. S. vol. 96 (1942) p. 471.

When the book was first published in 1917, experimentation in genetics was just beginning to produce its brilliant results and experimental embryology was approaching its most spectacular era. Here were exciting things to attract the young investigator with which the abstract ideas of the mathematician, whose language he all too frequently could not understand, could hardly compete. A rich harvest has been reaped in these fields with very slight if any influence of Thompson's book. Interestingly enough, on the other hand, almost every reviewer has complained that the new edition has not been influenced by this research, which is for the most part hardly mentioned. McClung[3] misses a treatment of cytogenetics, Buchanan[4] a treatment of physiological gradient ideas, and both Sinnott[2] and Hammett[5] list a number of important subjects that one might expect to have been treated.

The failure to take the results of modern biological research into consideration (Sinnott says the revision might just as well have been written twenty-five years ago) is related to the failure to make an important impact upon this research. They are both due to the antithesis of the fundamental ideas. *On growth and form* harks back to an older habit of thought, and may also as Wrinch[6] suggests be the herald of a new era, but it is not an essential part of contemporary biological advance.

There are only three essentially different ways of considering biological form. It may be considered as present from the beginning, existing either as an actual minute replica in the germ, or as an "idea" of some nonphysical formative entity like the entelechy of Aristotle and Driesch; or existing neither actually nor ideally, but merely potentially, in the organization of the germ. In older biological thinking form always pre-existed in some fashion or other. The adult form was in existence from the beginning of the life of the organism as an individual. The "preformationists" of the 18th century—their ideas culminating in Bonnet—believed that the germs of all organisms ever to exist were created at the original creation. Each germ was a minute replica of the adult that was to develop from it. It had merely to expand. There was no real development of form. It had existed from the beginning.

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[4] J. W. Buchanan, Physiological Zoology vol. 16 (1943) p. 135.

[5] F. S. Hammett, Growth vol. 7 (1943) p. 321.

[6] D. Wrinch, Isis vol. 25 (1943) p. 232.

The "epigeneticists," their opponents, believed that organisms are developed from unorganized and unformed matter under the influence of a vital, nonphysical entity. Here the form of the adult preexisted not actually, as in the germ of the preformationist, but as an idea of the entelechy or *vis essentialis.* The unformed material was moulded into shape by this agent—the actual form produced was new, an epigenesis.

With the development of modern techniques it became clear that the epigeneticists were right in that the germ contains no actual replica of the adult. The form develops rather by gradual stages. The notion of a nonphysical formative agent has not appealed to most biologists, however, although Driesch has made a valiant attempt to sustain it. The alternative chosen by most is a modified form of preformationism, in which the germ is considered to contain not a preformed replica of the adult, but an organization which determines the development of the adult form. The nature of this organization may be considered from two points of view: biologically, from the point of view of heredity, as with the trend begun by Weismann and leading to modern genetics, or mathematically and physically as in Thompson's book.

Thompson says (p. 1022) "To look on the hereditary or evolutionary factor as the *guiding principle* in morphology is to give to that science a one-sided and fallacious simplicity" and states his position unequivocally (p. 340): "The efforts to explain 'heredity' by the help of 'genes' and chromosomes, which have grown up in the hands of Morgan and others since this book was first written, stand by themselves in a category which is all their own and constitute a science which is justified of itself. To weigh or criticize these explanations would lie outside my purpose, even were I fitted to attempt the task. . . . I leave this great subject on one side not because I doubt for a moment the facts nor dispute the hypotheses nor decry the importance of one or other; but because we are so much in the dark as to the mysterious field of force in which the chromosomes lie, far from the visible horizon of physical science, that the matter lies (for the present) beyond the range of problems which this book professes to discuss, and the trend of reasoning which it endeavors to maintain."

Thompson's concept of organic form is that it is predetermined by the physical organization of the system in which it develops, and "it is in obedience to the laws of physics that their particles have been moved, moulded and conformed" (p. 10). The nature of the physical organization and of the physical processes by which the form is produced from it is to be arrived at by mathematical analysis and physical interpretation of the adult form (p. 16). "The form of an object is a 'diagram of forces,' in this sense, at least, that from it we can judge of or deduce the forces that are acting or have acted upon it." "In an organism, great or small, it is not merely the nature of the motions of the living substance which we must interpret in terms of force (according to kinetics), but also the conformation of the organism itself, whose permanence or equilibrium is explained by the interaction or balance of forces, as described in statics." His interest is not in the biological analysis of the organization of the germ, from the point of view of either the geneticist or the experimental morphologist. He approaches the problem from the opposite direction.

He thinks of form as a Platonist. In his discussion of the tortoise shell, for example (p. 517), he is obviously talking of an ideal tortoise, more valid than any actual specimen or species, and any variant from which may be considered an "accident." As a matter of fact in the Epilogue (p. 1097) he alines himself with the teaching of Plato and Pythagoras, and again (p. 1094) he says "In natural history Cuvier's 'types' " (which are Plato's ideas under another name) "may not be perfectly chosen nor numerous enough, but types they are; and to seek for stepping stones across the gaps is to seek in vain, forever." On this view his attitude toward embryology (p. 5) and his failure, pointed out by Julian Huxley,[7] to apply his method of transformation of coordinates to stages in the life history of a single organism, are readily explainable. He is not thinking of the development of form as a biological process at all. He takes the adult form as given and analyzes it.

To place Thompson thus among the Platonists and beside the transcendental morphologists of the last century is by no means derogatory. Man has been seeking systematically for answers to his great problems for only 5000 years at most, and on a conservative estimate will be doing so for some 5,000,000 years to come. To attempt to read off with finality any consistently developed line of thought would seem at this stage presumptuous. It needs to be strongly emphasized that none of the hypothetical entities proposed on the basis of experiments in embryology deal with form in the sense that Thompson does. The gradients of Child and the fields of Weiss serve to organize many significant facts about development, but do not tell us why specific form develops. Child[8] says that the specific form develops because the gradient operates in a substratum of specific protoplasm, but what does that mean? Weiss[9] says "the fact that each cell is bound to react exclusively in accordance with the standards of the species to which it belongs" constitutes "the principle of 'genetic limitation.'" What is the basis of that? In spite of all we have learned about the nature and action of organizers and evocators, Needham admits "that after all the larger part of the mystery remains in that we can as yet form little idea of what constitutes reactivity—competence to react to the morphogenetic inductor."[10] The specificity of protoplasm from the point of view of morphogenesis, and the competence of tissues to respond in a specific way to organizers are associated with the "form" in Thompson's *On growth and form*. The present day experimental embryologist is simply not investigating it at all. The geneticist, with his genom—the fundamental organization of the germ made up of genes arranged on chromosomes—comes nearer to it. It may well be, as Wrinch[6] suggests, that as the chemist learns more of the morphology of huge molecules and their aggregates, and as the biologist learns to apply this knowledge to the structure of the genom, a new place will be found, in our thinking, for the form of organisms as the morphologist, who descends from the line of Plato, thinks of it.

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[7] J. S. Huxley, *Problems of relative growth*, New York, 1932, p. 105.

Even before that time arrives, however, this book has a message for the biologist, whose idea of quantitative biology is frequently merely the statistical treatment of data, sometimes even forgetting that "measurements may be as empty of significance as any other kind of descriptive materials. The statistical answers that can be wrung out of such measurements may have very little meaning if the quantities measured depend on a multiplicity of causes. Statistical treatment may indicate that certain results are significant, but what they signify no man may know, or even surmise."[11] Such a man might gain from *On growth and form* an idea of what could be meant by mathematical biology.

For the mathematician who has an interest in the relationship between his abstract ideas and the phenomena of the natural world, the book should be a treasure house. The author attributes to Lobatchevsky the statement (p. 11) that "there is no branch of mathematics however abstract, which may not some day be applied to phenomena of the real world."

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[8] C. M. Child, *Individuality in organisms*, Chicago, 1915, p. 188.

[9] P. Weiss, *Principles of development*, New York, 1939, p. 362.

[10] J. Needham, *Biochemistry and morphogenesis*, Cambridge, 1942, p. xiii.

[11] T. H. Morgan, *Experimental embryology*, New York, 1927, p. 13.

The book is written in an elegant, almost poetic, style that makes it delightful to read, and its many references conveniently placed at the bottom of the page near the related subject matter serve as a lure to further reading.

J. WALTER WILSON

Partial differential equations. By Frederick H. Miller. New York, Wiley; London, Chapman and Hall, 1941. 9+259 pp. $3.00.

According to the author the book is intended to be a text in a first course in partial differential equations. The chapter on ordinary differential equations is intended for review and reference purposes and not as a first course in the subject. The author finds it advisable to include a chapter on direction cosines and partial derivatives, probably because so many exercises in the book are taken from geometry. In the main the book is concerned with the quest for solutions depending on arbitrary constants and arbitrary functions. The examples of Chapter III show very clearly why this viewpoint of the subject is much more complicated in the case of partial differential equations than it is in the case of ordinary equations. In ordinary equations the solution of an nth order equation depends on n arbitrary constants, and conversely, the elimination of n arbitrary constants leads to an equation of nth order. In general, the number of partial derivatives of a given order is higher than the number of independent variables. The elimination of two arbitrary functions may lead to a pair of third order equations in one unknown. Since the first order partial differential equation behaves more like an ordinary equation than do those of higher order, Chapter IV on the linear equation of first order and Chapter V on nonlinear equations of first order are almost entirely devoted to the quest for solutions depending on arbitrary functions and arbitrary constants.

Chapter VI on Fourier series and the boundary value problems in Chapter VII furnish an exception to the above viewpoint. In this work the author is, of course, not seeking solutions depending on arbitrary functions. Chapter VI on Fourier series contains a statement of the expansion theorem for a function continuous except for a finite number of jump discontinuities. Chapter VII on the linear equation of higher order is devoted largely to the consideration of operator methods, undetermined coefficients, and variation of parameter methods for obtaining the particular solution. In case the nth order differential operator can be factored into linear factors a complimentary function depending on n arbitrary functions can be obtained. As mentioned above, Chapter VII also contains separating variables and expansion of boundary conditions in Fourier series. The equations treated include the vibrating string, the cable, fluid flow, and heat flow. These equations have already been derived from physics in Chapter III.

Chapter VIII on nonlinear equations of second order is devoted largely to obtaining solutions of Monge's equation depending on arbitrary functions.

The book contains very few examples from physics except in the boundary value problems solved with Fourier series. However, the book contains an exceptionally large number of problems in which the student is asked to find a solution depending on arbitrary constants or functions. In a large proportion of these the partial differential equation is proposed as a problem in geometry.

The multiple integration approach to partial differential equations is not touched upon. The concept of characteristics of a second order equation and the classification of second order equations does not appear. Also there is no mention of successive integrations.

EDWIN W. TITT

The calculi of lambda-conversion. By Alonzo Church. (Annals of Mathematics Studies, no. 6.) Princeton, Princeton University Press; London, Humphrey Milford and Oxford University Press, 1941. 2+77 pp. $1.25.

This is a brief and attractively written introduction to the remarkable formal systems discovered by Church and called by him calculi of lambda-conversion. These systems were developed by Church in collaboration with his students, S. C. Kleene and J. B. Rosser. The present booklet, which is lithoprinted, is in most respects a considerable improvement over the same author's mimeographed Princeton lecture notes of 1936, of which it may be considered a revision. The notation has been simplified and improved, and the treatment of Gödel numbers is much simpler than in the former version. The proof of the fundamental Church-Rosser consistency theorem is now given in full detail, and the section on recursive arithmetic has been considerably expanded.

Nevertheless the text of the present version totals only 71 pages. This brevity is partly accounted for by the plan which the author has wisely adopted of considering only the calculus of lambda-conversion proper in full detail. In fact, the first four chapters are devoted to this, the most elementary of the lambda-calculi. The more complicated, but also more useful, systems called the calculus of λ-*K*-conversion, the calculus of restricted λ-*K*-conversion, and the calculus of λ-δ-conversion, as well as the applications of the latter to symbolic logic, are discussed only very briefly in the fifth and final chapter.

The first chapter is introductory, and explains the intended interpretation of the symbols by means of examples. The second chapter presents the lambda-calculus as a formal system; and a very simple formal system it is. There are three kinds of symbols (the symbol λ, parentheses, and letters for variables), one undefined operation called *application* and denoted by juxtaposition, and three rules of procedure called the rules of lambda-conversion. There are no axioms and no theorems, the subject being developed metatheoretically. (This is a matter of convenience; it would be possible to introduce axioms in terms of a symbol for interconvertibility of formulas, if that were considered desirable.) Chapter II also deals with the important notion of *normal form*, and contains the proof of the Church-Rosser consistency theorem. The author intends only those formulas which have a normal form to be meaningful, although there exist quite simple formulas with no normal form and hence with no interpretation. It has been shown that no effective procedure exists for determining in all cases whether a given formula has a normal form. This fact is related to the incompleteness property which, according to the results of Gödel, any system which is consistent and adequate for formal logic must have.

In the third chapter the positive integers are defined in terms of the lambda symbolism. This is possible because the system contains formulas which, applied to a function, have the effect of iterating it a given number of times. The familiar operations of addition, multiplication, and exponentiation of integers are also defined. It is shown that an integral-valued function of positive integers is general recursive if and only if it is lambda-definable, that is, definable in terms of the lambda symbolism. Reasons are given for identifying the notion of an effectively calculable function of positive integers with that of a lambda-definable function, or equivalently of a general recursive function.

Chapter four deals with Gödel numbers. It is convenient, in formal systems which contain a notation for the integers, to develop a method of numbering the formulas of the system which is definable in terms of the system itself. This enables the system to talk about itself, roughly speaking, and sometimes, indeed, even to contradict itself (but that cannot happen here). Since it is simpler to assign the same Gödel number to two formulas if they differ only in the letters used to represent bound variables, the author first assigns to every well-formed formula a special kind of formula called a *combination*, and then enumerates the combinations. Formulas which differ only in the symbols for bound variables are assigned the same combination. In connection with combinations, the close connection between the lambda-calculi and the combinatory logic of M. Schönfinkel and H. B. Curry is brought out.

The restricted calculus of lambda-conversion which is treated in the first four chapters is not an adequate basis for logic or mathematics. In particular, constant functions cannot be defined because of the rule that $\lambda x M$ (which is interpreted to mean the function that M is of x) is not well-formed unless x occurs as a free variable in M. The fifth chapter deals briefly with modified systems which avoid the objection mentioned (the lack of constant functions), but which give rise to new difficulties, as the author points out. Of these, the calculi of λ-K-conversion and of restricted λ-K-conversion arise by changing the definition of well-formed formula to allow more well-formed formulas. The λ-δ-calculus involves the introduction of a new symbol δ which behaves somewhat like a sign of equality.

Finally the author outlines his method of defining within the λ-δ-calculus a system of symbolic logic. This is accomplished by identifying the truth values *truth* and *falsity* with the formulas for the integers 2 and 1 respectively. The provable formulas of the logic are then those formulas of the calculus which are convertible to the formula for the integer 2, that is to the formula for truth, according to the rules of λ-δ-conversion. Operations of negation, conjunction, and disjunction are defined, as well as an existential quantifier and a selection operator. All these have properties differing from those of the corresponding operations of classical logic, and resembling somewhat those of the intuitionist logic. It is indicated that there are difficulties involved in defining a universal quantifier. Although this logic seems peculiar in some ways, it has the advantage over certain other systems that there exists for it an elementary consistency proof, namely the analogue for the λ-δ-calculus of the Church-Rosser theorem. The further elaboration of this system of logic is left for another book.

In this book Church has done a first-rate job of exposition, which should be examined by everyone interested in the foundations of mathematics. Not that it is all easy reading; that would be impossible in the nature of the subject. The fundamental notions of the lambda-calculus have what seems to be a simple interpretation, but this interpretation does not seem to help one to follow the details. Perhaps this is partly because of the long arrays of symbols required to define most of the important formulas. Verification that the formulas have the desired properties must proceed formally according to the conversion rules, and the work is often tedious.

Two interesting points concerning these calculi may be noted. One is that arithmetic is developed first, and logical notions are then defined in terms of arithmetic, whereas in most such formal systems, like the Whitehead-Russell system, logical notions come first and arithmetic is defined in terms of logic. The other point is that these calculi seem to support the views of the effectivists (including the intuitionists) who would exclude from mathematics the axiom of choice and in general those concepts which are not constructively definable. In particular the prominent role played here by the integers is suggestive of the viewpoint of Kronecker. However, it should be said that although the lambda-calculi make it possible to state more exactly what is meant by an effective definition, and to arrive at interesting results concerning the concept of effectiveness, one can use the methods of this book without committing oneself to the effectivist position. Presumably it is quite possible to erect a non-effectivist mathematics on the basis of these systems.

Church has suggested elsewhere that his lambda notation might be useful in avoiding the ambiguity which results (especially in dealing with function spaces) from the bad habit of using the notation $f(x)$ to stand sometimes for a function as a single entity, and sometimes for the value of the function when the argument is x. It is interesting to note that Whitehead and Russell use the notation $f(\hat{x})$ for the first of these concepts in place of Church's $\lambda x \cdot f(x)$ or $\lambda x(fx)$, and that our elementary textbooks at least provide for the distinction by the use of different letters for the argument; thus $f(x)$ stands for a function, while $f(a)$ or $f(x_0)$ stands for a functional value. Unfortunately the distinction is not always made. It should of course be said that Church's lambda notation is greatly superior in some ways to these other notations, and allows the construction of formulas not representable in them.

ORRIN FRINK, JR.

NOTES

The supply of certain issues of early volumes of the BULLETIN is almost exhausted. Any member who has copies of the particular issues listed below which he would be willing to contribute or sell is invited to communicate with the office of the Society at 531 West 116th Street, New York City, 27. Bulletin of the New York Mathematical Society, vol. 2, no. 3; Bulletin of the American Mathematical Society, vol. 1, nos. 3, 5; vol. 2, nos. 2, 8; vol. 3, no. 3; vol. 4, nos. 8, 9, 10; vol. 5, no. 6; vol. 8, nos. 6, 8, 9, 10; vol. 9, nos. 1, 3; vol. 10, nos. 1, 3, 4.

The BULLETIN has received from Professor A. Terracini and Professor F. Cernuschi, editors of Matematicas y Fisica Teorica, published in Tucuman, Argentina, an invitation for mathematical contributions. Accepted papers will be published in the language in which they are written.

The Sylvester Medal of the Royal Society has been awarded to Professor J. E. Littlewood of the University of Cambridge.

Dr. Vannevar Bush, President of the Carnegie Institution of Washington, has been awarded the Edison Medal for 1943 by the American Institute of Electrical Engineers.

Professor J. L. Synge of Ohio State University has been awarded the Henry Marshall Tory Medal of the Royal Society of Canada.

President H. N. Davis of the Stevens Institute of Technology has been elected an honorary member of the Institution of Mechanical Engineers of Great Britain.

Professor E. T. Whittaker has been elected president of the Royal Society of Edinburgh.

Professor Helen M. Walker of Teachers College, Columbia University, has been elected president of the American Statistical Association.

Professor J. L. Walsh of Harvard University has been elected vice president of the American Association for the Advancement of Science.

Assistant Professor A. F. Bernhart of Bucknell University has been appointed to an assistant professorship at the University of Oklahoma.

Professor G. D. Birkhoff of Harvard University is on leave of absence for the present winter term in order to give a series of lectures at the University of Mexico.

Assistant Professor M. G. Boyce of Western Reserve University has been promoted to an associate professorship.

Assistant Professor A. T. Brauer of the University of North Carolina has been promoted to an associate professorship.

Assistant Professor H. A. Dangel of the University of Cincinnati has been promoted to an associate professorship.

Miss Flora Dinkines of the University of South Carolina has been promoted to an adjunct professorship.

Mr. Edwin Eagle of Taft Junior College, Taft, California, has been appointed to an assistant professorship at Oregon State College.

Associate Professor Paul Eberhart of Washburn University, Topeka, Kansas, has been promoted to a professorship.

Mr. W. E. Ekman of the University of South Dakota has been promoted to a professorship.

Dr. W. S. Erickson of St. Olaf College, Northfield, Minnesota, has been appointed to a professorship at Minot State Teachers College, Minot, North Dakota.

Mr. M. P. Fobes of the College of Wooster, Wooster, Ohio, has been promoted to an assistant professorship.

Mr. W. A. Gager of the Junior College in St. Petersburg, Florida, has been appointed to an associate professorship at the University of Florida.

Dr. M. C. Hartley has been appointed to an assistant professorship at the University of Illinois.

Associate Professor H. H. Hartzler of Goshen College, Goshen, Indiana, has been promoted to a professorship.

Assistant Professor R. N. Haskell of the University of Texas has been promoted to an associate professorship.

Assistant Professor P. R. Hill of the University of Georgia has been promoted to an associate professorship.

Associate Professor J. W. Hurst of Montana State College has been promoted to a professorship.

Associate Professor J. A. Hyden of Vanderbilt University has been promoted to a professorship.

Assistant Professor F. B. Jones of the University of Texas has been promoted to an associate professorship.

Assistant Professor A. H. Knebel of the University of Cincinnati has been promoted to an associate professorship.

Dr. Fulton Koehler of the Northwest National Life Insurance Company has been appointed to an assistant professorship at the University of Minnesota.

Mr. J. E. LaFon of the University of Oklahoma has been promoted to an assistant professorship.

Assistant Professor R. G. Lubben of the University of Texas has been promoted to an associate professorship.

Dr. C. A. Ludeke of the University of Cincinnati has been promoted to an assistant professorship.

Mr. J. N. McClelland of Drake University has been promoted to an assistant professorship.

Dr. J. C. C. McKinsey of New York University has been appointed to an assistant professorship at Montana State College.

Assistant Professor W. T. MacCreadie of Bucknell University has been promoted to an associate professorship.

Professor A. E. Meder, Jr., of the New Jersey College for Women has been appointed secretary of Rutgers University.

Associate Professor C. E. Melville of Clark University has been promoted to a professorship.

Professor H. A. Meyer of Hanover College, Hanover, Indiana, has been appointed acting professor at Indiana University.

Assistant Professor R. R. Middlemiss of Washington University has been promoted to an associate professorship.

Mr. E. T. Miller of the University of Cincinnati has been promoted to an assistant professorship.

Associate Professor H. L. Miller of the University of Cincinnati has been promoted to a professorship.

Professor F. D. Murnaghan of Johns Hopkins University has been appointed to a visiting professorship at Brown University for the current year.

Associate Professor W. G. Pollard of the University of Tennessee has been promoted to a professorship.

Mr. W. E. Restemeyer of the University of Cincinnati has been promoted to an assistant professorship.

Mr. H. L. Rice of the University of Omaha has been promoted to an assistant professorship.

Assistant Professor E. B. Roessler of the College of Agriculture of the University of California has been promoted to an associate professorship.

Associate Professor J. B. Rosser of Cornell University has been promoted to a professorship.

Miss S. Grace Smyth of Knox College has been promoted to an associate professorship.

Mr. A. L. Starrett of Georgia School of Technology has been promoted to an assistant professorship.

Associate Professor W. R. Talbot of Lincoln University, Jefferson City, Missouri, has been promoted to a professorship.

Dr. H. C. Trimble of Iowa State Teachers College has been promoted to an assistant professorship.

Assistant Professor P. L. Trump of the University of Wisconsin has been promoted to an associate professorship in the teaching of mathematics.

Mr. R. N. Van Arnam of Lehigh University has been promoted to an assistant professorship.

Dr. G. L. Walker of the University of Delaware has been promoted to an assistant professorship.

Miss Fern Welker of the University of Toledo has been promoted to an assistant professorship.

Dr. Y. K. Wong of the University of Chicago has been appointed lecturer at the University of North Carolina.

The following individuals are assisting in the military training program at the University of Chicago: Professors J. M. Kinney and E. R. Breslich, Miss Evelyn R. Garbe, Messrs. Herman Meyer, Daniel Zelinsky, H. J. Zimmerberg, L. A. Blatz, Charles Brumfiel, Morris Friedman, Miss Marjorie Groves, Messrs. A. R. Jacoby and Charles Nichols.

The following appointments to instructorships are announced: University of Cincinnati: Mr. Robert Fopma; DePaul University: Mr. Paul D'Arco; Greenbrier Junior College, Lewisburg, W. Va.: Miss Margaret R. Davis; Heidelberg College, Tiffin, Ohio: Dr. Ruth G. Simond; Kansas State College: Mr. D. K. Brooks, Mr. L. E. Milleson, Mr. W. V. Unruh; University of Kentucky: Mrs. J. C. Lamb, Mrs. C. G. Latimer; Lehigh University: Mr. R. C. King, Mr. S. W. Smith; Michigan State College: Mr. Nicholas Musselman; Ohio State University: Mr. L. H. Miller; University of Oklahoma: Miss Phyllis Barclay, Mrs. Dorothea A. Sudduth; Oregon State College: Miss Florence Bakkum, Mr. L. R. Foote, Mr. Fred Young; Pennsylvania State College: Mr. J. R. Kinney, Mrs. Nellie M. Krall; University of Pennsylvania: Mr. R. C. Cambell; Purdue University: Mr. H. M. Anderson, Mr. F. C. Leone; Rice Institute: Mr. G. R. MacLane; Stanford University: Miss Marjorie L. Hoffman, Miss Mary V. Sunseri; Yale University: Mr. J. S. Blair, Mr. J. C. Caughlan, Dr. W. C. G. Fraser, Mr. C. R. Kossack, Mr. J. R. Lee, Mr. P. E. Poe, Mr. James Smillie.

The death in 1941 of Professor H. L. Lebesgue of the Collège de France is now reported.

Dean John Matheson of Queen's University died January 24, 1944, at the age of seventy-one years. He had been a member of the Society since 1918.

Professor Emeritus H. H. Dalaker of the University of Minnesota died May 20, 1943. He had been a member of the Society since 1909.

Professor Emeritus T. S. Fiske of Columbia University died January 10, 1944. Further notice of his death will appear in a forthcoming issue of the BULLETIN.

Dr. Edward Helly of the Illinois Institute of Technology died November 28, 1943.

Mr. H. L. Mintzer died November 24, 1943.

Associate Professor G. A. Pfeiffer of Columbia University died December 28, 1943. He had been a member of the Society since 1914.

Professor H. L. Rietz of the State University of Iowa died December 7, 1943. He had been a member of the Society since 1902.

ABSTRACTS OF PAPERS

SUBMITTED FOR PRESENTATION TO THE SOCIETY

The following papers have been submitted to the Secretary and the Associate Secretaries of the Society for presentation at meetings of the Society. They are numbered serially throughout this volume. Cross references to them in the reports of the meetings will give the number of this volume, the number of this issue, and the serial number of the abstract.

ALGEBRA AND THEORY OF NUMBERS

53. A. W. Jones: *The lattice isomorphisms of certain finite groups.*

Let G be any finite group whose lattice of subgroups is modular. The author gives an explicit procedure for obtaining defining relations for all groups whose lattices are isomorphic to the lattice of G. Since abelian groups have lattices which are modular, the procedure given includes a characterization, for the finite case, of the groups which are lattice isomorphic to abelian groups (cf. Baer, Amer. J. Math. vol. 61 (1939) pp. 1–44). (Received January 28, 1944.)

54. Oystein Ore: *Galois connexions.*

In numerous mathematical theories there occur order inverting correspondences between two structures P and Q, $p\rightarrow Q(p)$, $q\rightarrow P(q)$, such that $PQ(p)\supset p$, $QP(q)\supset q$. The basic properties of such Galois correspondences are derived. They correspond to a duality between structures or closure relations and can also be considered to be mappings of closure relations. They are closely connected with the theory of binary relations as discussed in the author's Colloquium Lectures given in Chicago in 1941. Among the applications should be mentioned a general Galois theory for binary relations, illustrated in detail for the case of equivalence relations. (Received December 7, 1943.)

55. Raphael Salem: *On a remarkable class of algebraic integers. Proof of a conjecture of Vijayaraghavan.*

Let us denote by C the set of all algebraic integers such that all their conjugates have moduli inferior to 1 ("Pisot-Vijayaraghavan numbers" or briefly "P. V. numbers"). It is proved that the set C is closed. C being enumerable it follows that it is (1) nowhere dense; (2) not dense in itself; (3) reducible. There exists a number larger than 1 which is the smallest "P. V. number." (Received December 29, 1943.)

56. P. M. Whitman: *Identities in lattices of equivalence relations.* Preliminary report.

It is shown that free lattices can be represented by equivalence relations. Therefore no lattice identity can hold in every lattice which consists of all equivalence relations on some set, unless it holds in every lattice. (Received December 27, 1943.)

57. P. M. Whitman: *Lattices and equivalence relations.* Preliminary report.

It is shown that any lattice is isomorphic to a sublattice of the lattice of all equivalence relations on some set. (Received January 28, 1944.)

ANALYSIS

58. Jesse Douglas: *Separable transformations with separable inverse.*

All transformations $X=f(x)+h(y)$, $Y=g(x)+k(y)$ are found whose inverses are of the same form. Six essentially different types are obtained. If x, y are interpreted as minimal coordinates $u+iv$, $u-iv$ (and X, Y similarly), we have all harmonic transformations whose inverses are harmonic. The paper will be published in full. (Received January 15, 1944.)

59. K. O. Friedrichs: *The identity of weak and strong extensions of differential operators.*

In applying the theory of linear operators in Hilbert spaces or spaces L_p to the solution of differential equation problems, it is impossible to retain the meaning of differentiation in the ordinary sense; the concept of differential operator must be extended. Two such extensions offer themselves, a "weak" and a "strong" one, that is, the adjoint of the "formal-adjoint" and essentially the closure. The purpose of the paper is to prove the identity of these two extensions for general linear differential operators. The main tool for the proof is a certain class of smoothening operators approximating unity. They yield the identity of both extensions immediately for differential operators with constant coefficients; they are a strong enough tool to yield this identity likewise for operators with non-constant coefficients. (Received December 3, 1943.)

60. B. M. Ingersoll: *On singularities of solutions of linear partial differential equations.*

Let $U(z, \bar{z})$, $z=x+iy$, $\bar{z}=x-iy$, x, y real, be a real solution of $L(U)=\Delta U+AU_x+BU_y+CU=0$, where A, B, and C are entire functions when x and y are extended to complex values. To every such solution corresponds uniquely a complex solution $u(z, \bar{z})=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}A_{mn}z^m\bar{z}^n$ of $L(U)=0$, with the property that $\sum_{n=0}^{\infty}A_{0n}\bar{z}^n=\pi U(0, 0)$ $\exp\left(-\int_0^{\bar{z}} a(0, \bar{z})d\bar{z}\right)$, where $a(z, \bar{z})=(1/4)\{A[((z+\bar{z})/2, (z-\bar{z})/2i)]+iB[((z+\bar{z})/2, (z-\bar{z})/2i)]\}$. These solutions were introduced by Bergman (Rec. Math. (Mat. Sbornik) N. S. vol. 2 pp. 1169–1198 and Trans. Amer. Math. Soc. vol. 53 pp. 130–155) who showed that the location of the singularities of $u(z, \bar{z})$ is determined by the sequence $\{A_{m0}\}$. Employing this result the author investigates the relations between sequences $\{A_{mk}\}$, k fixed, $m=0, 1, 2, \cdots$, and the positions of singularities of $u(z, \bar{z})$. For example, using a result of Mandelbrojt (C. R. Acad. Sci. Paris, 1937, pp. 1456–1458) he determines the arguments of the singularities on the circle of convergence of $u(z, \bar{z})$ in terms of the sequence $\{A_{mk}\}$, k fixed. In the last section of the paper, using explicitly an integral representation of the complex solutions $u(z, \bar{z})$, the author investigates the real solutions $U(z, \bar{z})=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}D_{mn}z^m\bar{z}^n$ of $L(U)=0$. He constructs, in terms of $\{D_{mk}\}$, k fixed, $m=0, 1, 2, \cdots$, and some of the derivatives of the coefficients of L, an analytic function $f(z/2)$ whose singularities coincide with those of $U(z, \bar{z})$. (Received January 28, 1944.)

61. Mark Kac: *On real zeros of some functions.*

The purpose of this note is to call attention to some applications of the formula for the number of real roots given recently by the author (*On the distribution of values of trigonometric sums with linearly independent frequencies*, Amer. J. Math. vol. 65 (1943) pp. 609–615, in particular §3). If $f(x)$ has a continuous first derivative and a finite number of turning points in (a, b) then the number of zeros of $f(x)$ in (a, b) is given by the formula $(\pi/2)\int_{-\infty}^{+\infty}\{\int_a^b \cos(\xi f(x))|f'(x)|dx\}d\xi$, with the understanding that multiple roots are counted only once and that end points (a and b) if they happen to be roots are counted as 1/2 each. This formula is a source of amusing identities. For instance if $f(x)=\cos x$, $a=0$, $b=2\pi$ one obtains easily that $4\pi^2=\int_{-\infty}^{\infty}\int_{-\infty}^{+\infty}((J_0(\xi)-J_0(\xi^2+\eta^2)^{1/2})/\eta^2)d\xi d\eta$. Also this formula is very convenient in computing average numbers of real roots of certain random functions. (Received January 31, 1944.)

62. A. T. Lonseth: *Dirichlet's principle for* $\Delta u = F_u(u; x, y)$.

This nonlinear elliptic partial differential equation (Δ = Laplacean operator) is the Euler-Lagrange equation for the minimization of $\iint\{u_x^2+u_y^2+2F(u; x, y)\}dx\,dy$. Here the simultaneous boundary-value and minimum problem solution is obtained by modifying Courant's method for self-adjoint linear elliptic equations (R. Courant and D. Hilbert, *Methoden der mathematischen Physik*, vol. II, Berlin, 1937, chap. 7). It is assumed that $F(u; x, y)$ is of class C'' and convex in u, and of class C' in x and y. Assumptions as to region and boundary values are as in Courant-Hilbert. (Received January 17, 1944.)

63. Szolem Mandelbrojt: *Some theorems connected with the theory of infinitely differentiable functions.*

The author gives a simple and elementary proof of the necessity of a well known necessary and sufficient condition for quasi analyticity, and an immediate proof of a theorem concerning Watson's problem. The author proves also an inverse to a theorem of S. Bernstein on the best approximation on the whole real axis. (Received January 27, 1944.)

64. H. E. Newell: *The solutions of a certain linear matric differential equation.*

The linear matric differential equation $dY(x, \lambda)/dx=\{\lambda(\delta_{ij}r_j(x))+(q_{ij}(x, \lambda))\}\cdot Y(x, \lambda)$, x and λ being complex, has been shown to possess under certain conditions solutions of the form $P(x, \lambda)E(x, \lambda)$, where $E(x, \lambda)=(\delta_{ij}\exp\{\lambda\int^x r_j(x)dx\})$ and $P(x, \lambda)$, analytic in x, reduces uniformly in x to the identity matrix when λ becomes infinite. The theory, originally developed by R. E. Langer (R. E. Langer, *The boundary problem of an ordinary linear differential system in the complex domain*, Trans. Amer. Math. Soc. vol. 46 (1939) pp. 151–162; correction, p. 467) and later extended by the author (H. E. Newell, Jr., *The asymptotic forms of the solutions of an ordinary linear matric differential equation in the complex domain*, Duke Math. J. vol. 9 (1942) pp. 245–258, and vol. 10 (1943) pp. 705–709), was applied for the most part to cases in which the elements $q_{ij}(x, \lambda)$ were analytic and bounded. The present paper treats of certain special cases in which the functions $q_{ij}(x, \lambda)$ may have poles on the boundary of the x region in question. (Received January 21, 1944.)

65. A. M. Peiser: *Some applications of Fourier analysis to the study of real roots of algebraic equations.*

Let $N_n(x) = N_n(x_0, \cdots, x_n)$ denote the number of real roots of $F = \sum_{j=0}^{n} x_j t^j = 0$. It is shown that at each continuity point of N_n, $N_n(x) = \lim_{h \to \infty} (h/(2\pi)^{1/2}) \int_{-\infty}^{\infty} A(t) \cdot | Y_n(x, t) | \exp [-h^2 X_n^2(x, t)/2] dt$ where $A(t)$ is a known positive function of t alone, and X_n and Y_n are orthogonal, normal linear forms in $x_0, \cdots, x_n$. Thus, if the x's are independent, normally distributed random variables, then X_n and Y_n are normally distributed and, because of the orthogonality, independent. For the average number (mathematical expectation = m.e.) of real roots of $F=0$ in this case we find m.e.$\{N_n(x)\} = \lim_{h \to \infty} (h/(2\pi)^{1/2}) \int_{-\infty}^{\infty} A(t)$m.e.$\{ | Y_n(x,t) | \}$m.e.$\{\exp [-h^2 X_n^2(x, t)/2]\} dt$. Suppose now that m.e. $\{x_j\} = c_j$ and that for each j, m.e. $\{x_j^2\} = s$. The author compares the number of real roots $N_n(c)$ of $F_1 = \sum_{j=0}^{n} c_j t^j = 0$ with the number of real roots $N_n(c+sx)$ of $F_2 = \sum_{j=0}^{n} (c_j + s x_j) t^j = 0$, and shows how to choose s so that m.e.$\{N_n(c+sx)\}$ is near $N_n(c)$. This may have an application to empirical equations, for we can consider $F_1 = 0$ as the theoretical, or "true" equation, and $F_2 = 0$ as an "observed" equation in which each coefficient is measured with precision s. (Received January 28, 1944.)

66. Harry Pollard: *Fourier series with coefficients in a Banach space.*

Let $f(t)$ be a function on (0, 1) to the complex Banach space B. Bochner has shown that a large part of the older theory of Fourier series carries over to functions of this character, but breaks down in the fundamental L^2 theory. In particular Bessel's inequality is no longer valid. It is proved in this paper that the inequality carries over if and only if B admits a scalar product. (Received January 24, 1944.)

67. C. E. Rickart: *Representation of linear transformations on summable functions.*

This paper contains an integral representation theorem for the general bounded linear transformation on summable functions to an arbitrary Banach space. The representation is obtained by means of an abstract Radon-Nikodym theorem proved earlier by the author (see abstract 49-11-270). (Received January 26, 1944.)

68. H. E. Robbins: *A note on the Riemann integral.*

Let $f(x)$ be a continuous function. If, in the usual definition of the Riemann integral of $f(x)$ from a to b as the limit of S_n equals the sum of terms of the form $f(\xi_i)(x_i - x_{i-1})$, the points of subdivision, $x_0 = a, x_1, \cdots, x_n = b$, are *not* assumed to have the property $x_i \geqq x_{i-1}$, then S_n need not tend to a limit as $d_n = \max | x_i - x_{i-1} |$ tends to 0. But if any constant $C \geqq (b-a)$ is given in advance and if the points of every subdivision satisfy the inequality, sum of $| x_i - x_{i-1} | \leqq C$, then S_n will tend to the Riemann integral of $f(x)$ as d_n tends to 0. (Received January 28, 1944.)

69. F. H. Safford: *Analysis of a non-harmonic wave.*

The present paper considers the problem of expressing a periodic function by means of sines and cosines up to p harmonics, when the values of the function are given as ordinates for equidistant abscissas, with no further information. This is a variation of the method of selected ordinates used by J. Fischer-Hinnen in 1901, but with p coefficients for the sine terms and $p+1$ coefficients for the cosine terms, corresponding terms being combined into the form $A \sin n(t+\theta)$. The period may be taken as 2π, and the values of n are from 0 to p. It is necessary to use a second set of ordinates at the distance $(\pi/2p)$ from the first set thus providing $2p+1$ disposable constants corresponding to the same number of points from the observed function. The presence of higher harmonics with composite indices is duly provided for. The theorem concerning the sums of equidistant ordinates of sine curves provides the components of the observed ordinates and the latter are linearly related to the disposable constants. Thus for each harmonic it is possible to obtain its amplitude and phase angle without trigonometric ambiguity, in terms of the observed values of the given ordinates. (Received January 24, 1944.)

70. Raphael Salem: *On a theorem of Bohr and Pál.*

The paper gives a short and simple proof of the following extension of a theorem of Pál, due to H. Bohr: given a function $\phi(t)$ continuous and of period 2π, there exists a function $t(\theta)$ $(t(0)=0,\ t(2\pi)=2\pi)$, continuous, strictly increasing, such that the Fourier series of $\phi(t(\theta))$ converges uniformly for $0 \leqq \theta \leqq 2\pi$. (Received December 29, 1943.)

71. Raphael Salem: *Sets of uniqueness and sets of multiplicity.* II.

This paper is a continuation of a paper which appeared under the same title in Trans. Amer. Math. Soc. vol. 54 (1943) pp. 218–228. It gives the necessary and sufficient condition for an unsymmetric perfect set of constant ratio to be a set of uniqueness. It deals also with some aspects of the classification in sets of uniqueness and sets of multiplicity of symmetrical perfect sets of the Cantor type and of variable ratio of dissection. (Received December 29, 1943.)

72. A. R. Schweitzer: *On functional equations with solutions containing arbitrary functions.* IV.

Whenever an iterative compositional functional equation has a solution containing an arbitrary function in two or more variables, the possibility of an associated representation of substitution groups by functional equations may be entertained. A simple example is $f(u, x_2, \cdots, x_n, v)=f(x_1, x_2, \cdots, x_{n+1})$ where $u=f(x_1, t_1, t_2, \cdots, t_n)$ and $v=f(x_{n+1}, t_1, t_2, \cdots, t_n)(n=1, 2, \cdots)$ with solution: $f(x_1, x_2, \cdots, x_{n+1})=x_1-x_{n+1}+\alpha(x_2, \cdots, x_n)$. A group of equations follows if $f(x_1, x_2, \cdots, x_{n+1})=f(x_1, x_{i_2}, \cdots, x_{i_n}, x_{n+1})$. In this and certain analogous representations of a substitution group G, the equation corresponding to the identical substitution has a solution containing an arbitrary function and the corresponding group of equations has a solution based on the invariance of the latter function under G. A generalized associative equation as inverse of the above equation corresponds to the identical element in the group of equations: $\phi\{\phi(x_1, x_2, \cdots, x_n, y_1), y_2, \cdots, y_n, y_{n+1}\}=\phi\{x_1, x_2, \cdots, x_n, \phi(y_1, y_{i_2}, \cdots, y_{i_n}, y_{n+1})\}$. Another example is based on $f(u, u, \cdots, u)=u$, where $u=f(x_1, x_2, \cdots, x_{n+1})$ with solution: $f(x_1, x_2, \cdots, x_{n+1})=\alpha(w_1, w_2, \cdots, w_n)+x_{n+1}$ where $w_i=x_i-x_{n+1}$ and α is arbitrary except for $\alpha(0, 0, \cdots, 0)=0$. (Received January 24, 1944.)

73. A. R. Schweitzer: *On functional equations with solutions containing arbitrary functions.* V.

The quasi-transitive equation, $f(u_1, u_2, \cdots, u_{n+1})=f(x_1, x_2, \cdots, x_{n+1})$,

$u_i=f(x_i, t_1, t_2, \cdots, t_n)$, is made the basis of an $(n+1)$-adic genesis of the group concept by postulating further: (1) Given $f(x_1, x_2, \cdots, x_{n+1})$, there exists $\phi(x_1, x_2, \cdots, x_{n+1})$ such that $f\{\phi(x, t_1, t_2, \cdots, t_n), t_1, t_2, \cdots, t_n\}=x$ and $\phi\{f(x, t_1, t_2, \cdots, t_n), t_1, t_2, \cdots, t_n\}=x$. (2) If $x\neq y$, then $f(x, x, \cdots, x)=f(y, y, \cdots, y)$. (3) The set S of elements x_i is closed under the compositions f and ϕ. The preceding postulates define a group under binary composition if in the functions $f(x_1, x_2, \cdots, x_{n+1})$ and $\phi(x_1, x_2, \cdots, x_{n+1})$, $x_2=x_3= \cdots =x_{n+1}$. The postulates are also satisfied in the domain of abstract groups under $(n+1)$-ary composition if $f(x_1, x_2, \cdots, x_{n+1}) =x_1\cdot x_{n+1}^{-1}\cdot \alpha(w_2, w_3, \cdots, w_n)$ where $w_i=x_i\cdot x_{n+1}^{-1}$ and α is an arbitrary "function" of group elements. Finally, the postulates are satisfied by a set of elements closed under the binary compositions of a commutative group and any one of a class of specialized operations represented by the function α, including multiplication of arguments. An analogous genesis based on a generalized associative equation is discussed. (Received January 24, 1944.)

74. M. F. Smiley: *An extension of metric distributive lattices with an application in general analysis.*

It is shown that a metric distributive lattice (Garrett Birkhoff, *Lattice theory*, Amer. Math. Soc. Colloquium Publications, vol. 25, 1940, pp. 41 and 74) may be embedded in a field of sets with a finitely additive measure provided that the modular functional of the basic lattice is bounded from below. Under the additional hypothesis that this functional is bounded, it is proved that the corresponding measure is *completely additive*. This result is applied to E. H. Moore's last instance of his second general analysis theory (*General analysis*, Part I, Memoirs of the American Philosophical Society, vol. 1, 1935, pp. 14–15). In this example the positive character of Moore's basic matrix is found to depend essentially on the distributive law. The proof of the major result employs theorems of H. Wallman (*Lattices and topological spaces*, Ann. of Math. (2) vol. 39 (1938) pp. 112–126) and of M. H. Stone (*Applications of the theory of Boolean rings to general topology*, Trans. Amer. Math. Soc. vol. 41 (1937) pp. 376–481) on the representation of Boolean rings, as well as methods of H. M. MacNeille (*Extension of a distributive lattice to a Boolean ring*, Bull. Amer. Math. Soc. vol. 45 (1939) pp. 452–455) and of S. Kakutani (*Concrete representation of abstract (L)-spaces and the mean ergodic theorem*, Ann. of Math. (2) vol. 42 (1941) p. 533). (Received January 10, 1944.)

75. F. T. Wang: *On Riesz summability of Fourier series.* IV.

Let $f(t)$ be an integrable periodic function with period 2π and $\phi_\alpha(t)$ be the fractional integral of $\phi(t)=2^{-1}\{f(x+t)+f(x-t)-2s\}$ of order α. Then we have the following results: (1) If $\phi_\alpha(t)=o(t^\alpha/\log t^{-1})$, $\alpha>0$, as $t\to 0$, then the Fourier series of $f(t)$ is summable by Riesz typical mean of type $\lambda_n=\exp(\log n)^{1+\alpha^{-1}}$ and of order $\alpha+1$, or simply, summable $\exp(\log n)^{1+\alpha^{-1}}$, $\alpha+1$, to sum s at $t=x$. (2) If $\sum_{n=2}^{\infty}(a_n^2+b_n^2)(\log n)^\alpha$ $(0<\alpha<1)$ converges then the series $\sum_{n=1}^{\infty}(a_n \cos nx+b_n \sin nx)$ is summable $\exp(\log n)^{\alpha+1}$, δ, $\delta>0$, almost everywhere. (Received January 5, 1944.)

76. F. T. Wang: *Some remarks on oscillating series.*

Let $\sigma_n^{(r)}$ be the rth Cesàro mean of the series $\sum_{n=0}^{\infty}a_n$ where r is a positive integer. Then we have the following result: If $\sigma_n^{(r)}-s=o(n^{-r\alpha})$, $0<\alpha<1$, as $n\to\infty$ and $a_n>-Kn^{\alpha-1}$, then the series $\sum_{n=0}^{\infty}a_n$ converges to sum s. For the case $r=1$ this has been proved by Boas. The author proves in this note the result for $r=2$, and gives an example from the theory of Fourier series that there exists a divergent series $\sum_{n=0} a_n$ such that $\sigma_n^{(2)} - s = 0(n^{-2\alpha'})$, $0 < \alpha' < \alpha < 1$ and $a_n = O(n^{\alpha-1})$. (Received January 5, 1944.)

77. D. V. Widder: *The iterates of the Laplace kernel.*

The iterates of the Laplace kernel $G_0(x, y) = e^{-xy}$ are defined by the recursion relation $G_n(x, y) = \int_0^\infty G_0(x, t) G_{n-1}(t, y) dt$, $n = 1, 2, \cdots$. In an earlier paper (Bull. Amer. Soc. vol. 43 p. 813) the author determined explicitly all these functions which have odd subscripts. They are rational functions of x, y and $\log (x/y)$. In the present paper the iterates with even subscripts are studied. They cannot be expressed in terms of the elementary functions. Complete asymptotic series for their behavior near $x = +\infty$ are obtained. For their behavior near the origin they are developed in power series which converge for all positive values of x. The method consists in expressing the nth iterate in terms of a function $h_n(x)$ whose bilateral Laplace transform turns out to be $\Gamma(-s)^{n+1}\,\Gamma(s+1)^n$. The function $h_n(x)$ is in turn expressible in terms of a new set of transcendental functions related to the familiar exponential integral $EI(x)$. (Received January 14, 1944.)

APPLIED MATHEMATICS

78. Stefan Bergman: *On solutions of certain partial differential equations in three variables.*

(I) Let $\mathrm{E}(x, y, z, t, \tau)$ be a solution of the equation $\mathrm{G}(\mathrm{E}) = \mathrm{L}(\mathrm{E}) + [\tau^{-1} U^{-1}(1-\tau^2)(\mathrm{E}_x + i\mathrm{E}_y \cos t + i\mathrm{E}_z \sin t)_\tau] + A[(1/2)\tau^{-1}(1-\tau^2)\mathrm{E}]_\tau$ which satisfies certain boundary conditions. Here $\mathrm{L}(\mathrm{E}) = \Delta\mathrm{E} + A(x\mathrm{E}_x + y\mathrm{E}_y + z\mathrm{E}_z) + C\mathrm{E}$, $U = x + iy \cos t + iz \sin t$. Then $\psi(x, y, z) = \int_0^{2\pi}\int_{-1}^{1} \mathrm{E}f[(1/2)u(1-\tau^2), t]/dt d\tau$, where $f(\zeta, t)$ is an arbitrary analytic function of ζ and t, will be a solution of $\mathrm{L}(\psi) = 0$. (II) If A and C are entire functions of $r^2 = x^2 + y^2 + z^2$ alone, then $\mathrm{G}(\mathrm{E})$ becomes $\mathrm{G}_0(\mathrm{E}) = (1-\tau^2)\mathrm{E}_{\tau\tau} - \tau^{-1}(1+\tau^2)\mathrm{E}_\tau + r\tau[\mathrm{E}_{rr} + 2r^{-1}\mathrm{E}_r + B\mathrm{E}] = 0$ where $B = [-(3/2)A - (1/2)rA_r + C - (1/4)r^2A^2]$. The author shows that in the case II, there always exists a solution $\mathrm{E} = \mathrm{H}(r, \tau)$ of $\mathrm{G}_0(\mathrm{E}) = 0$, which is an entire function of r. The author considers vectors $S = (\psi_1, \psi_2, \psi_3)$ where $\psi_1 = \int_0^{2\pi}\int_{-1}^{1} \mathrm{H}f dt d\tau$, $\psi_2 = i\int_0^{2\pi}\int_{-1}^{1} \mathrm{H}f \cos t dt d\tau$, $\psi_3 = i\int_0^{2\pi}\int_{-1}^{1} \mathrm{H}f \sin t dt d\tau$. Clearly $\mathrm{L}(\psi_K) = 0$, $K = 1, 2, 3$. Let c^1 be a simple closed curve which lies on a sphere $x^2 + y^2 + z^2 = \text{const.}$ If the ψ_K are regular in this sphere then $\int_{c^1} S \cdot dX = 0$. Here $X = (x, y, z)$, and $\cdot$ means the interior product. "Residue" theorems are derived if the ψ_K have singularities in the above sphere. Applications in the theory of waves propagation are indicated. (Received January 28, 1944.)

79. Nathaniel Coburn: *A boundary value problem in plane plasticity for the Coulomb yield condition.*

The following problem is studied: given a semi-plane $x > 0$, composed of plastic material which follows the Coulomb yield condition; the stresses σ_x, σ_y, σ_{xy} acting on the boundary $x = 0$ are considered as known; to find the stresses at any point in the interior of the semi-plane. The method of attack is a modification of that used in studying a similar problem for a perfectly plastic material. The stresses and the functions $\sin 2\gamma$, $\cos 2\gamma$ (where γ is the angle between the x-axis and a tangent to a line of principal shearing stress) are expanded in power series of the friction coefficient. Substituting these power series into the Levy equations, there results an infinite set of Levy equations for the various approximations to the stresses. By requiring that each set of approximations satisfy the equilibrium relations, an infinite sequence of Airy stress functions is obtained. Since each Airy stress function must satisfy two independent Levy equations, there results an infinite set of second order linear partial differential equations. The various approximations are determined and the convergence of σ_{xy} is demonstrated for the case $\sigma_{xy}(0, y) = 0$. (Received December 20, 1943.)

80. H. L. Garabedian: *The analogue of Bromwich's theorem for integral transformations.*

The main theorem of this paper provides sufficient conditions in order that a method of summation defined by an integral transformation of the type $z(x) = \int_0^\infty k(x, t)a(t)dt$ shall include Cesàro summability of positive integral order n, defined by the transformation $y(x) = \int_0^x (1-t/x)^n a(t)dt$. A kernel $k(x, t)$ is exhibited which fulfills the conditions of this theorem and thus provides a means cf solving in a rigorous fashion the problem of the flow of heat in an infinite rod whose surface is a non-conductor. (Received December 20, 1943.)

81. Michael Goldberg: *A three-space analog of a plane Kempe linkage.*

The following theorem is due to A. B. Kempe: Each of the sides of a plane quadrilateral linkage may be joined by a link to a common pivot without restricting the movability of the linkage. The new pivot associated with a side need not be collinear with the other pivots of that side; the three pivots may form a triangle in a rigid plate. Extensions and further studies on this theorem were made by Kempe (*On conjugate four-bar linkages*, Proc. London Math. Soc. vol. 9 (1878) pp. 133–147) and Darboux (*Recherches sur un système articulé*, Bull. Sci. Math. vol. 3 (1879) pp. 151–192). This paper is concerned with the analogous theorem in three-space. It is shown that it is possible for each of the four bodies of a movable hinged skew quadrilateral (a Bennett linkage) to be joined by a hinged link to a common hinge without restricting the movability. As in the case of the plane, the three hinges in a body need not have a common perpendicular. In general, these hinge lines are non-parallel and non-intersecting. (Received January 29, 1944.)

82. Max Herzberger: *The diapoint characteristic.*

To every point in an optical system with symmetry axis belongs a manifold of *diapoints*, the intersections of the image rays with the meridian plane. If the object point has a sharp image, all the diapoints coincide with it; if the object point has an image with a curved or straight symmetry axis, the diapoints form the axis. The author introduces the diapoint characteristic, a function giving the optical path between point and diapoint. Such a function has a two-dimensional manifold of values in the general case, a one-dimensional manifold in case of symmetry, and is constant if the object point is sharply imaged. The diapoint function $F(x, y, z, v, w)$ can be constructed as a function of the coordinates of the object point (x, y, z) and two parameters v, w. Its analysis for a given system gives at once all the points in space which are sharply or symmetrically imaged, and tells whether there are any surfaces sharply or symmetrically imaged. An example, refraction at a plane and at a sphere, is analyzed. (Received January 24, 1944.)

83. R. P. Isaacs: *Airfoil theory for flows of variable velocity.*

There arise many cases in aircraft practise—conspicuously that of the rotating wing—where the flow over the lift surface is of variable velocity. The first section of this paper deals with theory of an airfoil in a variable speed stream (but constant angle of attack). The result is an integrodifferential relation between the two functions involved: velocity and circulation. Next the case is treated: velocity = constant + sinusoidal variation. Finally the lift is studied and given explicitly for this latter case in the form of a Fourier series. (Received January 24, 1944.)

84. R. v. Mises: *Note on the numerical solution of linear partial differential equations.*

The usual method of numerical solution consists in replacing the differential equation $L(u)=0$ by a set of algebraic linear equations for the values $u(P_i)$ where P_i are the points of a rectangular lattice. This procedure has obvious inconveniences in the case of curved boundaries. Many questions arise if instead of this lattice more general types of point sets are used. The present paper deals with some of these questions, in particular with the number and the arrangement of points required for any order of approximation and with the possibility of applying the method of successive approximations (relaxation method) to the solution of the algebraic equations. Specified formulae are proposed for the case of the Laplace equation. (Received January 28, 1944.)

85. David Moskovitz: *Numerical solution of Laplace's and Poisson's equations.*

This paper considers the system of equations (1) $L_c u_j(i) - u_{j+1}(i) - u_{j-1}(i) = \phi_j(i)$ $(i=1, 2, \cdots, n-1; j=1, 2, \cdots, m-1)$, where L_c is the linear operator defined by $L_c f(i) = cf(i) - f(i+1) - f(i-1)$; c is a constant, the $\phi_j(i)$ are prescribed, and we seek the values $u_j(i)$. The solution of (1) is obtained in symbolic form which is interpreted to define actual solutions. Application is made to the numerical solution of Laplace's and Poisson's equations with prescribed values on a rectangular boundary. Tables of values are included to facilitate making the applications. (Received January 28, 1944.)

86. Isaac Opatowski: *Use of special functions in a problem of uniform strength.*

A cantilever is bent under the action of its own weight and a concentrated load F at the free end. If x is the distance of the cantilever's cross section from its free end, $A(x)$ the area of the cross section, $S(x)$ the section modulus, g the specific weight of the beam, s the maximum stress of each cross section, the condition of uniform strength is: (*) $Fx + g\int_0^x (x-q)A(q)dq = sS(x)$. If the surface bounding the cantilever is representable in the parametric form: $y = u(x)U(t)$, $z = v(x)V(t)$ where (x, y, z) are orthogonal Cartesian coordinates, with z-axis lying in the plane of the bending couple, the use of the functions $w(x) = u(x)[v(x)]^2$, $r(x) = 1/v(x)$ (that is, w directly proportional to the section modulus, r inversely proportional to the radius of gyration) reduces (*) to the differential equation (**) $w''(x) = ar(x)w(x)$, where a is a constant depending on the type of cross section. (**) gives a wide possibility of use of most of the special functions. Cases leading to hyperbolic and cylindrical functions are discussed in detail. The treatment is general, that is, independent of the type of cross section. (Received January 28, 1944.)

87. H. E. Salzer: *Table of coefficients for inverse interpolation with advancing differences.*

This table can be used in place of the similar one described in a previous abstract (49-9-224). In addition it has the advantage that it can be employed for inverse interpolation near the beginning or end of a table and also when only a few tabulated values are available (as is the case, for instance, when solving transcendental equations). The Mathematical Tables Project has computed the coefficients of the products of ratios of advancing differences of various order. These coefficients occur in the formula obtained by the inversion of the Gregory-Newton formula for direct interpolation, employing Lagrange's theorem. The polynomial expressions for those coefficients are given in H. T. Davis, *Tables of the higher mathematical functions*, vol. 1, pp. 80–81. (A slight addition to the formula was made to complete it as far as the eighth order.) The coefficients of the two fourth order and the two fifth order terms were calculated to ten decimals, at intervals of 0.001 of the argument $m=(u-u_0)/(u_1-u_0)$. The coefficients of the four sixth order terms were calculated at intervals of 0.01 and the four seventh order coefficients as well as the seven eighth order coefficients were computed at intervals of 0.1 (all to ten decimals). (Received December 2, 1943.)

88. Andrew Vazsonyi: *On two-dimensional rotational gasflows.*

The differential equation of an inviscous compressible fluid is determined under the condition that the conductivity of the gas is negligible. (By admitting discontinuities this includes flows with shock waves.) The equation of motion is (1) $\psi_{xx}(1-(u^2/a^2))-(2uv/a^2)\psi_{xy}+\psi_{yy}(1-(v^2/a^2))=\rho^2[\partial h_0/\partial\psi-((k-1)/kR)(h_0+q^2/2)\partial s/\partial\psi]$, $u=(1/\rho)\psi_y$, $v=-(1/\rho)\psi_x$, $\rho=e^{-s/R}(h_0-q^2/2)^{1/k-1}$, $q^2(h_0-q^2/2)^{2/k-1}=e^{2s/R}(\psi_x^2+\psi_y^2)$, $a^2=ke^{-(k-1)s/R}\cdot(h_0-q^2/2)$ where the notations are as follows: ψ streamfunction, q velocity, u and v velocity components, ρ density, a local speed of sound, h_0 stagnation enthalpy (Bernoulli constant), s specific entropy, R gas constant, k isentropic exponent. There are two arbitrary functions in this equation, namely: $h_0(\psi)$ and $s(\psi)$; these must be given by the boundary conditions (or by the nature of the discontinuities). The flow is rotational in general and the rotation is given by $\omega=\partial v/\partial x-\partial u/\partial y=-\rho\partial h_0/\partial\psi+(p/R)\partial s/\partial\psi$. For irrotational flow the right-hand side of (1) equals 0. (Received January 28, 1944.)

GEOMETRY

89. Stefan Bergman: *A Hermitian metric and its property.* Preliminary report.

Let the real analytic function $\Phi(z_1, z_2, \bar{z}_1, \bar{z}_2)$, $z_k=x_k+iy_k$, $\bar{z}_k=x_k-iy_k$, $k=1, 2$, of four real variables x_1, x_2, y_1, y_2, satisfy the equation $\Phi=c\{[\partial^2\Psi/\partial z_1\partial\bar{z}_1][\partial^2\Psi/\partial z_2\partial\bar{z}_2]-|\partial^2\Psi/\partial z_1\partial\bar{z}_2|^2\}$, $c=$constant, $\Psi=\log\Phi$, in a domain B of the (four-dimensional) space and become infinite on the boundary of B. The expression $ds_B^2(z_1, z_2)=\sum_{m,n=1}^{2}[\partial^2\Psi/\partial z_m\partial\bar{z}_n]dz_m d\bar{z}_n$ defines in B a Hermitian metric which is invariant with respect to transformations by pairs of analytic functions, $z_k^*=z_k^*(z_1, z_2)$, $k=1, 2$, of two complex variables which are regular in B. Using the methods of the theory of orthogonal functions (see Bergman, *Sur les fonctions orthogonales de plusiers variables complex avec les applications à la théorie des fonctions analytiques*, Interscience Publishers, New York, 1941, chap. 4) the author investigates the property of this metric in the large. The author introduces the system of orthogonal functions, $\{M^1(z_1, z_2), M^{01}(z_1, z_2), \cdots\}$ (see loc. cit. p. 28) which he obtains by replacing in the formula (9) (see loc. cit. p. 46) the kernel function K by Φ. Various applications to the theory of pseudo-conformal transformations are discussed. (Received December 15, 1943.)

90. S. S. Chern: *A simple intrinsic proof of the Gauss-Bonnet formula for Riemannian polyhedra.*

In an n-dimensional Riemannian space let R_{ijkl} be the Riemann-Christoffel tensor. Let $\epsilon^{i_1 \cdots i_n}$ equal 1 or -1, according as the indices form an even or odd permutation of $1, \cdots, n$, and be otherwise zero. Define $S = \epsilon^{i_1 \cdots i_n} \epsilon^{j_1 \cdots j_n} R_{i_1 i_2 j_1 j_2} \cdots R_{i_{n-1} i_n j_{n-1} j_n}$ when n is even and $S = 0$ when n is odd. Let the Riemannian space be orientable and closed. The Gauss-Bonnet formula in its simplest form asserts that the integral of S over the volume element is equal to a constant multiple of the Euler characteristic of the space. (Cf. C. B. Allendoerfer and André Weil, Trans. Amer. Math. Soc. vol. 53 pp. 101–129.) A simple intrinsic proof of this formula is given, based on the following two facts: (a) the integrand in question is a null form in the $(2n-1)$-dimensional space formed by the unit vectors of the Riemannian space; (b) a field of unit vectors can be defined in the space with a finite number of singular points. Of these two facts (a) is proved by an explicit formula, while (b) is well known. An application of Stokes' formula in the space of unit vectors immediately yields the desired result. Incidentally the index theorem on vector fields follows as a consequence of the present proof. Suitable modifications give the formula for a Riemannian polyhedron. Under reasonable assumptions the formula holds for an infinite Riemannian space. (Received December 13, 1943.)

91. J. M. Feld: *On a representation in space of groups of circle and turbine transformations in the plane.*

In a previous paper (Bull. Amer. Math. Soc. vol. 48 (1942) pp. 783–790) the author showed that the oriented lineal elements of the euclidean plane can be mapped continuously and 1-1 upon the points of quasi-elliptic three-space, Q_3, so that the group of whirl-similitudes in the plane is isomorphic with the group of automorphic projectivities of Q_3. In this paper the mapping is extended to a 1-1 continuous representation of the lineal elements of the Moebius plane upon the points of projective S_3. The Laguerre, Moebius, and Lie groups of circle transformations are then shown to be isomorphic with groups of projective transformations in S_3 which leave certain linear complexes and congruences of lines invariant, and Kasner's group of turbine transformations is shown to be isomorphic with the projective group of S_3. (Received January 29, 1944.)

92. C. C. Hsiung: *Some invariants of certain pairs of hypersurfaces.*

The purpose of this paper is to generalize the results of Buzano (Bollettino della Unione Matematica Italiana vol. 14 (1935) pp. 93–98), Bompiani (Bollettino della Unione Matematica Italiana vol. 14 (1935) pp. 237–243), and the author (*Projective invariants of a pair of surfaces*, to appear in Amer. J. Math.). Two pairs of hypersurfaces V_{n-1}, V_{n-1}^* in a space S_n of n ($\geqq 3$) dimensions are considered: (a) The tangent hyperplanes t_{n-1}, t_{n-1}^* at two ordinary points O, O^* are coincident. (b) The tangent hyperplanes t_{n-1}, t_{n-1}^* at two ordinary points O, O^* are distinct, and the common tangent flat space t_{n-2} of t_{n-1}, t_{n-1}^* contains the line OO^*. It is shown that determined by the neighborhoods of the second order of the hypersurfaces V_{n-1}, V^*_{n-1} at the points O, O^* there exist a unique and two projective invariants for the two cases respectively; and each invariant is given metrically as well as projectively geometric characterizations. (Received December 29, 1943.)

93. G. B. Huff: *The completion of a theorem of Kantor.*

Let a complete and regular linear system Σ be defined by its order x_0 and its multiplicities $x_1, \cdots, x_\rho$ at a set, P^2_ρ, of ρ general points in the plane. A planar Cremona transformation C whose fundamental points fall at P^2_ρ will send Σ into a system Σ' of order x'_0 and multiplicities $x'_1, \cdots, x'_\rho$ at the fundamental points of C^{-1}. The characteristic $\{x\}=\{x_0; x_1, \cdots, x_\rho\}$ of Σ is related to $\{x'\}=\{x'_0; x'_1, \cdots, x'_\rho\}$ by a linear transformation L which has integral coefficients and the invariant forms $(xx)=x_1^2+\cdots+x_\rho^2-x_0^2$ and $(lx)=x_1+\cdots+x_\rho-3x_0$. In his prize memoir of 1884, S. Kantor incorrectly asserted the following proposition: any L with integral coefficients, leaving (xx), (lx) invariant, and having the coefficients of x_0 positive and all others negative, must be the description of some planar Cremona transformation. The author approaches the problem by obtaining further properties of an L associated with a C and proves the theorem: *An L is the description of some C if and only if the coefficients are integers, (xx), (lx) are invariants, and the sign of x_0 in all P-characteristics is unchanged.* Some results are listed. (Received January 26, 1944.)

94. Edward Kasner and John DeCicco: *Scale curves in cartography.*

In a non-conformal map T of a surface upon a plane, the scale function $\sigma=ds/dS$ depends upon the lineal-element (x, y, y'). The scale curves defined by $\sigma(x, y, y')=\text{const.}$ are ∞^2 in number. If the osculating circles are constructed to the ∞^1 scales passing through a fixed point P, at P, then the locus of the centers of curvature is a general cubic curve which has a node at P, the directions of the tangent lines at P coinciding with the characteristic directions. A complete analytic characterization of the ∞^2 scales is obtained. A new class of surfaces is found whose ∞^2 scales are all straight. In this class, the only surfaces of constant curvature are the developables. If the ∞^2 scales are of the Lie-Liouville cubic type, then they must form a velocity system. The cubical locus degenerates into a straight line. The velocity systems of scales define a new class of surfaces which are studied in detail. (Received January 25, 1944.)

95. Edward Kasner and John DeCicco: *Scale curves in conformal maps.*

In any conformal map Γ of a surface upon a plane, the scale function $\sigma=ds/dS$ depends upon (x, y) only. Thus there are ∞^1 scale curves $\sigma(x, y)=\text{const.}$ which may be any simple family. If it be demanded that an isothermal (or parallel) system be scale curves of a conformal map Γ such that the gaussian curvature is constant along the scales, then the surface must be developable (Δ) or applicable upon a surface of revolution (Σ). For (Δ), the scales can be any isothermal family, whereas for (Σ), the scales are concentric circles or parallel lines. If the scales are straight or circular such that the gaussian curvature is constant along them, then the surface is a (Δ) or a (Σ), and the scales must be pencils of lines or circles. The application of these results to any conformal map of a sphere upon a plane lead to new characterizations of the Mercator, stereographic, and the general Lambert conical projections. Thus the only conformal map with straight scale curves is the Mercator; and the only circular cases are the stereographic and Lambert maps. (Received January 25, 1944.)

Statistics and Probability

96. Benjamin Epstein and C. W. Churchman: *On the statistics of sensitivity data.*

"Sensitivity data" is a general term for that type of experimental data for which the measurement at any point in the scale destroys the sample. The paper is a generalization of a method of treating such data due to Spearman. (C. Spearman, *The method of "right and wrong cases" (constant stimuli) without Gauss' formulae*, British Journal of Psychology vol. 2 (1908) pp. 227–242.) Formulae for the moments and their standard sampling errors are given. Certain minimization problems are also discussed. (Received January 26, 1944.)

97. E. J. Gumbel: *The observed return period.*

The theoretical return period $T(x)$ of a value equal to, or greater than, x is defined as the inverse of the probability $1-F(x)$. The question is how to calculate, for n observations, the return period $T(x_m)$ of the mth observation $x_m (m=1, 2, \cdots, n)$, and especially $T(x_n)$ of the largest observation x_n for an unlimited variate. This problem is important in probability papers where the variate is plotted as a function of the return period. Engineers use a compromise between the exceedance interval $'T(x_m)=n/(n-m)$ and the recurrence interval $''T(x_m)=n/(n-m+1)$, namely $T(x_m)=n/(n-m+1/2)$. In this case $T(x_n)=2n$. If, however, the probability $F(\ddot{x}_n)$ of the median $\ddot{x}_n$ of the largest value is attributed to x_n, $T(x_n)=1.44n+1/2$. Both methods can hardly be justified. The author attributes the probability $F(\tilde{x}_n)$ of the most probable largest value $\tilde{x}_n$ to x_n. Then $T(x_n)$, as is to be expected, converges toward n, and equals n for the exponential distribution, and $n+1$ for the logistic distribution. In the same way, the probability $F(\tilde{x}_1)$ of the most probable smallest value $\tilde{x}_1$ is used, for an unlimited variate, as frequency of the smallest observation x_1. The frequencies $F(\hat{x}_m)$ of the intermediate $n-2$ observations are obtained by linear interpolation between $F(\tilde{x}_1)$ and $F(\tilde{x}_n)$. Thus the return periods may be determined for all observations. (Received January 27, 1944.)

98. H. E. Robbins: *On the measure of a random set.*

Let X, a measurable subset of Euclidean n-dimensional space E, be a random variable (for example, X may be the set-theoretical sum of N possibly overlapping and independently chosen unit intervals on a line with a given probability distribution for their centers). Let $m(X)$ denote the measure of X, and for any point x of E let $p(x)$ denote the probability that X contain x. Then under very general hypotheses on X it is shown that the expected value of $m(X)$ is equal to the integral over E of $p(x)$. More generally, the expected value of the rth power of $m(X)$ is equal to the integral over rn-dimensional space of the function $p(x_1, \cdots, x_r)=$ probability that X contain all the points $x_1, \cdots, x_r$. (Received January 28, 1944.)

TOPOLOGY

99. C. L. Clark: *Arc reversing transformations.*

If A and B are separable metric spaces, B nondegenerate, a single-valued continuous transformation $f(A)=B$ is said to be arc reversing provided the inverse of every simple arc in B is a simple arc in A. After several basic results are obtained, it is shown that if $f(A)=B$ is arc reversing, with A locally compact and B a locally connected generalized continuum, then the sets $A-A_1$ and $B-B_1$ are homeomorphic, where B_1 is the set of all points b in B whose inverses $f^{-1}(b)$ are nondegenerate and $A_1=f^{-1}(B_1)$. In particular if $f(A)=B$ is (1-1), A and B are homeomorphic. A characterization of arc reversing transformations is afforded by the result that a single-valued continuous transformation $T(A)=B$, where A and B are locally connected continua, is arc reversing if and only if the set of inverses $[T^{-1}(b)]$, b in B, consists of single points and at most a countable number of free arcs whose end points are of Urysohn-Menger order at most 2 in A. Further results are obtained concerning local separating points and continua having homeomorphs of finite linear measure. (Received December 29, 1943.)

100. Mariano García: *Component orbits under pointwise recurrent homeomorphisms.*

A point x of a separable metric space X on which a homeomorphism $f(X)=X$ is defined is called recurrent under f if, given any neighborhood U of x, there exists an N such that $f^N(x)\in U$, and an invariant set L in X whose components can be ordered in a sequence $\cdots, A_{-2}, A_{-1}, A_0, A_1, A_2, \cdots$ such that $f(A_i)=A_{i+1}$ is defined as a component orbit. Using methods analogous to those used by Whyburn in proving the results that Hall and Schweigert obtained relative to periodic component orbits (component orbits having a finite number of components) under a pointwise periodic homeomorphism on X, this paper establishes extensions of these results to non-pointwise periodic mappings. It is shown for example that if $f(X)=X$ is a homeomorphism on a compact space X and $G_1, G_2, \cdots$ is a sequence of component orbits whose limit inferior contains a periodic component orbit Q, and if (either) each point of $\limsup G_i - Q$ is recurrent under both f and f^{-1} or each point of $\sum_{i=1}^{\infty} G_i + \limsup G_i$ is recurrent under f, then $\limsup G_i$ is a periodic component orbit. (Received December 27, 1943.)

101. W. H. Gottschalk: *Powers of homeomorphisms with almost periodic properties.*

Let X be a topological space and let $f(X)=X$ be a homeomorphism. A point x of X is said to be *recurrent* under f provided that to each neighborhood U of x there corresponds a positive integer n such that $f^n(x)\in U$. A point x of X is said to be *almost periodic* under f provided that to each neighborhood U of x there corresponds a monotone increasing sequence $n_1, n_2, \cdots$ of positive integers with the properties that the numbers $n_{i+1}-n_i$ $(i=1, 2, \cdots)$ are uniformly bounded and $f^{n_i}(x)\in U$ $(i=1, 2, \cdots)$. A subset Y of X is said to be *minimal* under f provided that Y is nonvacuous, closed and invariant under f, and furthermore Y does not contain a proper subset with these properties. The following theorems are established: (1) If $x\in X$ is recurrent under f, then x is also recurrent under f^n for every positive integer n. (2) If X is minimal under f but not under f^k, where k is a nonzero integer, then there exists an integer n, $n>1$, such that n divides $|k|$ and f^n gives a finite minimal-set decomposition of X which contains exactly n elements. These theorems are developed and applied to yield further results, among them the following: If X is a compact metric space and if $x \in X$ is almost periodic under f, then x is also almost periodic under f^n for every integer n. (Received December 31, 1943.)

102. D. W. Hall: *On rotation groups of plane continuous curves under pointwise periodic homeomorphisms.*

This note makes use of the work of G. T. Whyburn on light interior transformations and orbit decompositions of certain spaces to obtain a theorem by means of which a certain subset of the orbits of points under a pointwise periodic transformation may be given a linear ordering. This theorem is then used to obtain an accessibility theorem for plane continuous curves similar to one previously published by L. Whyburn (Fund. Math. vol. 28 (1937) p. 127), but having the emphasis in the hypothesis placed upon the type of the transformation rather than upon the order of the rotation group under consideration. (Received December 18, 1943.)

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ON THE BERGMAN OPERATORS FOR LINEAR PARTIAL DIFFERENTIAL EQUATIONS

KAJ L. NIELSEN

1. Introduction. The frequent occurrence of the partial differential equation

$$L(U) = U_{z\bar{z}} + aU_z + bU_{\bar{z}} + cU = 0, \tag{1}$$

where a, b, c are functions of $z=x+iy$ and $\bar{z}=x-iy$ and $U_z=\partial U/\partial z$, $U_{\bar{z}}=\partial U/\partial \bar{z}$, $U_{z\bar{z}}=\partial^2 U/\partial z\partial \bar{z}$, in the theory of dynamics has led to a considerable amount of investigation. This investigation has in part been directed along the lines of solving boundary value and characteristic value problems and of finding particular solutions. These problems are of course classical questions in the theory of partial differential equations and have been widely discussed in the literature. Among the recent developments on these problems is the work done by Bergman [1–5],[1] who in his investigations has introduced operators

$$P(f) = \int_{-1}^{+1} E(z, \bar{z}, t) f(z[1 - t^2]/2) dt/(1 - t^2)^{1/2} \tag{2}$$

where f is an arbitrary analytic function of one complex variable. These operators shall be called the Bergman operators. They transform the class A of analytic functions of one complex variable into a certain class of functions $\mathbf{C}(E)$, which has the following properties:

(1°) If E is a solution of the equation

$$\begin{aligned} G(E) = (1 - t^2)(E_{\bar{z}t} + aE_t) - t^{-1}(E_{\bar{z}} + aE) \\ + 2zt(E_{z\bar{z}} + aE_z + bE_{\bar{z}} + cE) = 0 \end{aligned} \tag{3}$$

which satisfies certain conditions, then every function U of the class $\mathbf{C}(E)$ will be a particular solution of $L(U)=0$ and further there always exist solutions E satisfying the mentioned conditions.

(2°) It is possible to determine two functions, say E_1 and E_2, so that $\mathbf{C}(E_1)+\mathbf{C}^*(E_2)$ represents the totality of solutions of $L(U)=0$ (see [2, 5]). $\mathbf{C}^*$ is an analogous class of functions, the f being an analytic function of $\bar{z}$.

The general problem of finding the Bergman operators for a given partial differential equation (1) has been thoroughly discussed by

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[1] The numbers in the brackets refer to the bibliography.

Bergman [1, 2, 3] and by the present author and Ramsay [7]. The singularities of the operators have been treated by Bergman [1] and the present author [6]. In [4] Bergman discussed a method of solution of the classical problems consisting of approximations by expressions $W_n = \sum_{\nu=1}^{n} \alpha_\nu^{(n)} \phi_\nu(x, y)$ where $\phi_\nu(x, y)$ are particular solutions of the considered differential equations. Here the $\alpha_\nu^{(n)}$ are constants which are to be determined by the requirements that the values of W_n on the boundary approximate the given data. (This method is in a certain sense the reverse of the Rayleigh-Ritz method in which the approximating expressions satisfy the boundary conditions but do not satisfy the given equation.) The operators give a simple procedure for the construction of the particular solutions. To further expedite the numerical computation in practical problems it is desirable that the function $E(z, \bar{z}, t)$ have a simple form; for example,

$$E(z, \bar{z}, t) = \exp \sum_{k=1}^{n} C_k(z, \bar{z}) t^k. \tag{4}$$

The coefficients a, b, c, and $C_k(z, \bar{z})$ are connected by equation (3). This problem was considered in [7] and in the investigations which lead to that paper the converse problem suggested itself; namely, that of finding partial differential equations for which the Bergman operators have a particular form. In [7] it became apparent that a systematic approach to this converse problem would be to consider in turn expressions of the form $\exp(Nt^n)$, $\exp(Nt^n + Mt^m)$, $\exp(Nt^n + Mt^m + Rt^r)$, and so on, where N, M, and R are functions of z and $\bar{z}$. It is the purpose of this paper to give a general procedure for both problems and to complete the discussion for the case $E = \exp(Nt^n + Mt^m)$.

2. **General procedure.** The equation $L(U) = 0$ can always be transformed into the equation

$$L'(V) = V_{z\bar{z}} + BV_{\bar{z}} + CV = 0 \tag{5}$$

(see [2, p. 1172]). Thus it suffices to consider this equation. The equation (3) reduces to

$$G(E) = (1 - t^2)(E_{t\bar{z}}) - E_{\bar{z}}/t + 2zt(E_{z\bar{z}} + BE_{\bar{z}} + CE) = 0. \tag{6}$$

A given form of E will determine a particular solution

$$V = P(f) = \int_{-1}^{+1} E(z, \bar{z}, t) f(z[1 - t^2]/2) dt/(1 - t^2)^{1/2} \tag{7}$$

of the partial differential equation $L'(V)=0$ provided the coefficients B and C satisfy the equation (6) (see [2, p. 1171]). Thus for any given form of E we determine the required derivatives

$$E_{\bar{z}},\ E_{z\bar{z}},\ E_{t\bar{z}}. \tag{8}$$

Substituting these values into (6) we obtain an equation which is zero for an arbitrary t; thus the coefficients of each power of t must vanish. This yields a system of equations for B, C, M, and N. Upon solving this system we obtain the values for these unknowns and thus determine the partial differential equation (5) for which (7) with the given form of E is a particular solution.

The converse problem then reduces to a discussion of this system. A systematic approach to such a discussion would be to consider in turn the cases in which N is a function of z only (that is, $N_{\bar{z}}=0$), M is a function of z only, and so on. If all functions M, N, and so on, are functions of z only, then we have a trivial case. This is easily seen, for consider this case. We have

$$E = \exp \sum_{\nu=1}^{n} C_\nu(z)t^\nu. \tag{9}$$

The required derivatives (8) for the equation (6) are then all zero as each has a derivative with respect to $\bar{z}$. If we further choose $C=0$, the equation (6) is satisfied. This would give the result that the function

$$V = \int_{-1}^{+1}\left[\exp \sum_{\nu=1}^{n} C_\nu(z)t^\nu\right] f(z[1-t^2]/2)dt/(1-t^2)^{1/2}$$

is a particular solution of $L'(V)=V_{z\bar{z}}+B(z,\ \bar{z})V_{\bar{z}}=0$ for all values of $B(z,\ \bar{z})$, $C_\nu(z)$, and ν. But this is evident, as any analytic function $f(z)$ is a solution of this partial differential equation. We shall therefore say that if $N_{\bar{z}}=0$, $M_{\bar{z}}=0$, and so on, simultaneously, we have a *trivial case.*

3. **Case I.** $E=\exp[N(z,\ \bar{z})t^n]$. For simplicity of notation let $v=N(z,\ \bar{z})t^n$. In general we shall let $v=\sum_{k=1}^{n}C_k(z,\ \bar{z})t_k$ for each case; no ambiguity should arise in its meaning. The required derivatives (8) for the equation (6) are

$$\begin{aligned} E_{\bar{z}} &= e^v[N_{\bar{z}}t^n], \\ E_{t\bar{z}} &= e^v[nNN_{\bar{z}}t^{2n-1} + nN_{\bar{z}}t^{n-1}], \\ E_{z\bar{z}} &= e^v[N_zN_{\bar{z}}t^{2n} + N_{z\bar{z}}t^n]. \end{aligned} \tag{10}$$

The system of equations obtained by substituting (10) into (6) and setting the coefficients of the various powers of t equal to zero is

$$
\begin{array}{llll}
 & t^{n-1}: & (n-1)N_{\bar z} = 0, & \\
(11) & t^{2n-1}: & nNN_{\bar z} = 0, & t^{2n+1}: \quad (2zN_z - nN)N_{\bar z} = 0, \\
 & t^{n+1}: & (2zB - n)N_{\bar z} + 2zN_{z\bar z} = 0, & t: \quad 2\bar{z}C = 0.
\end{array}
$$

THEOREM 1. *Exclusive of the trivial case, the only possible values of n in Case* I *for which* (7) *is a particular solution of $L'(V)=0$ are $n=0$, $n=1$, and $n=2$.*

PROOF. To consider any case other than the trivial case N must be a function of $\bar z$ as well as z; that is, $N_{\bar z} \neq 0$. By (11) $N_{\bar z} \neq 0$ only if $2n-1=X$ where X denotes any exponent of t in (11). Thus

$$
\begin{array}{ll}
2n-1 = n-1 \quad \text{or} \quad n=0; & 2n-1 = 2n+1, \quad \text{impossible}; \\
2n-1 = n+1 \quad \text{or} \quad n=2; & 2n-1 = 1 \quad \text{or} \quad n=1. \quad \text{q.e.d.}
\end{array}
$$

THEOREM 2. *When $n=0$, Case* I *reduces to the trivial case.*

PROOF. If $n=0$, the system (11) reduces to

$$
\begin{array}{ll}
t^{-1}: & -N_{\bar z} = 0, \\
t: & 2\bar{z}N_{\bar z}N_z + 2\bar{z}N_{z\bar z} + 2B\bar{z}N_{\bar z} + 2\bar{z}C = 0.
\end{array}
$$

From the first of these equations $N_{\bar z}=0$ which is the trivial case.

The cases for $n=1$ and $n=2$ have been discussed in [7] and the solutions for B, C, and N were obtained there.

4. **Case II.** $E=\exp[Nt^n+Mt^m]=e^v$. If $N=0$, or $M=0$, or $n=m$ this case reduces to Case I; thus we consider only $N\neq 0$, $M\neq 0$, $n\neq m$. In order to avoid further duplication we may, without any loss of generality, order m and n in the following manner

$$
(12) \qquad \begin{array}{l}
n < m \text{ if } n \text{ and } m \text{ have different signs;} \\
|n| > |m| \text{ in all other cases.}
\end{array}
$$

The required derivatives (8) are

$$
\begin{aligned}
E_{\bar z} &= e^v[N_{\bar z}t^n + M_{\bar z}t^m], \\
E_{\bar z t} &= e^v[mM_{\bar z}t^{m-1} + mMM_{\bar z}t^{2m-1} + nN_{\bar z}t^{n-1} + nNN_{\bar z}t^{2n-1} \\
(13) \qquad &\quad + (mMN_{\bar z} + nNM_{\bar z})t^{n+m-1}], \\
E_{z\bar z} &= e^v[M_{z\bar z}t^m + M_zM_{\bar z}t^{2m} + N_{z\bar z}t^n + N_zN_{\bar z}t^{2n} \\
&\quad + (M_zN_{\bar z} + M_{\bar z}N_z)t^{n+m}].
\end{aligned}
$$

The system of equations obtained by substituting (13) into (6) and setting the coefficients of the various powers of t equal to zero is

$$
\begin{aligned}
t^{2n-1}&: \quad nNN_z = 0,\\
t^{n-1}&: \quad (n-1)N_z = 0,\\
t^{2n+1}&: \quad (2zN_z - nN)N_z = 0,\\
t^{n+1}&: \quad (2zB - n)N_z + 2zN_{zz} = 0,\\
t^{2m-1}&: \quad mMM_z = 0,\\
t^{m-1}&: \quad (m-1)M_z = 0,\\
t^{2m+1}&: \quad (2zM_z - mM)M_z = 0,\\
t^{m+1}&: \quad (2zB - m)M_z + 2zM_{zz} = 0,\\
t^{n+m-1}&: \quad mMN_z + nNM_z = 0,\\
t^{n+m+1}&: \quad (2zM_z - mM)N_z + (2zN_z - nN)M_z = 0,\\
t&: \quad 2zC = 0.
\end{aligned}
\tag{14}
$$

As was indicated in §2 the procedure of analysis is to first consider $N_z=0$, $M_z\neq 0$; then $M_z=0$, $N_z\neq 0$; and finally neither equal to zero. If $N_z=0$ and $M_z=0$, we have of course the trivial case. From (14) it is clear that $N_z=0$ unless $2n-1=X$ where X denotes any exponent of t in the system (14). Checking all these possibilities and using (12), we obtain the following property.

PROPERTY 1. *$N_z=0$ for all values of m and n except $n=1$ and $n=2$.*

If $N_z=0$, the system (14) reduces to

$$
\begin{aligned}
t^{2m-1}&: \quad mMM_z = 0, && \\
t^{m-1}&: \quad (m-1)M_z = 0, & t^{2m+1}&: \quad (2zM_z - mM)M_z = 0,\\
t^{m+1}&: \quad (2zB - m)M_z + 2zM_{zz} = 0, & t^{n+m-1}&: \quad nNM_z = 0,\\
t^{n+m+1}&: \quad (2zN_z - n)NM_z = 0, & t&: \quad 2zC = 0.
\end{aligned}
\tag{15}
$$

If $M_z=0$, the system (14) reduces to

$$
\begin{aligned}
t^{2n-1}&: \quad nNN_z = 0, && \\
t^{n-1}&: \quad (n-1)N_z = 0, & t^{2n+1}&: \quad (2zN_z - nN)N_z = 0,\\
t^{n+1}&: \quad (2zB - n)N_z = 0, & t^{n+m-1}&: \quad mMN_z = 0,\\
t^{n+m+1}&: \quad (2zM_z - mM)N_z = 0, & t&: \quad 2zC = 0.
\end{aligned}
\tag{16}
$$

THEOREM 3. *Case* II *reduces to the trivial case*

(i) *if* $n<0$ *and* $m\geqq 0$;

(ii) *if* $n<0$ *and* $m<0$;
(iii) *if* $m>2$;
(iv) *if* $n>2$ *and* $m=0$;
(v) *if* $n>3$ *and* $m=1$;
(vi) *if* $n>4$ *and* $m=2$;
(vii) *if* $n=3$ *and* $m=2$.

PROOF. By definition of the trivial case it is necessary to show only that under these conditions $N_s=0$ and $M_s=0$.

(i) and (ii). If $n<0$, $N_s=0$ by Property 1. The consideration then reduces to an analysis of the exponents of t in (15). In (15) $M_s=0$ unless $2m-1=X$ and $n+m-1=X$ simultaneously, X denoting any other exponent of t in the system (15). If $n<0$ and $m\geqq 0$, $2m-1=X$ only if $m=0$, $m=1$, or $m=2$; $n+m-1=X$ only if $m>2$. By a comparison it is easily seen that none of these conditions are satisfied simultaneously, therefore M_s is also zero. If $n<0$ and $m<0$, $n+m-1=X$ only if $n=m+2$; $2m-1=X$ only if $n=m-2$. These conditions can not be satisfied simultaneously, therefore $M_s=0$.

(iii). In (14) $M_s=0$ unless $m-1=X$; but by (i), (ii), and (12) this is impossible for $m>2$; thus $M_s=0$. If $M_s=0$, the system (14) reduces to (16) and $N_s=0$ unless $2n-1=X$, $n-1=X$, and $n+m-1=X$ simultaneously in (16). Checking these conditions one easily sees that this is impossible if $m>2$. Thus N_s is also equal to zero.

We now turn to (iv), (v), and (vi). If $n>2$, $N_s=0$ by Property 1. The system (14) then reduces to (15).

(iv). If $m=0$, the exponents of t in (15) become

$$-1,\ 1,\ n-1,\quad \text{and}\quad n+1.$$

Further $M_s=0$ unless $-1=X$ and $n-1=X$ simultaneously. This is impossible if $n>2$. Therefore M_s is also equal to zero.

(v). If $m=1$, the exponents of t in (15) become

$$1,\ 3,\ 2,\ n,\quad \text{and}\quad n+2.$$

Further $M_s=0$ unless $n=X$ which is impossible if $n>3$. Therefore M_s is also zero.

(vi). If $m=2$, the exponents of t in (15) become

$$3,\ 1,\ 5,\ n+1,\ n+3.$$

Further $M_s=0$ unless $n+1=X$ which is impossible if $n>4$. Therefore M_s is also zero.

(vii). If $n=3$ and $m=2$, it is easily seen that the system (14) yields $N_s=0$ and $M_s=0$.

THEOREM 4. *The only possible integral values for m and n which yield solutions different from those of the trivial case are*

(i) $m=0$ *and* $n=1$ *or* $n=2$;
(ii) $m=1$ *and* $n=2$ *or* $n=3$;
(iii) $m=2$ *and* $n=4$.

PROOF. This theorem follows directly from Theorem 3.

The cases of Theorem 5 have been discussed in [7] and solutions for B, C, M, and N were obtained there.

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ON THE EQUATION $\chi\alpha=\gamma\chi+\beta$ OVER AN ALGEBRAIC DIVISION RING

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1. Introduction and notation. The main purpose of this paper is to give necessary and sufficient conditions in order that the equation

$$\chi\alpha = \gamma\chi + \beta \tag{1}$$

have a solution χ over an algebraic division ring. In case a solution exists, it is given explicitly if it is unique; otherwise, a method of obtaining one of the solutions is given. The application of the results to a quaternion algebra is discussed in the final section.

Let R be a division ring algebraic over its separable[1] center F, and λ a commutative indeterminate over R. Using the notation of Ore,[2] a polynomial $a(\lambda)\in R[\lambda]$ of degree n,

$$a(\lambda) = \alpha_n\lambda^n + \alpha_{n-1}\lambda^{n-1} + \cdots + \alpha_0, \tag{2}$$

will be called reduced if $\alpha_n=1$. The unique reduced polynomial $m(\lambda)\in F[\lambda]$ of minimum degree for which $m(\alpha)=0$ will be labelled $m_\alpha(\lambda)$. It is apparent that $m_\alpha(\lambda)$ is irreducible over $F[\lambda]$. The ring of all elements of R which commute with α will be denoted by R_α.

The substitution of an element of R for λ in the polynomial (1) is not well defined, as λ commutes with elements of R, whereas the elements of R do not all commute among themselves. However, unilateral substitution is well defined. We shall use the symbol $a^r(\beta)$ to mean that β has been substituted for λ on the right in (2), so that

$$a^r(\beta) = \alpha_n\beta^n + \alpha_{n-1}\beta^{n-1} + \cdots + \alpha_0. \tag{3}$$

Left substitution is defined similarly—as there is a complete duality between left and right substitution in our case, we shall discuss right substitution only. If $a^r(\beta)=0$, β is called a right root of $a(\lambda)$. The notation $a(\lambda)\,|^r b(\lambda)$ is used to mean that $a(\lambda)$ is a right factor of $b(\lambda)$. As is well known, β is a right root of $a(\lambda)$ if and only if $(\lambda-\beta)\,|^r a(\lambda)$.

2. Preliminary lemmas. A division algorithm exists over $R[\lambda]$. The particular case of interest here is given by

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[1] That is, no irreducible polynomial in $F[\lambda]$ has a multiple root in R.

[2] O. Ore, *Theory of noncommutative polynomials*, Ann. of Math. vol. 34 (1933) pp. 481–508.

$$b(\lambda) = q(\lambda)(\lambda - \alpha) + b^r(\alpha). \tag{4}$$

That is, the remainder on dividing a polynomial $b(\lambda)$ on the right by $(\lambda-\alpha)$ is $b^r(\alpha)$. For any two elements $a(\lambda)$, $b(\lambda)$ of $R[\lambda]$, there exists a unique reduced greatest common right divisor and a unique reduced least common left multiple.

The following lemma is true for any ring[3] R. It is frequently proven for special cases.

Lemma A. *If $c(\lambda)=a(\lambda)b(\lambda)$, than $b^r(\alpha)=0$ implies $c^r(\alpha)=0$.*

To prove this for a general ring, let $a(\lambda)=\sum_{i=0}^{n}\alpha_i\lambda^i$, $b(\lambda)=\sum_{i=0}^{m}\beta_i\lambda^i$: then

$$c(\lambda) = \sum_{i=0}^{n}\alpha_i\left(\sum_{j=0}^{m}\beta_j\lambda^j\right)\lambda^i. \tag{5}$$

From this form, it is apparent that $c^r(\alpha)=0$ if $b^r(\alpha)=0$.

In any polynomial ring which possesses a division algorithm, the following lemma holds.

Lemma B. *If $c(\lambda)=a(\lambda)b(\lambda)$, then $(\lambda-\alpha)\,|^r c(\lambda)$ if and only if $(\lambda-\alpha)\,|^r a(\lambda)b^r(\alpha)$.*

From (4), $c(\lambda)=a(\lambda)q(\lambda)(\lambda-\alpha)+a(\lambda)b^r(\alpha)$, and the lemma follows. Over a division ring, this lemma can be put in the following form.

Lemma B′. *If $c(\lambda)=a(\lambda)b(\lambda)$ and $\tau=b^r(\alpha)\neq 0$, then $a^r(\tau+\tau^{-1})=0$ if and only if $c^r(\alpha)=0$.*

This result was obtained by Wedderburn,[4] and later by Richardson[5] and, in a more general form, by Ore.[2] Another result of Wedderburn's[4] is the following lemma.

Lemma C. *If $a^r(\tau\alpha\tau^{-1})=0$ for all nonzero elements $\tau\in R$, then $m_\alpha(\lambda)\,|\,a(\lambda)$.*

The following fundamental theorem was obtained by Wedderburn[4] for division algebras and holds equally well for algebraic division rings.

Lemma D. *If $m_\alpha(\lambda)$ is of degree n, then there exist elements $\alpha_1(=\alpha)$, $\alpha_2, \cdots, \alpha_n$ in R such that*

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[3] See C. C. MacDuffee, *Vectors and matrices* (Carus Mathematical Monographs, No. 7), Mathematical Association of America, 1943, Theorem 36.

[4] J. H. M. Wedderburn, *On division algebras*, Trans. Amer. Math. Soc. vol. 22 (1921) pp. 130–131.

[5] A. R. Richardson, *Equations over a division algebra*, Messenger of Mathematics vol. 57 (1928) pp. 1–6.

$$m_\alpha(\lambda) = (\lambda - \alpha_n)(\lambda - \alpha_{n-1}) \cdots (\lambda - \alpha_1). \tag{6}$$

A particular factorization of $m_\alpha(\lambda)$ is needed in the proof of Theorem 2. To obtain this, we establish the following lemma.

LEMMA D′. *There exist elements* $\sigma_{11}, \sigma_{12}, \cdots, \sigma_{1n} \in R$ *such that, if*

$$\sigma_{i+1\,j} = \sigma_{ij}\alpha - \sigma_{i\,i-1}\alpha\sigma_{i\,i-1}^{-1}\sigma_{ij}, \qquad i, j = 1, 2, \cdots, n-1, \tag{7}$$

where $\sigma_{i0} = 1$ *for all* i, *then* $m_\alpha(\lambda)$ *has the factorization* (6) *for*

$$\alpha_i = \sigma_{i\,i-1}\alpha\sigma_{i\,i-1}^{-1}, \qquad i = 2, 3, \cdots, n. \tag{8}$$

That this is true can be seen inductively. Assume that $\sigma_{11}, \sigma_{12}, \sigma_{13}, \cdots, \sigma_{1\,k-1}$ exist, $k < n$, so that

$$m_\alpha(\lambda) = a_{k+1}(\lambda)(\lambda - \alpha_k)(\lambda - \alpha_{k-1}) \cdots (\lambda - \alpha_1),$$

where $\alpha_1, \alpha_2, \cdots, \alpha_k$ are given by (7) and (8). Let

$$a_i(\lambda) = a_{k+1}(\lambda)(\lambda - \alpha_k)(\lambda - \alpha_{k-1}) \cdots (\lambda - \alpha_i), \qquad i = 1, 2, \cdots, k,$$

and

$$b_i(\lambda) = (\lambda - \alpha_k)(\lambda - \alpha_{k-1}) \cdots (\lambda - \alpha_i), \qquad i = 1, 2, \cdots, k.$$

From Lemma C, there must exist an element $\sigma_{1k} \in R$ such that $b_1^r(\sigma_{1k}\alpha\sigma_{1k}^{-1}) \neq 0$. Then $\sigma_{1k}\alpha\sigma_{1k}^{-1} - \alpha_1 \neq 0$, and by Lemma B′, $a_2^r[(\sigma_{1k}\alpha\sigma_{1k}^{-1} - \alpha_1)\sigma_{1k}\alpha\sigma_{1k}^{-1}(\sigma_{1k}\alpha\sigma_{1k}^{-1} - \alpha_1)^{-1}] = 0$ or $a_2^r(\sigma_{2k}\alpha\sigma_{2k}^{-1}) = 0$. Now, as $b_1(\lambda) = b_2(\lambda)(\lambda - \alpha_1)$, and $b_1^r(\sigma_{1k}\alpha\sigma_{1k}^{-1}) \neq 0$, $b_2^r(\sigma_{2k}\alpha\sigma_{2k}^{-1}) \neq 0$ from Lemma B′. Thus $\sigma_{2k}\alpha\sigma_{2k}^{-1} - \alpha_2 \neq 0$, so that $a_3^r[(\sigma_{2k}\alpha\sigma_{2k}^{-1} - \alpha_2)\sigma_{2k}\alpha\sigma_{2k}^{-1}(\sigma_{2k}\alpha\sigma_{2k}^{-1} - \alpha_2)^{-1}] = 0$ or $a_3^r(\sigma_{3k}\alpha\sigma_{3k}^{-1}) = 0$. By induction, $a_i^r(\sigma_{ik}\alpha\sigma_{ik}^{-1}) = 0$, $i = 1, 2, \cdots, k+1$, so that we can select $\alpha_{k+1} = \sigma_{k+1\,k}\alpha\sigma_{k+1\,k}^{-1}$.

It is apparent that $m_\alpha(\tau\alpha\tau^{-1}) = 0$ for all nonzero $\tau \in R$. That all roots of $m_\alpha(\lambda)$ are of this form is given by the following lemma.

LEMMA E. *If* $m_\alpha(\beta) = 0$, *then* β *is a transform of* α.

To prove this, let $m_\alpha(\lambda) = (\lambda - \beta)a(\lambda)$. From Lemma C, there must exist an element $\tau \in R$ such that $a^r(\tau\alpha\tau^{-1}) \neq 0$. Thus, in view of Lemma B′, $\beta = \sigma\alpha\sigma^{-1}$, where $\sigma = a^r(\tau\alpha\tau^{-1})$.

3. **Principal theorems.** If either α or γ is in F, equation (1) becomes trivial. Therefore we shall assume that both α and γ are not in F. Define $\nu_0 = \beta$, and, in general,

$$\nu_i = \gamma^i\beta + \gamma^{i-1}\beta\alpha + \cdots + \gamma\beta\alpha^{i-1} + \beta\alpha^i, \qquad i = 1, 2, \cdots.$$

Then, if $m(\lambda) = \sum_{i=0}^{n} \mu_i\lambda^i$ is any polynomial in $F[\lambda]$, any χ which is a solution of (1) is also a solution[6] of

$$\chi\alpha m(\alpha) = \gamma m(\gamma)\chi + \sum_{i=0}^{n} \mu_i\nu_i. \tag{9}$$

The discussion of (1) is divided quite naturally into two cases. The first case, which is the easier of the two, is for γ not a transform of α. The second case is for γ a transform of α.

Case 1. As α and γ are not transforms of each other, $m_\alpha(\gamma)\neq 0$ in view of Lemma E. Thus, if we let $m(\lambda)$ of (9) be $m_\alpha(\lambda)$, we obtain

$$\chi = - [m_\alpha(\gamma)]^{-1}\gamma^{-1}\left(\sum_{i=0}^{n} \mu_i\nu_i\right) \tag{10}$$

as the unique solution of (9). A substitution of this value of χ in (1) shows that it is also a solution of (1). As any solution of (1) is also a solution of (9), (10) gives the unique solution of (1). We have thus established the following theorem:

THEOREM 1. *If α and γ are not transforms of each other, then*

$$\chi\alpha = \gamma\chi + \beta$$

has a unique solution. If γ is not zero, this solution is given by (10).

Case 2. The remaining considerations are for $\gamma=\tau\alpha\tau^{-1}$. It is apparent that the methods of Case 1 now fail, as $m_\alpha(\gamma)=0$. Thus a new approach must now be made.

Equation (1) can now be put in the form

$$\chi\alpha = \tau\alpha\tau^{-1}\chi + \beta.$$

This equation has a solution if and only if the equation

$$\tau^{-1}\chi\alpha = \alpha\tau^{-1}\chi + \tau^{-1}\beta$$

has a solution. Therefore we need only consider an equation of the form

$$\chi\alpha = \alpha\chi + \beta. \tag{11}$$

The existence of solutions of this equation is given by the following theorem.

THEOREM 2. *Let α be an element of R not in F with minimum polynomial $m_\alpha(\lambda)=a(\lambda)(\lambda-\alpha)$ and β be a nonzero element of R. Then the equation*

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[6] See M. H. Ingraham and H. C. Trimble, *On the matric equation $TA=BT+C$*, Amer. J. Math. vol. 63 (1941) p. 13.

$$\chi\alpha = \alpha\chi + \beta$$

has a solution χ *in* R *if and only if* $a^r(\beta\alpha\beta^{-1}) = 0$.

PROOF. We shall first assume that there exists an element $\chi \in R$ such that (11) is satisfied. Then, as $m_\alpha(\chi\alpha\chi^{-1}) = 0$ and $\chi\alpha \neq \alpha\chi$, we have by Lemma B′ that $a^r([\chi\alpha - \alpha\chi]\alpha[\chi\alpha - \alpha\chi]^{-1}) = 0$. Thus $a^r(\beta\alpha\beta^{-1}) = 0$, and the first part of the theorem is established.

On the other hand, suppose that $a^r(\beta\alpha\beta^{-1}) = 0$. We shall now use the particular factorization of $m_\alpha(\lambda)$ given in Lemma D′. Let the polynomials $b_{ij}(\lambda)$ be defined by

$$b_{ij}(\lambda) = (\lambda - \alpha_i)(\lambda - \alpha_{i-1}) \cdots (\lambda - \alpha_j), \qquad i \geqq j = 1, 2, \cdots, n.$$

Also, let $\beta_1 = \beta$, and recursively,

$$\beta_i = \beta_{i-1}\alpha - \alpha_i\beta_{i-1}, \qquad i = 2, 3, \cdots, n. \tag{12}$$

There must exist an integer k such that $b^r_{k2}(\beta\alpha\beta^{-1}) \neq 0$, $b^r_{k+1\,2}(\beta\alpha\beta^{-1}) = 0$. As in the proof of Lemma D′, the successive application of Lemma B′ yields

$$b^r_{k+1\;i+1}(\beta_i\alpha\beta_i^{-1}) = 0, \qquad i = 1, 2, \cdots, k.$$

The last application gives $b^r_{k+1\;k+1}(\beta_k\alpha\beta_k^{-1}) = 0$, so that $\alpha_{k+1} = \beta_k\alpha\beta_k^{-1}$. From (8), $\alpha_{k+1} = \sigma_{k+1\;k}\alpha\sigma^{-1}_{k+1\;k}$: thus there must exist an element $\delta_k \in R_\alpha$ such that $\beta_k = \sigma_{k+1\;k}\delta_k$. Now let us assume that there exist elements $\delta_j \in R_\alpha$ and an integer m such that

$$\beta_i = \sum_{j=i}^{k} \sigma_{i+1\;j}\delta_j, \qquad i = m, m+1, \cdots, k.$$

Then it follows from (7), (8), and (12) that

$$\beta_{m-1}\alpha - \alpha_m\beta_{m-1} = \sum_{j=m}^{k} (\sigma_{mj}\alpha - \alpha_m\sigma_{mj})\delta_j,$$

so that

$$\left(\beta_{m-1} - \sum_{j=m}^{k} \sigma_{mj}\delta_j\right)\alpha = \alpha_m\left(\beta_{m-1} - \sum_{j=m}^{k} \sigma_{mj}\delta_j\right).$$

As $\alpha_m = \sigma_{m\;m-1}\alpha\sigma^{-1}_{m\;m-1}$, there must exist an element $\delta_{m-1} \in R_\alpha$ such that

$$\beta_{m-1} = \sum_{j=m-1}^{k} \sigma_{mj}\delta_j.$$

By induction,

$$\beta = \sum_{j=1}^{k} \sigma_{2j}\delta_j.$$

From (7),

$$\beta = \sum_{j=1}^{k} \sigma_{1j}\delta_j\alpha - \alpha\sum_{j=1}^{k}\sigma_{1j}\delta_j$$

and thus, for any $\delta \in R_\alpha$,

$$\chi = \sum_{j=1}^{k} \sigma_{1j}\delta_j + \delta \tag{13}$$

is a solution of (11).

4. Special considerations. As a special case of Theorem 2, consider R as the ring of quaternions over a formally real field F, $R=F(1, i, j, k)$. If we let $\bar{\alpha}$ denote the conjugate of α, α not in F, then

$$m_\alpha(\lambda) = (\lambda - \bar{\alpha})(\lambda - \alpha).$$

Thus $a(\lambda)=(\lambda-\bar{\alpha})$, and Theorem 2 can be written in the following form.

COROLLARY 1. *If R is a quaternion algebra over a formally real field F and α is an element of R not in F, then*

$$\chi\alpha = \alpha\chi + \beta$$

has a solution if and only if

$$\beta\alpha = \bar{\alpha}\beta. \tag{14}$$

Having obtained one solution of (11) from (13), say χ_1, then all solutions are given by $\chi_1+\delta$, $\delta \in R_\alpha$. It is observed that (11) cannot have a solution if $\beta \in R_\alpha$—as $a^r(\beta\alpha\beta^{-1})=a^r(\alpha)$ in this case, and $a^r(\alpha)$ cannot be zero due to the separability of F. However, it is not true that (11) always has a solution if β is not in R_α. A simple example to show this is as follows: let R be the ring of quaternions over a formally real field F. For $\beta=i+j$ and $\alpha=i$, (14) is not satisfied, and thus (11) can have no solution.

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A RECURRENCE FORMULA FOR THE SOLUTIONS OF CERTAIN LINEAR PARTIAL DIFFERENTIAL EQUATIONS

MORRIS MARDEN

1. Introduction. In a number of recent papers, Bergman[1] has developed the theory of operational methods for transforming analytic functions of a complex variable into solutions of the linear partial differential equation

$$(1.1)\qquad L(U) = U_{z\bar z} + a(z,\bar z)U_z + b(z,\bar z)U_{\bar z} + c(z,\bar z)U = 0,$$

where $z = x+iy$, $\bar z = x-iy$,

$$U_z = \frac{1}{2}\left(\frac{\partial U}{\partial x} - i\,\frac{\partial U}{\partial y}\right),\qquad U_{\bar z} = \frac{1}{2}\left(\frac{\partial U}{\partial x} + i\,\frac{\partial U}{\partial y}\right),$$

$$U_{z\bar z} = \frac{1}{4}\left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2}\right) = \frac{\Delta U}{4},$$

and where the coefficients $a(z,\bar z)$, $b(z,\bar z)$ and $c(z,\bar z)$ are analytic functions of both variables z and $\bar z$. The equation (1.1) is equivalent to the system of two real equations

$$\begin{aligned}\Delta U^{(1)} + 2AU_x^{(1)} + 2BU_y^{(1)} + 2CU_x^{(2)} + 2DU_y^{(2)}&\\ + 4c_1U^{(1)} - 4c_2U^{(2)} &= 0,\\ \Delta U^{(2)} - 2CU_x^{(1)} - 2DU_y^{(1)} + 2AU_x^{(2)} + 2BU_y^{(2)}&\\ + 4c_2U^{(1)} + 4c_1U^{(2)} &= 0,\end{aligned}$$

where

$$\begin{aligned}&U = U^{(1)} + iU^{(2)}; &&2A = (a+\bar a)+(b+\bar b); &&2B = i[(\bar a - a) - (\bar b - b)];\\ &c = c_1 + ic_2; &&2D = (a+\bar a) - (b+\bar b); &&2C = i[(a-\bar a) + (b - \bar b)].\end{aligned}$$

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[1] S. Bergman, (a) *Zur Theorie der Funktionen, die eine lineare partielle Differentialgleichung befriedigen*, Rec. Math. (Mat. Sbornik) N.S. vol. 44 (1937) pp. 1169–1198; (b) *The approximation of functions satisfying a linear partial differential equation*, Duke Math. J. vol. 6 (1940) pp. 537–561; (c) *Linear operators in the theory of partial differential equations*, Trans. Amer. Math. Soc. vol. 53 (1943) pp. 130–155; (d) *On the solutions of partial differential equations of the fourth order*, to appear later.

Furthermore, if $a=\bar{b}$ and c is real, then $C=D=c_2=0$, and the two differential equations become real and identical.

For equations (1.1), Bergman proved the existence of two functions $E_1(z, \bar{z}, t)$ and $E_2(z, \bar{z}, t)$, called by him "generating functions of the first kind,"[2] with the following properties:

(1) They have the forms

$$E_1(z, \bar{z}, t) = \exp\left(-\int_0^{\bar{z}} a(z, \bar{z})d\bar{z}\right)[1 + z\bar{z}tE_1^*(z, \bar{z}, t)],$$

$$E_2(z, \bar{z}, t) = \exp\left(-\int_0^{z} b(z, \bar{z})dz\right)[1 + z\bar{z}tE_2^*(z, \bar{z}, t)],$$

where each $E_k^*(z, \bar{z}, t)$ has continuous first partial derivatives in z, $\bar{z}$ and t for $|t| \leqq 1$ and for z and $\bar{z}$ within a certain four-dimensional region.

(2) The classes $C(E_1)$ and $C(E_2)$ of functions $U_1(z, \bar{z})$ and $U_2(z, \bar{z})$ defined by the formulas

$$(1.2)\qquad U_1(z, \bar{z}) = \int_{-1}^{1} E_1(z, \bar{z}, t)f(z(1 - t^2)/2)dt/(1 - t^2)^{1/2},$$

$$(1.3)\qquad U_2(z, \bar{z}) = \int_{-1}^{1} E_2(z, \bar{z}, t)g(\bar{z}(1 - t^2)/2)dt/(1 - t^2)^{1/2},$$

where $f(\zeta)$ and $g(\zeta)$ are arbitrary analytic functions of ζ, form subsets of solutions of (1.1).

(3) Every solution $U(z, \bar{z})$ of (1.1) may be written in the form

$$U(z, \bar{z}) = U_1(z, \bar{z}) + U_2(z, \bar{z}),$$

with $f(\zeta)$ and $g(\zeta)$ suitably chosen analytic functions.

As was proved by Bergman, to many theorems about analytic functions of a complex variable correspond analogous theorems about functions belonging to classes $C(E)$ generated by functions E of the first kind. In particular, if we define as "basic solutions" those corresponding to $f(z)=z^p$, that is

$$(1.4)\qquad u_p(z, \bar{z}) = \int_{-1}^{1} E(z, \bar{z}, t)[z(1 - t^2)/2]^p dt/(1 - t^2)^{-1/2},$$

then every function U of class $C(E)$ which is regular in $|z| \leqq r$ may be expanded in a series $U=\sum\alpha_p u_p$ which is uniformly and absolutely convergent in $|z|\leq r$.

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[2] Generating functions which are considered as not of the first kind are those failing to satisfy property (1). When E is a generating function not of the first kind, the integration in (1.2) and (1.3) must be taken along a rectifiable curve joining the points $t=\pm 1$, but not passing through $t=0$.

For example, in the case of the equation

$$\Delta U + U = 0, \tag{1.5}$$

$E(z, \bar{z}, t)=e^{itr}$, where $z=re^{i\theta}$ and thus $r=(z\bar{z})^{1/2}$. Because of the well known formula for the Bessel function of the first kind

$$J_p(r) = (2/\pi)(1/\Gamma(p + 1/2))\int_{-1}^{1} e^{rti}(r/2)^p(1 - t^2)^{p-1/2}dt,$$

the basic solutions are

$$u_p(r, \theta) = (\pi^{1/2}/2)\Gamma(p + 1/2)e^{p\theta i}J_p(r). \tag{1.6}$$

But for (1.5), successive terms in the expansion $U=\sum\alpha_p u_p$ can be computed from earlier terms by the use of some recurrence relation satisfied by the Bessel's functions, as for example the relation

$$J_p'(r) = (p/r)J_p(r) - J_{p+1}(r). \tag{1.7}$$

It would likewise be of practical value in the case of other differential equations $L(U)=0$ to determine what recurrence relations, if any, are satisfied by the basic solutions $u_p(r, \theta)$.

In the present note, recurrence formulas connecting the basic solutions $u_p(r, \theta)$ are found in the case of differential equations $L(U)=0$ for which at least one of the corresponding "generating functions" $E(z, \bar{z}, t)$ is of the form $E(z, \bar{z}, t)=\exp f(r, \theta, t)$ where $f(r, \theta, t)$ is a polynomial in t containing either only even powers of t or only odd powers of t. Obviously, the equation (1.5) is an example of such an equation. Other examples can be found by requiring the coefficients a, b and c in the equation $L(U)=0$ to satisfy certain differential relations.[3]

Our first main result may be stated as follows:

THEOREM 1. *Let $L(U)=0$ be a partial differential equation of the type* (1.1) *for which there exists a generating function having one of the forms*:

$$E(z, \bar{z}, t) = \exp P(r, \theta, t), \tag{I}$$

$$E(z, \bar{z}, t) = \exp tP(r, \theta, t), \tag{II}$$

where $P(r, \theta, t)=a_0(r, \theta)+a_1(r, \theta)t^2+ \cdots +a_n(r, \theta)t^{2n}$, and where the coefficients $a_j(r, \theta)$ are of class C' in r and θ. Let $u_p(r, \theta)$ be the corresponding "basic solutions" of equation $L(U)=0$ and let

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[3] See reference in footnote 1a, pp. 1194–1195, and also p. 158 of the following article: K. L. Nielsen and B. P. Ramsay, *On particular solutions of linear partial differential equations*, Bull. Amer. Math. Soc. vol. 49 (1943) pp. 156–162.

$$(1.8)\qquad \alpha_q(r,\theta) = (-1)^q\left[\sum_{j=q}^{n} C_{j,q}\frac{\partial a_j}{\partial r}\right]\left(\frac{2}{re^{i\theta}}\right)^q,$$

$$(1.9)\qquad \beta_q(r,\theta) = (-1)^q\left[\sum_{j=q}^{n}(2j+1)C_{j,q}a_j\right]\left(\frac{2}{re^{i\theta}}\right)^q.$$

Then, if E has form (I),

$$(1.10)\qquad \frac{\partial u_p}{\partial r} = \frac{p}{r}u_p + \sum_{j=0}^{n}\alpha_j u_{p+j};$$

whereas, if E has form (II),

$$(1.11)\qquad \frac{\partial u_p}{\partial r} = \frac{p}{r}u_p + \frac{2}{re^{i\theta}}\sum_{j,k=0}^{n}\frac{\alpha_j\beta_k u_{j+k+p+1}}{2j+2p+1}.$$

The above theorem will be derived as an immediate consequence of two lemmas that are given in the next section. In the third section the theorem will be applied to a few specific equations of form (1.1).

2. **Two lemmas.** First we shall derive a result for polynomials $P(r, \theta, t)$ involving only even powers of t.

LEMMA 1. *Let*

$$P(r,\theta,t) = a_0(r,\theta) + a_1(r,\theta)t^2 + \cdots + a_n(r,\theta)t^{2n},$$

where the $a_j(r, \theta)$ *are functions of class* C' *in* r *and* θ. *Then the function*

$$(2.1)\qquad u_p(r,\theta) = \int_{-1}^{1} e^{P(r,\theta,t)}(1-t^2)^{p-1/2}(re^{i\theta}/2)^p dt$$

satisfies the recurrence formula

$$(2.2)\qquad \partial u_p/\partial r = [p/r + P_r(r,\theta,(1-T)^{1/2})]u_p,$$

where T *is the operator such that,* T *acting* k *times upon* u_p,

$$(2.3)\qquad T^k u_p = (2/re^{i\theta})^k u_{p+k}$$

for $k=0, 1, \cdots$.

PROOF. From (2.1) we obtain by differentiating with respect to r:

$$(2.4)\qquad \begin{aligned}\frac{\partial u_p}{\partial r} = \frac{p}{r}u_p + \int_{-1}^{1} e^P\Big[\frac{\partial a_0}{\partial r} + \frac{\partial a_1}{\partial r}t^2 + \cdots \\ + \frac{\partial a_n}{\partial r}t^{2n}\Big](1-t^2)^{p-1/2}\left(\frac{re^{i\theta}}{2}\right)dt.\end{aligned}$$

To evaluate the latter integral, let us note that

$$\begin{aligned}\int_{-1}^{1} t^{2m}e^{P}(1-t^2)^{p-1/2}(re^{i\theta}/2)^p dt &= \int_{-1}^{1}[1-(1-t^2)]^m e^{P}(1-t^2)^{p-1/2}(re^{i\theta}/2)^p dt\\ &= \sum_{k=0}^{m}(-1)^k C_{m,k}\int_{-1}^{1} e^{P}(1-t^2)^{k+p-1/2}(re^{i\theta}/2)^p dt\\ &= \sum_{k=0}^{m}(-1)^k C_{m,k}(2/re^{i\theta})^k u_{p+k}.\end{aligned}$$

Hence,

$$\int_{-1}^{1} t^{2m}e^{P}(1-t^2)^{p-1/2}(re^{i\theta}/2)^p dt = (1-T)^m u_p. \tag{2.5}$$

Substituting now from (2.5) into (2.4), we find

$$\begin{aligned}\frac{\partial u_p}{\partial r} &= \frac{p}{r}u_p + \frac{\partial a_0}{\partial r}u_p + \frac{\partial a_1}{\partial r}(1-T)u_p + \cdots + \frac{\partial a_n}{\partial r}(1-T)^n u_p\\ &= [p/r + P_r(r,\theta,(1-T)^{1/2})]u_p,\end{aligned}$$

as was to be proved.

The corresponding result for a polynomial that involves only odd powers of t may be stated as follows.

LEMMA 2. *Let* $Q(r,\theta,t)\equiv a_0(r,\theta)t + a_1(r,\theta)t^3 + \cdots + a_n(r,\theta)t^{2n+1} \equiv tP(r,\theta,t)$, *where the* $a_j(r,\theta)$ *are functions of class* C' *in* r *and* θ. *Then the functions* $u_p(r,\theta)$ *defined by* (2.1) *satisfy the recurrence formula*:

$$\begin{aligned}\partial u_p/\partial r = (p/r)u_p + \Big\{Q_t(r,\theta,(1-T)^{1/2})&\\ \cdot\int_0^{T^{1/2}} t^{2p}P_r(r,\theta,(1-t)^{1/2})dt\Big\}&T^{-p+1/2}u_p,\end{aligned} \tag{2.6}$$

where T *is the operator defined by equation* (2.3).

PROOF. In place of (2.4), we now have

$$\begin{aligned}\frac{\partial u_p}{\partial r} = \frac{p}{r}u_p + \int_{-1}^{1} e^{Q}\Big[\frac{\partial a_0}{\partial r}t + \frac{\partial a_1}{\partial r}t^3 + \cdots&\\ + \frac{\partial a_n}{\partial r}t^{2n+1}\Big](1-t^2)^{p-1/2}\left(\frac{re^{i\theta}}{2}\right)^p dt.&\end{aligned} \tag{2.7}$$

In order to evaluate the latter integral, let us first integrate by parts:

$$\begin{aligned}\int_{-1}^{1} te^{Q}(1-t^2)^{p-1/2}(re^{i\theta}/2)^p dt &= -\frac{1}{2}\int_{-1}^{1} e^{Q}(1-t^2)^{p-1/2}(re^{i\theta}/2)^p d(1-t^2)\\ &= (1/(2p+1))\int_{-1}^{1} e^{Q}(\partial Q/\partial t)(1-t^2)^{p+1/2}(re^{i\theta}/2)^p dt\\ &= (1/(2p+1))\int_{-1}^{1} e^{Q}\big[a_0+3a_1t^2+\cdots\\ &\qquad +(2n+1)a_nt^{2n}\big](1-t^2)^{p+1/2}(re^{i\theta}/2)^p dt.\end{aligned}$$

Hence, by equation (2.5),

$$\begin{aligned}\int_{-1}^{1} te^{Q}(1-t^2)^{p-1/2}(re^{i\theta}/2)^p dt &= (1/(2p+1))Q_t(r,\theta,(1-T)^{1/2})Tu_p\\ &= Q_t(r,\theta,(1-T)^{1/2})\left(\int_0^{T^{1/2}} t^{2p}dt\right)T^{-p+1/2}u_p.\end{aligned}$$

Let us then assume that the formula

$$\begin{aligned}&\int_{-1}^{1} t^{2m+1}e^{Q}(1-t^2)^{p-1/2}(re^{i\theta}/2)^p dt\\ &\qquad = Q_t(r,\theta,(1-T)^{1/2})\left(\int_0^{T^{1/2}} t^{2p}(1-t^2)^m dt\right)T^{-p+1/2}u^p\end{aligned}\tag{2.8}$$

has already been verified for $m=0, 1, 2, \cdots, N$ and proceed to verify the formula for $m=N+1$, as follows.

$$\begin{aligned}\int_{-1}^{1} t^{2N+3}e^{Q}(1-t^2)^{p-1/2}(re^{i\theta}/2)^p dt &= \int_{-1}^{1} t^{2N+1}e^{Q}(1-t^2)^{p-1/2}(re^{i\theta}/2)^p dt\\ &\quad - \int_{-1}^{1} t^{2N+1}e^{Q}(1-t^2)^{p+1/2}(re^{i\theta}/2)^p dt\\ &= Q_t(r,\theta,(1-T)^{1/2})\left\{\left[\int_0^{T^{1/2}} t^{2p}(1-t^2)^N dt\right]T^{-p+1/2}u_p\right.\\ &\quad \left. - \left[\int_0^{T^{1/2}} t^{2p+2}(1-t^2)^N dt\right](2/re^{i\theta})T^{-p-1/2}u_{p+1}\right\}\\ &= Q_t(r,\theta,(1-T)^{1/2})\left(\int_0^{T^{1/2}} t^{2p}(1-t^2)^{N+1}dt\right)T^{-p+1/2}u_p.\end{aligned}$$

Thus, formula (2.8) has been established by mathematical induction.

We now substitute from formula (2.8) into expression (2.7), thus obtaining

$$\frac{\partial u_p}{\partial r}=\frac{p}{r}\,u_p+\Bigg(Q_t(r,\theta,(1-T)^{1/2})\cdot\int_0^{T^{1/2}} t^{2p}\left\{\frac{\partial a_0}{\partial r}+\frac{\partial a_1}{\partial r}(1-t^2)+\cdots+\frac{\partial a_n}{\partial r}(1-t^2)^n\right\}\Bigg)T^{-p+1/2}u_p$$

$$=\frac{p}{r}\,u_p+Q_t(r,\theta,(1-T)^{1/2})\left\{\int_0^{T^{1/2}} t^{2p}P_r(r,\theta,(1-t)^{1/2})\right\}T^{-p+1/2}u_p,$$

as was to be proved.

Proof of Theorem 1. Formula (2.2) may be reduced to formula (1.10) if, using (1.8) and (2.3), we set

$$P_r(r,\theta,(1-T)^{1/2})u_p=\sum_{j=0}^{n}\frac{\partial a_j}{\partial r}(1-T)^j u_p=\sum_{q=0}^{n}\alpha_q\left(\frac{re^{i\theta}}{2}\right)^q T^q u_p$$
$$=\sum_{q=0}^{n}\alpha_q u_{p+q}.$$

To reduce formula (2.6) to (1.11), let us set

$$Q_t(r,\theta,(1-T)^{1/2})=\sum_{j=0}^{n}(2j+1)a_j(1-T)^j=\sum_{j=0}^{n}\beta_j\left(\frac{re^{i\theta}}{2}\right)^j T^j$$

with the β_j defined as in formula (1.9). Since

$$P_r(r,\theta,(1-t^2)^{1/2})=\sum_{j=0}^{n}\alpha_j\left(\frac{re^{i\theta}}{2}\right)^j t^{2j},$$

we may write the second term of the left side of (2.6) as

$$\sum_{k=0}^{n}\beta_k\left(\frac{re^{i\theta}}{2}\right)^k T^k\left[\int_0^{T^{1/2}} t^{2p}\sum_{j=0}^{n}\alpha_j\left(\frac{re^{i\theta}}{2}\right)^j t^{2j}dt\right]T^{-p+1/2}u_p$$
$$=\sum_{j,k=0}^{n}\frac{\alpha_j\beta_k}{2p+2j+1}\left(\frac{re^{i\theta}}{2}\right)^{j+k}T^{k+j+1}u_p=\frac{2}{re^{i\theta}}\sum_{j,k=0}^{n}\frac{\alpha_j\beta_k u_{p+j+k+1}}{2p+2j+1}.$$

Thus the proof of our main theorem is completed.

3. **Examples.** Let us first verify that the recurrence relation (2.6) is a generalization of that for Bessel's functions as given in formula (1.7). Here $Q(r,\theta,t)=rti$ and thus (2.6) becomes

$$\partial u_p/\partial r = (p/r)u_p + ri\left[\int_0^{T^{1/2}} t^{2p} i\,dt\right] T^{-p+1/2}u_p \tag{3.1}$$
$$= (p/r)u_p - (2e^{-i\theta}/(2p+1))u_{p+1}.$$

If now we set

$$u_p = (\pi^{1/2}/2)\Gamma(p+1/2)J_p(r)e^{ip\theta},$$
$$u_{p+1} = (\pi^{1/2}/2)(p+1/2)\Gamma(p+1/2)J_{p+1}(r)e^{i(p+1)\theta},$$

formula (3.1) reduces at once to formula (1.7).

As our second example, let us consider the differential equation $L(U)=0$ in which the expression $F=c-ab-a_z\neq 0$ satisfies the two equations

$$F_z = 0, \qquad 2F - a_z + b_z = 0. \tag{3.2}$$

As shown by Bergman,[4] one of the possible corresponding generating functions is $E(z, \bar z, t)=\exp P(r, \theta, t)=\exp\,(a_0+a_1t^2)$, where

$$a_0 = -\int_0^z a\,d\bar z, \qquad a_1 = 2z\int_0^z F\,d\bar z. \tag{3.3}$$

According to our theorem, the recurrence relation satisfied by the basic solutions is in this case

$$(\partial u_p/\partial r) = (p/r)u_p + \alpha_0 u_p + \alpha_1 u_{p+1}, \tag{3.4}$$

where

$$\alpha_0 = \partial a_0/\partial r + \partial a_1/\partial r = -ae^{-i\theta} + 2rF + 2e^{i\theta}\int_0^z F\,d\bar z,$$

$$\alpha_1 = -(2/re^{i\theta})(\partial a_1/\partial r) = -4e^{-i\theta}F - (4/r)\int_0^z F\,d\bar z.$$

A partial differential equation which satisfies conditions (3.2) is

$$U_{z\bar z} - 2(z+\bar z)U_z + U = 0. \tag{3.5}$$

Here $F(z)=1$, $a_0=0$, $a_1=2r^2$, and therefore in the recurrence relation (3.4) $\alpha_0=4r$, and $\alpha_1=-8e^{-i\theta}$. Setting $U=U^{(1)}+iU^{(2)}$, we see that equation (3.5) is equivalent to the system of two partial differential equations

$$\Delta U^{(1)} - 8xU_x^{(1)} + 8xU_y^{(2)} + 4U^{(1)} = 0,$$
$$\Delta U^{(2)} - 8xU_x^{(2)} - 8xU_y^{(1)} + 4U^{(2)} = 0,$$

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[4] See p. 1194, reference in footnote 1a.

and that recurrence relation (3.4) for $u_p = u_p^{(1)} + iu_p^{(2)}$ is in this case equivalent to the system of recurrence relations

$$\partial u_p^{(1)}/\partial r = (p/r)u_p^{(1)} + 4ru_p^{(1)} - 8u_{p+1}^{(1)} \cos\theta - 8u_{p+1}^{(2)} \sin\theta,$$

$$\partial u_p^{(2)}/\partial r = (p/r)u_p^{(2)} + 4ru_p^{(2)} - 8u_{p+1}^{(2)} \cos\theta + 8u_{p+1}^{(1)} \sin\theta.$$

Other examples of partial differential equations $L(U)=0$ for which $\log E$ is an even or odd polynomial in t may be found in the articles referred to in footnotes 1a and 3. For these differential equations also, a recurrence relation may be derived by use of Lemmas 1 and 2.

4. **Generalization.** By means of formulas (2.5) and (2.8), the theorem given in the introduction may be extended to partial differential equations of type (1.1) for which a generating function exists that has the form $E = g \exp f$ with both f and g suitably chosen polynomials in t. The generalization may be stated as follows.

THEOREM 2. *Let $L(U)=0$ be a partial differential equation of type* (1.1) *for which a generating function $E(z, \bar{z}, t)$ exists that has one of the forms*

$$\text{I.}\quad E(z, \bar{z}, t) = R(r, \theta, t) \exp P(r, \theta, t),$$

$$\text{II.}\quad E(z, \bar{z}, t) = R(r, \theta, t) \exp tP(r, \theta, t),$$

$$\text{III.}\quad E(z, \bar{z}, t) = tR(r, \theta, t) \exp tP(r, \theta, t),$$

where

$$P(r, \theta, t) = a_0(r, \theta) + a_1(r, \theta)t^2 + \cdots + a_m(r, \theta)t^{2m},$$

$$R(r, \theta, t) = b_0(r, \theta) + b_1(r, \theta)t^2 + \cdots + b_n(r, \theta)t^{2n},$$

and where the $a_j(r, \theta)$ and $b_j(r, \theta)$ are of class C' in r and θ. Let $u_p(r, \theta)$ be the corresponding basic solutions and let $R(\partial p/\partial r) = \sum_0^{m+n} c_j(r, \theta)t^{2j}$,

$$\alpha_k = (-1)^k \left(\frac{2}{re^{i\theta}}\right)^k \sum_{j=k}^{m} (2j+1)C_{j,k}a_j;$$

$$\beta_k = (-1)^k \left(\frac{2}{re^{i\theta}}\right)^k \sum_{j=k}^{n} C_{j,k} \frac{\partial b_j}{\partial r}; \quad \gamma_k = (-1)^k \left(\frac{2}{re^{i\theta}}\right)^k \sum_{j=k}^{n+m} C_{j,k}c_j.$$

Then the recurrence relation satisfied by these basic solutions is

$$\frac{\partial u_p}{\partial r} = \frac{pu_p}{r} + \sum_{k=0}^{n} \beta_k u_{p+k} + \sum_{k=0}^{n+m} \gamma_k u_{p+k}$$

if E has the form I;

$$\frac{\partial u_p}{\partial r} = \frac{pu_p}{r} + \sum_{k=0}^{n} \beta_k u_{p+k} + \left(\frac{2}{re^{i\theta}}\right) \sum_{k=0}^{m} \sum_{\nu=0}^{m+n} \frac{\alpha_k \gamma_\nu u_{p+k+\nu+1}}{2p+2\nu+1}$$

if E has the form II; *and*

$$\frac{\partial u_p}{\partial r} = \frac{pu_p}{r} + \left(\frac{2}{re^{i\theta}}\right) \sum_{k=0}^{m} \sum_{\nu=0}^{n} \left(\frac{\alpha_k \beta_\nu}{2p+2\nu+1}\right) u_{p+k+\nu+1}$$
$$+ \gamma_0 u_p + \sum_{k=1}^{m+n} \left(\gamma_k - \frac{2}{re^{i\theta}} \gamma_{k-1}\right) u_{p+k}$$

if E has the form III.

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ON GAUSS' AND TCHEBYCHEFF'S QUADRATURE FORMULAS

J. GERONIMUS

The well known Gauss' Quadrature Formula

$$(1) \qquad \int_{-\infty}^{\infty} G_k(x) d\psi(x) = \sum_{i=1}^{n} \rho_i^{(n)} G_k(\xi_i^{(n)})$$

is valid for every polynomial $G_k(x)$, of degree $k \leqq 2n-1$, the $\{\xi_i^{(n)}\}$ being the roots of the polynomial $P_n(x)$, orthogonal with respect to the distribution $d\psi(x)$ $(i=1, 2, \cdots, n;\ n=1, 2, \cdots)$.[1] If the sequence $\{P_n(x)\}$ is that of Tchebycheff (trigonometric) polynomials, then the Christoffel numbers $\rho_i^{(n)}$, $i=1, 2, \cdots, n$, are equal, and the two quadrature formulas of Gauss and Tchebycheff coincide:

$$(2) \qquad \int_{-\infty}^{\infty} G_k(x) d\psi(x) = \rho_n \sum_{i=1}^{n} G_k(\xi_i^{(n)}), \qquad k \leqq 2n-1;\ n = 1, 2, \cdots.$$

The converse—that this is the only case of coincidence of these formulas—was proved by R. P. Bailey [1a] and, under more restrictive conditions, by Krawtchouk [1b] (cf. also [2]).[2]

We shall give here four distinct proofs of this statement, without imposing any restrictions on $\psi(x)$.

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[1] $\psi(x)$ is a bounded non-decreasing function, with infinitely many points of increase, for which all moments exist: $c_n = \int_{-\infty}^{\infty} x^n d\psi(x)$; $n = 0, 1, 2, \cdots$.

[2] Numbers in brackets refer to the bibliography at the end of the paper.

Consider the sequence of complex numbers $\{c_n\}$, $n=0, 1, 2, \cdots$, subject to the conditions

$$\Delta_{n+1} \equiv \left| c_{i+k} \right|_{i,k=0}^{n} \neq 0; \qquad n = 0, 1, 2, \cdots . \tag{3}$$

Consider also the Stieltjes linear functional σ, with

$$\sigma(x^k) = c_k; \qquad k = 0, 1, 2, \cdots , \tag{4}$$

and the polynomials $\{P_n(x)\}$, $n=0, 1, 2, \cdots$, orthogonal relative to the sequence $\{c_n\}$, that is [3, 4]

$$\sigma\{P_n(x)P_m(x)\} = \begin{cases} 0, & m \neq n, \\ h_n \equiv \Delta_{n+1}/\Delta_n \neq 0, & m = n. \end{cases} \tag{5}$$

Let $\{\xi_i^{(n)}\}$, $i=1, 2, \cdots, n$, be the roots, distinct or not, of $P_n(x)$, $n=1, 2, \cdots$.

I. The first method of proving our statement consists in proving the following theorem.

THEOREM. *From the validity of the formula*

$$p_n \sum_{i=1}^{n} \left(\xi_i^{(n)}\right)^k = c_k, \qquad n \geqq \left[\frac{k+1}{2}\right], \tag{6}$$

for $k \leqq 2$, *it follows that the* $\{P_n(x)\}$ *are the Tchebycheff polynomials, so that* (6) *holds for all integral* k.

Introduce the mean of order ν of the numbers $\{\xi_i^{(n)}\}$, $i=1, 2, \cdots, n$:

$$\mu_\nu^{(n)} = \left[\frac{1}{n} \sum_{i=1}^{n} \left(\xi_i^{(n)}\right)^\nu\right]^{1/\nu}, \qquad \nu = 1, 2, \cdots ; n = 1, 2, \cdots . \tag{7}$$

On equating the coefficients of x^{n-1} and x^{n-2} on both sides of the recurrence relation

$$\begin{aligned} P_n(x) = (x - \alpha_n)P_{n-1}(x) - \lambda_n P_{n-2}(x), \\ \lambda_n \neq 0, \quad P_{-1} \equiv 0, n = 1, 2, \cdots , \end{aligned} \tag{8}$$

we get

$$\delta_1^{(n)} = - \sum_{k=1}^{n} \alpha_k, \qquad \delta_2^{(n)} = - \sum_{k=2}^{n} \lambda_k + \sum_{r=2}^{n} \alpha_r \sum_{k=1}^{r-1} \alpha_k, \tag{9}$$

where we let

$$P_n(x) = x^n + \delta_1^{(n)} x^{n-1} + \delta_2^{(n)} x^{n-2} + \cdots + \delta_n^{(n)}. \tag{9'}$$

On putting $k=0, 1, 2$ in (6) we obtain

$$\rho_n = c_0/n; \quad \mu_1^{(n)} = c_1/c_0, \ n \geqq 1; \quad \mu_2^{(n)} = c_2/c_0, \ n \geqq 2. \tag{10}$$

Since, by (9) and (9′),

$$\begin{aligned} n\mu_1^{(n)} &= -\delta_1^{(n)} = \sum_{k=1}^{n} \alpha_k, \\ n(\mu_2^{(n)})^2 &= (\delta_1^{(n)})^2 - 2\delta_2^{(n)} = \left(\sum_{k=1}^{n} \alpha_k\right)^2 + 2\sum_{k=1}^{n} \lambda_k - 2\sum_{r=2}^{n}\sum_{k=1}^{r-1} \alpha_k, \end{aligned} \tag{11}$$

we find:

$$\alpha_1 = (\alpha_1 + \alpha_2)/2 = (\alpha_1 + \alpha_2 + \alpha_3)/3 = \cdots ,$$

whence we have the fundamental result:

$$\alpha_1 = \alpha_2 = \cdots = \alpha, \tag{12}$$

and similarly,

$$\lambda_3 = \lambda_4 = \cdots = \lambda_2/2 = \lambda. \tag{12′}$$

Here α and λ are independent of n. The solution of (8) under these conditions is

$$\begin{aligned} P_n(x) = [(x - \alpha + ((x - \alpha)^2 - 4\lambda)^{1/2})^n \\ + (x - \alpha - ((x - \alpha)^2 - 4\lambda)^{1/2})^n]/2^n, \quad n = 1, 2, \cdots , \end{aligned} \tag{13}$$

which shows that the $\{P_n(x)\}$ are the Tchebycheff polynomials, whence the validity of (6) for $k=3, 4, \cdots$ follows.

We have found incidentally the following property of the Tchebycheff polynomials.

COROLLARY. *If the arithmetic mean and the root-mean-square of the roots* $\{\xi_i^{(n)}\}$, $i=1, 2, \cdots, n$, *of the orthogonal polynomials* $P_n(x)$, $n=1, 2, \cdots$, *do not depend on* n, *the means of all orders* $\nu \leqq 2n-1$ *possess the same property, and the* $\{P_n(x)\}$ *are the Tchebycheff polynomials* (13).

II. Define

$$R_{n-1}(y) = \sigma\{(P_n(y) - P_n(x))/(y - x)\},$$

a polynomial of degree $n-1$; $n=1, 2, \cdots$. Our second proof consists in showing that (6) implies

$$R_{n-1}(y) \equiv (c_0/n)P_n'(y), \qquad n \geqq 2. \tag{14}$$

From (6), wherein ρ_n necessarily equals c_0/n, we get, by virtue of (4),

$$\sigma(x^k) = \frac{c_0}{n}\sum_{i=1}^{n}(\xi_i^{(n)})^k; \quad \sigma(G_k(x)) = \frac{c_0}{n}\sum_{i=1}^{n}G_k(\xi_i^{(n)}), \qquad n \geqq \left[\frac{k+1}{2}\right].$$

The desired relation

$$R_{n-1}(y) = \frac{c_0}{n}\sum_{i=1}^{n}\frac{P_n(y) - P_n(\xi_i^{(n)})}{y - \xi_i^{(n)}} = \frac{c_0}{n}P_n(y)\sum_{i=1}^{n}\frac{1}{y - \xi_i^{(n)}} \equiv \frac{c_0}{n}P_n'(y)$$

follows. Now there are different methods of showing that (14) implies (13).

II$_1$. This statement is proved in a note [2] as a particular case of more general theorems.

II$_2$. Introduce the polynomials $\{P_n^{(i)}(x)\}$, $i=1, 2, \cdots, n$—the denominators of the convergents of order n of the continued fractions[3]

$$(15) \qquad \frac{\lambda_i|}{|x - \alpha_i} - \frac{\lambda_{i+1}|}{|x - \alpha_{i+1}} - \cdots, \qquad i = 1, 2, \cdots.$$

We have $P_n^{(1)}(x) \equiv P_n(x)$; $P_n^{(2)}(x) \equiv (1/c_0)R_n(x)$, $n=0, 1, 2, \cdots$.

It is easy to show that

$$(16) \qquad P_n(x) = (x - \alpha_1)P_{n-1}^{(2)}(x) - \lambda_2 P_{n-2}^{(3)}(x), \qquad n = 2, 3, \cdots.$$

On the other hand

$$(17) \qquad R_{n-1}(x) = (x - \alpha_n)R_{n-2}(x) - \lambda_n R_{n-3}(x), \qquad n = 3, 4, \cdots.$$

Using (8), (14) and (17), we find that

$$(18) \qquad P_{n-1}(x) = (x - \alpha_n)P_{n-2}^{(2)}(x) - 2\lambda_n P_{n-3}^{(2)}(x), \qquad n = 3, 4, \cdots,$$

which gives, in conjunction with (16),

$$(19) \qquad (\alpha_1 - \alpha_n)P_{n-2}^{(2)}(x) = 2\lambda_n P_{n-3}^{(2)}(x) - \lambda_2 P_{n-3}^{(3)}(x), \qquad n = 3, 4, \cdots.$$

Hence

$$(20) \qquad \begin{gathered} \alpha_3 = \alpha_4 = \cdots = \alpha_1 = \alpha; \qquad \lambda_3 = \lambda_4 = \cdots = \lambda_2/2 = \lambda; \\ P_{n-3}^{(2)}(x) \equiv P_{n-3}^{(3)}(x). \end{gathered}$$

This identity, for $n=4$, gives $\alpha_2=\alpha$, and thus we have arrived again at (12, 12′).

II$_3$. On putting

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[3] They were introduced by Stieltjes [5]; cf. also Perron [6].

$$R_n(y) = c_0(y^n + d_1^{(n)} y^{n-1} + \cdots + d_n^{(n)}), \tag{21}$$

we find from (17) the relations

$$d_1^{(n-1)} = -\sum_{k=2}^{n} \alpha_k, \qquad d_2^{(n-1)} = \sum_{k=3}^{n} \alpha_k \sum_{r=2}^{k-1} \alpha_r - \sum_{k=3}^{n} \lambda_k \tag{22}$$

analogous to (9). From (14) we find

$$n d_1^{(n-1)} = (n-1)\delta_1^{(n)}, \tag{23}$$

whence again $\alpha_1 = \alpha_2 = \cdots = \alpha$; and in the same way, the condition

$$n d_2^{(n-1)} = (n-2)\delta_2^{(n)} \tag{24}$$

again implies $\lambda_3 = \lambda_4 = \cdots = \lambda_2/2 = \lambda$.

We have found incidentally the following property of the Tchebycheff polynomials.

COROLLARY. *If the three highest coefficients of the polynomials* $\{nR_{n-1}(x)\}$ *and* $\{c_0 P_n'(x)\}$, $n = 1, 2, \cdots$, *coincide, then these polynomials are identical, and the* $\{P_n(x)\}$ *are the Tchebycheff polynomials* (13).

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POWERS OF HOMEOMORPHISMS WITH ALMOST PERIODIC PROPERTIES

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Let X be a topological space (an "accessible space," a "1-space," or a "T_1-space" in the terminology of Fréchet, Kuratowski, or Alexandroff-Hopf, respectively) and let $f(X)=X$ be a homeomorphism. We use the following terminology, which was suggested by G. A. Hedlund and which is to be carefully distinguished from those terminologies used by Birkhoff, Ayres, Whyburn, and others. A point x of X is said to be *recurrent* under f provided that to each neighborhood U of x there corresponds a positive integer n such that $f^n(x)\in U$. The mapping f is said to be *pointwise recurrent* provided that each point of X is recurrent under f. A point x of X is said to be *almost periodic* under f provided that to each neighborhood U of x there corresponds a monotone increasing sequence $n_1, n_2, \cdots$ of positive integers with the properties that the numbers $n_{i+1}-n_i$ $(i=1, 2, \cdots)$ are uniformly bounded and $f^{n_i}(x)\in U$ $(i=1, 2, \cdots)$. The mapping f is said to be *pointwise almost periodic* provided each point of X is almost periodic under f. Following Birkhoff [1, p. 198],[2] a subset Y of X is said to be *minimal* under f provided that Y is nonvacuous, closed and invariant under f, that is, $f(Y)=Y$, and furthermore Y does not contain a proper subset with these properties. For $x\in X$, the set $\sum_{n=-\infty}^{+\infty}f^n(x)$ is called the *orbit* of x under f and the set $\sum_{n=0}^{+\infty}f^n(x)$ is called the *semi-orbit* of x under f. A *decomposition* of X is a collection of nonvacuous pairwise disjoint closed subsets of X which fill up X.

THEOREM 1. *If $x\in X$ is recurrent under f, then x is also recurrent under f^n for every positive integer n.*

PROOF. We make use of an induction. The theorem is true for $n=1$. Let m be any positive integer. Assume the theorem is true for $n\leqq m$. We now show the theorem is true for $n=m+1=k$.

We may suppose without loss of generality that X is the closure of the semi-orbit of x under f, for this set is invariant under f. Define X_i $(i=0, 1, \cdots, k)$ to be the closure of the semi-orbit of $f^i(x)$ under f^k. It is readily verified that $f(X_i)=X_{i+1}$ $(i=0, 1, \cdots, k-1)$,

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[1] I wish to thank Professor G. A. Hedlund for his genial interest in the development of these results.

[2] Numbers in brackets refer to the bibliography at the end of the paper.

$f^k(X_i)\subset X_i$ ($i=0, 1, \cdots, k$), and $X=\sum_{i=0}^{k-1}X_i$. We may suppose that $x\in\sum_{i=1}^{k-1}X_i$, for otherwise the conclusion follows. Let p, $1\leqq p\leqq k-1$, be the smallest integer such that $x\in X_p$. Then, the semi-orbit of x under f^k is contained in X_p. Hence, $X_0\subset X_p=f^p(X_0)$ and p is the smallest positive integer such that $X_0\subset f^p(X_0)$. Since $f^k(X_0)\subset X_0$, there exists a smallest positive integer t such that $f^t(X_0)\subset X_0$. Now $f^t(X_0)\subset X_0\subset f^p(X_0)$. Since $f^{t-p}(X_0)\subset X_0$ and $X_0\subset f^{p-t}(X_0)$, p is neither less than nor greater than t. Hence, $p=t$ and $f^p(X_0)=X_0$. Write $k=pq+r$, $0\leqq r<p$, where q and r are integers. Now

$$X_0\supset f^k(X_0)=f^r(f^{pq}(X_0))=f^r(X_0).$$

Thus, $r=0$ and $k=pq$. If $p=1$, then $X_0=X_1=\cdots=X_k$ whence $x\in X_k$ and the conclusion follows. We may suppose, therefore, that $p>1$. Now $p\leqq m$ and $q\leqq m$. By the induction assumption, x is recurrent under f^p and, applying the induction assumption to f^p, x is recurrent under $(f^p)^q=f^k$.

COROLLARY 1. *Every positive power of a pointwise recurrent homeomorphism on a topological space is itself pointwise recurrent.*

Theorem 1 can be used to provide a different proof of the following theorem, due to Birkhoff and Smith [2, p. 358, Theorem 3].

THEOREM. *If X is a compact metric space and if $f(X)=X$ is a homeomorphism, then for every nonzero integer n the central orbits under f^n are identical with the central orbits under f.*

This follows from Theorem 1 and the result, due to Birkhoff and Smith [2, p. 353, Theorem 2], that the sum of the central orbits under a homeomorphism h on a compact metric space is characterized as the closure of the set of points recurrent under both h and h^{-1}. Although their results on central orbits [2, pp. 350–355, 356–360] are stated for closed surfaces, their proofs are actually valid for compact metric spaces.

THEOREM 2. *If X is a compact connected metric space and if the recurrent points are dense in X, then every recurrent cut point x of X is periodic.*

PROOF. Express $X=A+B$, where A and B are nondegenerate continua such that $A\cdot B=x$. By a theorem due to Kelley [4, p. 194, Theorem 3.4] there exists an F-set (that is, either a simple link, or cut point, or end point) F of X such that $f(F)=F$. (For properties of simple links, see Whyburn [5, pp. 64–65].) Now F is contained in either A or B, say A, and $A \cdot f^n(A) \neq \Lambda$ for every integer n. Since some point of $B-x$ is recurrent under f, there exists a positive integer n such that $B \cdot f^n(B) \neq \Lambda$. By Theorem 1, x is recurrent under f^n. Applying a lemma due to Whyburn [5, p. 247, Lemma 4.21], it follows that $f^n(x)=x$. The proof is completed.

Theorem 2 and its proof are partial generalizations of Whyburn [5, p. 248, Theorem 4.6], but the original conclusion—that the mapping is elementwise periodic on all simple links—is no longer valid without semi-local connectedness, even though the mapping be regularly almost periodic in the sense of Whyburn [5, p. 250]. Theorem 1, however, may be used as an aid in the proof of the cited theorem.

Remarks. 1. If the subset Y of X is minimal under f, then Y is minimal also under f^{-1}. 2. A nonvacuous subset Y of X is minimal if and only if the closure of the orbit of every point of Y is Y. 3. The collection of sets minimal under f is a decomposition of X if and only if the closure of the orbits under f is a decomposition of X; and, in either case, these two collections coincide. In other words, f *gives a minimal-set decomposition* if and only if f *gives an orbit-closure decomposition.*

THEOREM 3. *If X is minimal under f but not under f^k, where k is a nonzero integer, then there exists an integer n, $n>1$, such that n divides $|k|$ and f^n gives a finite minimal-set decomposition which contains exactly n elements.*

PROOF. By Remark 1, it is sufficient to prove the theorem when k is positive. There exists a point x of X such that the orbit of x under f^k is not dense in X, by Remark 2. Define X_i $(i=0, 1, \cdots, k-1)$ to be the closure of the orbit of $f^i(x)$ under f^k. Clearly, $f(X_i)=X_{i+1}$ $(i=0, 1, \cdots, k-2)$, $f(X_{k-1})=X_0$, and $f^k(X_i)=X_i$ $(i=0, 1, \cdots, k-1)$. Let p be the maximum positive integer such that there exist integers $i_1, i_2, \cdots, i_p$ with the properties that $0 \leqq i_1 < i_2 < \cdots < i_p \leqq k-1$ and $\prod_{j=1}^{p} X_{i_j} \neq \Lambda$. Choose integers $i_1, i_2, \cdots, i_p$ with these properties. Define $Y=\prod_{j=1}^{p} X_{i_j}$. Clearly, $f^k(Y)=Y$. Let n be the smallest positive integer such that $f^n(Y)=Y$. Define $Y_j=f^j(Y)$ $(j=0, 1, \cdots, n-1)$. The sets Y_j $(j=0, 1, \cdots, n-1)$ are closed and pairwise disjoint. Choose $y \in Y$. Then,

$$X = \overline{{}_j\sum_{-\infty}^{+\infty} f^j(y)} = \overline{{}_j\sum_{-\infty}^{+\infty} f^j(Y)} = \sum_{j=0}^{n-1} f^j(Y) = \sum_{j=0}^{n-1} Y_j.$$

Thus, $D \equiv [Y_j \mid j=0, 1, \cdots, n-1]$ is a decomposition of X.

We show $n>1$. Suppose $n=1$. Then, $X=Y_0=Y=X_{i_1}$, and thus

$X = f^{k-i_1}(X_{i_1}) = f^k(X_0) = X_0$ whence the orbit of x under f^k is dense in X, contrary to the second statement of the proof.

We show that n divides k. Write $k = qn + r$, $0 \leqq r < n$, where q and r are integers. Then,

$$Y = f^k(Y) = f^r(f^{qn}(Y)) = f^r(Y).$$

Hence, $r = 0$.

In order to show that each element Y_j of D is minimal under f^n, it is sufficient to observe that for $y \in Y_j$, the orbit of y under f is dense in X, D is a decomposition of X whose elements are invariant under f^n, and the subset of the orbit of y under f which is contained in Y_j is actually the orbit of y under f^n.

COROLLARY 2. *If X is connected and minimal under f, then X is also minimal under f^n for every nonzero integer n.*

COROLLARY 3. *If X has only finitely many, say k, components and if X is minimal under f, then for every nonzero integer n the mapping f^n gives a finite minimal-set decomposition, the number of whose elements is the greatest common divisor of k and $|n|$.*

If $k = 1$, Corollary 3 reduces to Corollary 2. If $k > 1$, Corollary 3 may be proved by first of all considering the case when $k = k'$ and $n = n' > 0$ are relatively prime and then extending the result to $k = ak'$ and $n = an'$, where a is any positive integer. Corollary 3 essentially combines Corollary 2 with a property of cyclic counting or, what is the same, a property of periodic orbits.

THEOREM 4. *If X is minimal under f, then for every nonzero integer n the mapping f^n gives a finite minimal-set decomposition of X into at most $|n|$ elements.*

PROOF. By Remark 1, it is sufficient to prove the theorem when n is positive. We make use of an induction. The theorem is true for $n = 1$. Let m be any positive integer. Assume the theorem is true for $n \leqq m$. We now show the theorem is true for $n = m + 1 = k$.

If X is minimal under f^k, the conclusion follows. Suppose now that X is not minimal under f^k. By Theorem 3, there exist integers p and q such that $p > 1$, $k = pq$, and f^p gives a finite minimal-set decomposition D of X into exactly p elements. Let Y be any element of D. Now apply the induction assumption to $f^p(Y) = Y$ and $n = q \leqq m$. Thus $(f^p)^q = f^k$ gives a finite minimal-set decomposition of Y into at most q elements. Hence, $(f^p)^q = f^k$ gives a finite minimal-set decomposition of X into at most $pq = k$ elements.

COROLLARY 4. *If the mapping f gives an orbit-closure decomposition of X, then for every integer n the mapping f^n also gives an orbit-closure decomposition of X.*

PROOF. By virtue of Remark 3, it is sufficient to apply Theorem 4 to the elements of the orbit-closure decomposition given by f.

LEMMA 1. *If X is a metric space and if $x \in X$ is almost periodic under f, then the closure Y of the orbit of x under f is minimal under f.*

PROOF. Suppose Y is not minimal. Then, there exists a nonvacuous closed invariant subset Z of Y such that $x \notin Z$. Choose $z \in Z$. Let 2ϵ be the distance from x to Z. There exists a positive integer N such that in every set of N consecutive positive integers appears an integer n so that $\rho(x, f^n(x)) < \epsilon$, where ρ is the metric in X. Choose $\delta > 0$ so small that $x' \in X$ with $\rho(z, x') < \delta$ implies $\rho(f^i(z), f^i(x')) < \epsilon$ $(i = 1, 2, \cdots, N)$. There exists an integer $p \geqq 0$ such that $\rho(z, f^p(x)) < \delta$. Also it is possible to find an integer q, $1 \leqq q \leqq N$, so that $\rho(x, f^{p+q}(x)) < \epsilon$. Furthermore, $\rho(f^q(z), f^{p+q}(x)) < \epsilon$. Hence, $\rho(x, f^q(z)) < 2\epsilon$ which is impossible because $f^q(z) \in Z$.

LEMMA 2. *If X is a compact metric space and if f gives an orbit-closure decomposition, then f is pointwise almost periodic.*

PROOF. Suppose that some point x of X is not almost periodic. Then there exist a neighborhood U of x and a sequence $m_1, m_2, \cdots$ of positive integers such that $U \cdot \sum_{j=-i}^{+i} f^{m_i + j}(x) = \Lambda$ $(i = 1, 2, \cdots)$. We may suppose that the sequence $\{f^{m_i}(x)\}$ converges to some point, say y, of X. It is easy to show that the orbit of y is contained in $X - U$. Hence, the closure of the orbit of y is a proper subset of the closure of the orbit of x. This is impossible.

THEOREM 5. *If X is a metric space, then in order that f give an orbit-closure decomposition it is sufficient that f be pointwise almost periodic; and in case X is compact, this condition is also necessary.*

The proof follows easily from Lemmas 1 and 2 and Remark 3. Theorem 5 and Lemmas 1 and 2 are closely related to Hall and Kelley [3, p. 628, Theorem 4] and to Birkhoff [1, p. 199].

THEOREM 6. *Every power (including negative powers) of a pointwise almost periodic homeomorphism on a compact metric space is itself pointwise almost periodic.*

The proof follows readily from Theorem 5 and Corollary 4.

THEOREM 7. *If X is a compact metric space and if $x \in X$ is almost* *periodic under f, then x is also almost periodic under* f^n *for every integer n.*

The proof proceeds easily from Lemma 1, Remark 2, and Theorems 5 and 6.

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SOME PROPERTIES OF SUMMABILITY. II[1]

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1. Summability of bounded sequences. It follows from a well known result of H. Steinhaus[2] that no regular matrix method of summability can be effective for (that is, assign a finite limit to) every element in the space (m) of bounded sequences. The object of this note is to consider some questions suggested by this fact. The first of these may be formulated as follows. If A is a given regular matrix method let J_A denote the set of all A-summable bounded sequences. We then ask what are necessary and sufficient conditions on a subset E of (m) in order that there exist a regular A such that $E \subset J_A$? In Theorem 1 below it is shown that the separability of E is a sufficient condition. It seems unlikely that this condition is necessary although we have been unable to decide the question. It is clearly equivalent to the question of whether every J_A is separable.

THEOREM 1. *Let E be an arbitrary separable subset of (m). Then every regular matrix $A = (a_{mk})$ contains a (necessarily regular) row-submatrix $B = (a_{m_i k})$ such that $E \subset J_B$.*

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[1] This note is in the nature of an appendix to the paper cited in footnote 4.

[2] H. Steinhaus, *Some remarks on the generalizations of the notion of limit* (in Polish), Prace Matematyczno-Fizyczne vol. 22 (1921) pp. 121–134. See also I. Schur, *Über lineare Transformationen in der Theorie der unendlichen Reihen*, J. Reine Angew. Math. vol. 151 (1921) pp. 79–111.

PROOF. The proof depends on an application of the so-called diagonal process. Let D denote a dense denumerable subset of E of points $x_n \equiv \{s_k^n\}$ for $n=1, 2, 3, \cdots$, and let $A=(a_{mk})$ be an arbitrary regular matrix. By the Silverman-Toeplitz conditions the sequence $\{t_m^1 \equiv \sum_k a_{mk} s_k^1\}$ is bounded and therefore contains a convergent subsequence $\{t_{m_i}^1\}$, where we may suppose that $m_1=1$. Let $a_{m_i k} \equiv a_{ik}^1$ for all i and k, and select a convergent subsequence $\{t_{i_j}^2\}$ from the bounded sequence $\{t_i^2 \equiv \sum_k a_{ik}^1 s_k^2\}$ so that $i_1=1$ and $i_2=2$. Next let $a_{i_j k}^1 \equiv a_{jk}^2$ for all j and k, proceed to $\{s_k^3\}$, and so forth. We define in this way a sequence of regular matrices (a_{mk}), (a_{mk}^1), (a_{mk}^2), $\cdots$, (a_{mk}^n), $\cdots$, each a row-submatrix of the preceding. The "diagonal matrix" $(a_{mk}^m) \equiv B$ is evidently regular and by its construction it is clearly effective for each x_n in D. To show that B is also effective in $E-D$ we first observe that the operation $y=B(x)$, where $B(x) \equiv \{\sum_k a_{mk}^m s_k\}$ and $x \equiv \{s_k\}$, is linear on (m) to (m). Then if x_0 is any point of $E-D$ there exists a subsequence $\{x_{n_i}\}$ of D that converges to x_0. Setting $y_i \equiv B(x_{n_i})$ we have $y_0 \equiv \lim_i y_i = \lim_i B(x_{n_i}) = B(x_0)$. Since y_i belongs to (c), a closed subset of (m), it follows that y_0 belongs to (c). This completes the proof.

2. **A property of reversible methods.** Let us first recall the following definitions. The method A corresponding to the matrix (a_{mk}) is said to be *reversible*[3] if the system of equations

$$(2.1) \qquad t_m = \sum_{k=1}^{\infty} a_{mk} s_k \qquad (m = 1, 2, 3, \cdots)$$

has a unique solution $\{s_k\}$ corresponding to each convergent sequence $\{t_m\}$. The method A is said to be of[3] *type M* if the conditions

$$\sum_{m=1}^{\infty} |u_m| < \infty, \qquad \sum_{m=1}^{\infty} u_m a_{mk} = 0 \qquad (k = 1, 2, 3, \cdots)$$

always imply $u_m=0$ $(m=1, 2, 3, \cdots)$. Finally, every method A which is simultaneously regular, reversible, and of type M, is said to be *perfect*[3].

We now ask if every regular method of summability which is stronger than convergence is necessarily effective for some *unbounded* sequences. For the class of reversible matrix methods an affirmative answer is supplied by the following theorems. These theorems bring to light the fact that while perfect methods of summability may be equivalent to convergence, reversible regular methods not of type M are always stronger than convergence.

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[3] S. Banach, *Théorie des opérations linéaires*, Warsaw, 1932, p. 90.

THEOREM 2. *Every perfect method of summability which is stronger than convergence is effective for some unbounded sequences.*

PROOF. Let $A=(a_{mk})$ be an arbitrary perfect matrix, stronger than convergence, and let F_A denote the set of all A-summable sequences. Then the transformation (2.1) of F_A into (c) has an inverse of the form[4]

$$s_k = \sum_{m=0}^{\infty} A_{km}t_m \qquad (k = 1, 2, 3, \cdots), \tag{2.2}$$

where $t_0 \equiv \lim_m t_m$ and $\sum_{m=0}^{\infty}|A_{km}| < \infty$ $(k=1, 2, 3, \cdots)$. If we let $x\equiv\{s_k\}$ and $y\equiv\{t_m\}$ we may express (2.1) as $y=A(x)$ and (2.2) as $x=A^{-1}(y)$.

Since A is of type M it follows from a lemma of Banach[5] that as x ranges over (c) the set G_A of all points $y=A(x)$ is dense in (c).

To give an indirect proof we assume now that $F_A\subset(m)$, that is, that every A-summable sequence is bounded. As a consequence it follows easily that the operation $x=A^{-1}(y)$ is then linear on (c) to F_A. Let x_0 be an arbitrary point of F_A and set $y_0=A(x_0)$. Since y_0 is in (c) there exists, by the foregoing remark, a sequence $\{y_n\}$ in G_A, where $y_n=A(x_n)$ for certain x_n in (c), such that $y_n\rightarrow y_0$. But then $x_n=A^{-1}(y_n)\rightarrow A^{-1}(y_0)=x_0$, and since (c) is complete we see that x_0 belongs to (c). Thus A is equivalent to convergence, and this contradiction completes the proof.

THEOREM 3. *Every reversible regular method A which is not of type M is effective for some unbounded sequences.*

PROOF. As x ranges over (m) we denote by H_A the set of all points $y=A(x)$. By an easy extension of the previously mentioned lemma of Banach, it has been shown[6] that A is of type M if and only if the points $z_n\equiv\{\delta_k^n\}$ for $n=1, 2, 3, \cdots$ (where δ_k^n is the Kronecker symbol) are points of $\overline{H}_A$, the closure of H_A. Hence if A is not of type M at least one of the points z_n, say z_r, is not in $\overline{H}_A$. Since A is reversible and z_r belongs to (c), the equation $A(x)=z_r$ has a unique solution $x=x_r$. It is clear that x_r can not be a point of (m) and this fact establishes the theorem.

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[4] See J. D. Hill, *Some properties of summability*, Duke Math. J. vol. 9 (1942) pp. 373–381; in particular, p. 376.

[5] See footnote 3, p. 93, Lemma 2.

[6] See J. D. Hill, *On perfect methods of summability*, Duke Math. J. vol. 3 (1937) pp. 702–714; in particular p. 704, Theorem 4.

An alternative proof of this theorem may be given as follows. Under the assumed hypotheses on A we conclude from known theorems that[7] there exists a regular method B, not weaker than A, with which A is inconsistent; and that[8] A and B are consistent for bounded sequences. Hence the inconsistency must occur for unbounded sequences.

3. **Methods effective for no bounded divergent sequences.** As our final question we ask if there exist regular methods, stronger than convergence, which are effective for *no* bounded divergent sequences. An example will suffice to show that the answer is affirmative.

Example. We define matrices $B=(b_{mk})$ and $C=(c_{mk})$ as follows: $b_{11}=b_{k+1,k}=1$ $(k=1, 2, 3, \cdots)$, and $b_{mk}=0$ otherwise; $c_{mm}=1/m$ $(m=1, 2, 3, \cdots)$, and $c_{mk}=0$ otherwise. The matrix B is a regular matrix that leaves invariant the limit points of each bounded sequence. The matrix C, on the other hand, assigns the limit 0 to every bounded sequence. Consequently, the sum matrix $A\equiv B+C$ leaves invariant the limit points of each bounded sequence, and is therefore effective for no bounded divergent sequence. One easily verifies, however, that A assigns the limit 0 to the unbounded sequence $\{(-1)^{k-1}k!\}$.

The matrix A of this example is a simple instance of a Raff matrix,[9] namely, a matrix that leaves invariant the limit points of each bounded sequence. Every Raff matrix which is stronger than convergence will therefore possess the property in question, for example, every reversible Raff matrix which is not of type M (see Theorem 3 above).

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[7] See footnote 4, p. 380, Theorem 5.

[8] See footnote 3, p. 95, Théorème 11.

[9] Hermann Raff, *Zur Theorie der lineare Transformationen*, Math. Zeit. vol. 37 (1933) pp. 572–577.

THE TRANSFORMATION OF ČECH

V. G. GROVE

1. **Introduction.** The purpose of this paper is to give a simple construction of the general transformation of Čech [1, p. 192].[1]

Let the differential equations of a surface S be written in the Fubini canonical form [2, p. 123]

$$(1)\qquad \begin{aligned} x_{uu} &= \theta_u x_u + \beta x_v + p x, \\ x_{vv} &= \gamma x_u + \theta_v x_v + q x, \end{aligned} \qquad \theta = \log (\beta\gamma).$$

Let the differential equation defining a conjugate net N on S be written in the form

$$(2)\qquad dv^2 - \lambda^2 du^2 = 0.$$

The ray and the associate ray intersect in *the canonical point* [3, p. 7] of N. The line joining the point x to the canonical point intersects the reciprocal of the Green-Fubini projective normal in a point whose coordinates are

$$(3)\qquad (\beta/\lambda^2) x_u - \gamma\lambda^2 x_v.$$

We shall call this point *the conjugal point* of N at x.

2. **Conjugal quadrics.** Let the coordinates X of a point X be written in the form

$$X = x_1 x + x_2 x_u + x_3 x_v + x_4 x_{uv}.$$

Then with properly selected unit point, (x_1, x_2, x_3, x_4) are the coordinates of X referred to the tetrahedron (x, x_u, x_v, x_{uv}). The equation of the three-parameter family of quadrics each of which has second order contact [2, p. 142] with S at x is

$$(4)\qquad x_2 x_3 + x_4(-x_1 + k_2 x_2 + k_3 x_3 + k_4 x_4) = 0.$$

The equation of any plane through the conjugal point (3) is

$$(5)\qquad x_1 - k(\gamma\lambda^2 x_2 + (\beta/\lambda^2) x_3) - 2l x_4 = 0.$$

We shall speak of this plane as *the conjugal plane* of N at x.

If we impose the condition that the polar plane of the covariant point (0, 0, 0, 1) with respect to the quadric (4) be the conjugal plane

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[1] Numbers in brackets refer to the references cited at the end of the paper.

(5), we find that the equation of the quadric (4) assumes the form

$$(6)\qquad x_2x_3 + x_4[-x_1 + k(\gamma\lambda^2 x_2 + (\beta/\lambda^2)x_3) + lx_4] = 0.$$

We shall call the quadric so determined *a conjugal quadric* of S at x with respect to the conjugate net N. There is a two-parameter family of such quadrics associated with a given conjugate net. It follows from (6) that *a given conjugal quadric may also serve as the conjugal quadric of the associate conjugate net of N only if it is a quadric of Darboux.* Such a quadric serves as the conjugal quadric of any conjugate net.

Davis has defined [3, p. 12] a pencil of quadrics by demanding that the tangents to the curves of intersection of a quadric of the family (4) with S be apolar to the tangents to the curves of a conjugate net. *The quadric of Davis for the conjugate net defined by* (2) *is the conjugal quadric* (6) *with* $k=1$. The quadric of Davis for the associate conjugate net of N is the conjugal quadric (6) with $k=-1$. We may call this quadric the *conjugate quadric* of Davis.

If one demands that a quadric of the family (4) have third order contact with each of the curves of N, the quadric must be a conjugal quadric with $k=-1/3$. The quadric with $k=1/3$ plays the same role with the associate conjugate net.

It will be recalled [2, pp. 185, 187] that the lines joining the points whose general coordinates are

$$x_u - bx, \qquad x_v - ax,$$

wherein

$$(7)\quad a = (\theta_v + \lambda_v/\lambda + m\beta/\lambda^2)/2, \qquad b = (\theta_u - \lambda_u/\lambda + m\gamma\lambda^2)/2,$$

are respectively the flex-ray, ray, associate ray, principal join and associate principal join according as

$$m = 0,\ -1,\ 1,\ 5/3,\ -5/3.$$

The lines joining x to the point whose coordinates are $x_{uv}-a'x_u-b'x_v$ with a', b' defined by

$$(8)\qquad a' = (\theta_v + \lambda_v/\lambda + n\beta/\lambda^2)/2, \quad b' = (\theta_v - \lambda_u/\lambda + n\gamma\lambda^2)/2$$

are respectively the axis, associate axis, cusp-axis, polar reciprocals of the principal join and associate principal join with respect to any quadric of Darboux according as $n=1,\ -1,\ 0,\ 5/3,\ -5/3$.

Imposing the condition that the lines determined by (7) be the polar lines of those defined by (8) with respect to the quadric (4) we find that quadric must be a conjugal quadric with

$$k = (n - m)/2. \tag{9}$$

It follows that the conjugal quadrics may be defined as quadrics having second order contact with S at x with respect to which canonical [3, p. 7] *lines of the first kind are reciprocal polars of canonical lines of the second kind.*

Using the various values of m and n for the well known canonical lines above we find interpretations for the conjugal quadrics for the particular values $k=0$, $\pm 1/3$, $\pm 1/2$, $\pm 5/6$, ± 1, $\pm 4/3$, $\pm 5/3$.

In particular *the quadric of Davis is a quadric having second order contact with S at x and for which the axis of N is the polar of the ray of N.* Thus a new characterization of this quadric is obtained.

By means of the first characterization of the conjugal quadric with $k=-1/3$, we may give characterizations of some of the canonical lines associated with the net N. In particular *the principal join of the curves of N may be characterized as the polar line of the axis of the net with respect to the conjugal quadric $k=-1/3$; or it is the reciprocal of the associate ray of N with respect to the quadric of Davis.*

3. The transformation of Čech. In this section we shall show how the conjugal quadrics (6) may be considered as inducing the general transformation of Čech.

The coordinates of any point ρ on the tangent to the curve defined by $dv-\lambda du=0$ of the net N may be written in the form

$$\rho = x_u + \lambda x_v + \mu x.$$

The polar plane of ρ with respect to a conjugal quadric (6) has local coordinates given by the formulas

$$u_1 = 0, \quad u_2 = \lambda^2, \quad u_3 = \lambda, \quad u_4 = -\lambda\mu + k(\beta + \gamma\lambda^2). \tag{10}$$

If the local coordinates of ρ be written in the form $(x_1, x_2, x_3, 0)$, equations (10) assume the form

$$\begin{aligned} u_1 &= 0, \quad u_2 = x_2x_3^2, \quad u_3 = x_2^2x_3, \\ u_4 &= -x_1x_2x_3 + k(\beta x_2^3 + \gamma x_3^3) \end{aligned} \tag{11}$$

of general transformation of Čech. We shall say that *the conjugal quadrics* (6) *induce the general transformation of Čech.* In particular *the quadrics of Darboux induce the polarity of Lie*, $k=0$, and *the quadrics of Davis induce the correspondence of Segre*, $k=1$.

The associate quadric of Davis may be said to induce *the associate correspondence of Segre*, $k=-1$. *The conjugal quadric $k=-1/3$ induces the correspondence of Moutard.* We may call the transformation

(11) with $k=1/3$ *the associate correspondence of Moutard.* The quadric (6) with $k=1/3$ induces this latter correspondence. The other particular conjugal quadrics characterized in the previous section of course induce the corresponding transformations of Čech.

Lane has given [2, p. 203] certain properties of the general transformation of Čech. We call attention to one additional property. We may easily verify that the most general linear transformation which leaves the form of the equations of the family of conjugal quadrics (6) invariant is the transformation

$$\begin{gathered} \sigma x_1 = \bar{x}_1 - b\bar{x}_2 - a\bar{x}_3 - c\bar{x}_4, \\ \sigma x_2 = \bar{x}_2 - a\bar{x}_4, \qquad \sigma x_3 = \bar{x}_3 - b\bar{x}_4, \qquad \sigma x_4 = \bar{x}_4. \end{gathered} \tag{12}$$

The transformation (12) and the corresponding transformation in plane coordinates leave the form of each of the transformations of Čech (11) absolutely invariant. *It follows therefore from the form of* (12) *that the transformation of the form* (11) *obtained by using any R-harmonic line* [4, p. 584] *and its reciprocal in the definition of the conjugal quadrics is the transformation of Čech for the same value of k.*

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DISTANCE GEOMETRY NOTES

L. M. BLUMENTHAL

Introduction. This paper is concerned with several disconnected developments in distance geometry. §1 deals with the congruent imbedding of metric spaces in euclidean or Hilbert spaces. By showing that *the validity of the pythagorean theorem insures the essentially euclidean character of the metric*, the basic role this theorem plays in euclidean geometry is seen to be fully justified. In §2 the circle and n-dimensional sphere are considered with respect to the property of covering euclidean subsets. The concluding section presents an algebraic-geometric proof of the quasi congruence order property of the E_n which, by making use of determinants, achieves a considerable abbreviation of the two proofs of this important result hitherto published. The desired purely algebraic proof has not yet been obtained.

1. Metric spaces and the theorem of Pythagoras. If a metric space contains a line L (that is, a set congruent with the euclidean straight line) and a point p not on L, then the line contains a point p_0 nearest p. The pythagorean theorem is valid in the space provided for each element x of L, $pp_0^2+p_0x^2=px^2$. It is clear that this property insures the uniqueness of the point p_0.

THEOREM 1.1. *A separable, complete, convex, externally convex metric space in which the theorem of Pythagoras is valid is congruent with a euclidean or Hilbert space.*

PROOF. From an early result of the writer it suffices to show that the space has the weak euclidean four-point property; that is, each quadruple of points containing a linear triple is congruently contained in the plane.[1] If p, q, r, s are four such points, L a line containing q, r, s, and p_0 the point of L nearest p, let q', r', s', p_0' be points of an E_1 congruent with the set q, r, s, p_0. In the plane E_2 formed by this E_1 and the line perpendicular to it at p_0', let p' denote a point on this perpendicular with $p'p_0'=pp_0$. Since the theorem of Pythagoras is valid in both the given space and in E_2, it is clear that the distances of p' from the points q', r', s' equal, respectively, the distances of p from q, r, s, and the theorem is proved.

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[1] See Amer. J. Math. vol. 57 (1935) pp. 51–61; in particular, p. 61.

It is worth remarking that the proof of the theorem, trivial when based upon the *weak* euclidean four-point property, is by no means immediate in terms of the ordinary four-point property which necessitates showing that *every* quadruple is congruently contained in E_3. It serves, therefore, as a good example of the usefulness of the weaker property.

If any condition leading to finite dimensionality is adjoined to those in the theorem (for example the absence of equilateral $(n+2)$-tuples or the compactness of bounded sets) the condition of separability may be suppressed (completeness also in the second alternative, of course) and congruence with a euclidean space results.

2. Covering indices. A given space is said to have congruence indices (n, k) with respect to a prescribed class of spaces provided any space of the class, containing more than $n+k$ points, is congruently contained in the given space whenever each of its n-tuples has this property. The indices (n, k) are "best" in case neither can be reduced.[2]

If the given space is a subset of a euclidean space and the spaces of the prescribed class are also euclidean subsets (though not necessarily of the same space of which the given space is a subset) the possession of congruence indices (n, k) means that any member of the class, containing more than $n+k$ points, can be covered by the given space whenever each n of its points is coverable by the space. This suggests the term "covering indices" as a specialization of the more general concept. Thus, for example, a circular disc has covering indices (3, 0) with respect to the class of subsets of the plane, since any plane set may be covered by such a disc whenever each three of its points are so coverable. Denoting by $C_{n,r}$ the surface of the n-dimensional sphere of radius r, with euclidean (chord) metric, we have:

LEMMA 2.1. *If each three of four distinct coplanar points are coverable by the circle $C_{1,r}$, but the four points are not so coverable, then they form an orthocentric quadruple.*

PROOF. Let p_1, p_2, p_3, p_4 denote the four points, and q_i the circumcenter of the triple obtained by omitting the ith point $(i=1, 2, 3, 4)$ (clearly no three of the points are on a line). It follows at once from the hypotheses that q_1, q_2, q_3 are reflections of q_4 in the sides p_2p_3, p_1p_3, p_1p_2, respectively, of the triangle $p_1p_2p_3$, and $p_4q_1=p_4q_2=p_4q_3=r$. Thus p_4 is the isogonal conjugate of the circumcenter q_4

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[2] The concept of congruence indices was introduced in Bull. Amer. Math. Soc. vol. 47 (1941) pp. 435–443.

of triangle $p_1p_2p_3$ and hence is the orthocenter of the triangle.

If four points form an orthocentric quadruple, the four triples have circumcircles of the same radius. But the four points are surely not concyclic, for since p_4, for example, is the isogonal conjugate of the circumcenter q_4 of the triangle $p_1p_2p_3$, it follows that q_4 is the isogonal conjugate of p_4 which cannot then lie on the circumcircle of $p_1p_2p_3$ since no point on this circle has an isogonal conjugate with respect to $p_1p_2p_3$. *Hence a necessary and sufficient condition that four distinct coplanar points be not coverable by* $C_{1,r}$, *though each triple of the points be coverable, is that the four points form an orthocentric quadruple.*

THEOREM 2.1. *The* $C_{1,r}$ *has best covering indices* (3, 1) *with respect to subsets of the plane.*

PROOF. To prove the validity of indices (3, 1) it suffices to show that any planar quintuple is coverable by $C_{1,r}$ whenever each three of its points are. If, now, $p_1, \cdots, p_5$ be such a quintuple, an assumption that it is not coverable by $C_{1,r}$ implies that at least two of its quadruples be not coverable. According to the foregoing, each of these quadruples forms an orthocentric set, and since they have a triple in common, the fourth points in each quadruple coincide, for each is the orthocenter of the common triple.

That the indices (3, 1) are best is obvious since it is trivial to observe that the first index cannot be reduced, while the example of an orthocentric quadruple shows that the second index cannot be replaced by zero.

LEMMA 2.2. *If four noncoplanar points have each triple coverable by* $C_{1,r}$, *then they are the vertices of an isosceles tetrahedron.*

PROOF. Let O denote the center of the sphere circumscribing the four points and R its radius. The feet of the perpendiculars from O to the faces of the tetrahedron are the circumcenters of these faces and so the distances of O from each face is $(R^2-r^2)^{1/2}$. Thus O is also the center of the inscribed sphere and the assertion of the lemma follows.

LEMMA 2.3. *The* $C_{1,r}$ *has best covering indices* (3, 1) *with respect to subsets of* E_3.

PROOF. Let S be any subset of E_3 containing more than four points, each three of which are coverable by $C_{1,r}$, and let $p_1, \cdots, p_5$ be any five points of S. If these five points are not coverable by $C_{1,r}$ then, by Theorem 2.1, the five points are not coplanar and hence they contain at most one planar quadruple. Examining the (at least) four nonplanar quadruples in the light of Lemma 2.2, one sees that all the ten distances determined by the five points are equal. Since the E_3 contains no equilateral five-point, this is impossible and the lemma follows. (Since the indices (3, 1) are best with respect to subsets of E_2 they are a fortiori best for subsets of E_3.)

It is noteworthy that $C_{1,r}$ has the same set of best indices with respect to subsets of E_2 as it has with respect to subsets of E_3.

THEOREM 2.2. *The* $C_{1,r}$ *has best covering indices* $(3, n-2)$ *with respect to subsets of* E_n, $n>2$.

PROOF. This has been proved for $n=3$. We make the inductive assumption of the theorem's validity in E_n, $n>2$. Let $p_1, p_2, \cdots, p_{n+3}$ be any $(n+3)$-tuple of a subset S of E_{n+1} containing more than $n+2$ points, with each triple coverable by $C_{1,r}$. If this $(n+3)$-tuple be supposed not coverable by $C_{1,r}$ it follows (by the inductive hypothesis) that the $(n+3)$-tuple is not contained in E_n.

Case 1. *No* $(n+2)$*-tuple of* $p_1, p_2, \cdots, p_{n+3}$ *is in* E_n.

Then no quadruple of these points is in E_2 and hence (Lemma 2.2) for each quadruple p_i, p_j, p_k, p_l, "opposite" distances are equal. Then the $n+3$ points form an equilateral set, which is impossible since they are contained in E_{n+1}.

Case 2. *At least one* $(n+2)$*-tuple of* $p_1, p_2, \cdots, p_{n+3}$ *is contained in* E_n.

Since the $(n+3)$-tuple is not contained in E_n, it follows that in this case exactly one $(n+2)$-tuple is contained in E_n, say $p_2, \cdots, p_{n+3}$. By the inductive hypothesis these $n+2$ points are coverable by $C_{1,r}$, and hence they lie in a plane. But then the $n+3$ points lie in E_3, which (since $n>2$) contradicts the conclusion reached above that $p_1, p_2, \cdots, p_{n+3}$ are not contained in E_n.

Hence the $C_{1,r}$ has covering indices $(3, n-2)$ with respect to subsets of E_n, $n>2$. The presence in E_n of equilateral $(n+1)$-tuples with each three points coverable by $C_{1,r}$ shows that the indices $(3, n-2)$ are best.

THEOREM 2.3. *The* $C_{n,r}$ *has covering indices* $(n+2, 1)$ *with respect to subsets of* E_{n+1}.

PROOF. Let S be a subset of E_{n+1} containing more than $n+3$ points and consider the semimetric space $S'+(a')$, with S' congruent with S and $a'p'=r$ for each element p' of S'.

It is seen at once that each set of $n+3$ points of this semimetric space is congruently contained in E_{n+1}, for if the $n+3$ points are in S' this follows from the congruence of S' with S, while if $p_1', p_2', \cdots, p_{n+2}'$ are any $n+2$ points of S', the $(n+3)$-tuple

$p_1', p_2', \cdots, p_{n+2}', a'$ is congruent with $p_1, p_2, \cdots, p_{n+2}, a$, where the points $p_1, p_2, \cdots, p_{n+2}$ are selected from S and the point a is the center of the $C_{n,r}$ passing through them.

Since $S'+(a')$ contains more than $n+4$ points, and E_{n+1} has, by a fundamental theorem (proof of which is given in §3), congruence indices $(n+3, 1)$ with respect to the class of all semimetric spaces, it follows that this semimetric space is congruently contained in E_{n+1}, and as each point of S' has distance r from a', it is coverable by $C_{n,r}$ and the theorem is proved.

It seems quite likely that the indices $(n+2, 1)$ are best, though this has not as yet been completely established except for $n=1, 2$.

It is expected that these covering properties will lead to new characterizations of the figures concerned. It has already been shown, for example, that the circular disc is characterized among all connected, simply connected domains of the plane by the property of possessing covering indices (3, 0) with respect to plane sets. In an analogous manner, the circular rim $C_{1,r}$ might be characterized among all simple closed curves of the plane by possessing best covering indices (3, 1) with respect to planar subsets (or, perhaps, among all Jordan curves of E_n by having as best indices $(3, n-2)$).[3]

3. Congruence indices of the E_n with respect to semimetric spaces. It was shown by Menger that any semimetric space of more than $n+3$ points is congruently contained in E_n whenever each $n+2$ of its points has this property. Thus E_n has congruence indices $(n+2, 1)$ with respect to the class of semimetric spaces or, in the older terminology, quasi congruence order $n+2$. This theorem is fundamental in the metric study of euclidean space.

The literature contains only two proofs of this important result, both of which are quite lengthy and necessitate considerable indoctrination of the reader in the preceding theory.[4] This is due, in part, to the fact that the argument is entirely geometrical and demonstrates more than is actually needed in order to arrive at the desired result. Since the theorem is easily formulated in the language of determinant or quadratic form theory, attempts have been made to furnish a completely algebraic proof of the theorem. The developments presented here do not accomplish this but they do employ determinant theory to considerably shorten the proofs heretofore obtained, and they hew close to the line by proving only those facts which are needed to establish the result. Thus the end is achieved more speedily though the route travelled is not essentially different.[5]

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[3] In this connection it would be of interest to prove or disprove the writer's conjecture that corresponding to each plane simple closed curve there exists a circle with each three of its points coverable by the simple closed curve. In the event that the curve is a triangle, the inscribed circle serves as the circle of the conjecture.

[4] The proofs are by Karl Menger, Math. Ann. vol. 100 (1928) pp. 120–130 and Amer. J. Math. vol. 53 (1931) pp. 730–737.

Calling a set pseudo-E_n provided it is not congruently contained in E_n though each of its $(n+2)$-tuples is, we show first the following lemma.

LEMMA 3.1. *A pseudo-E_n $(n+4)$-tuple P contains at least $n+2$ pseudo-E_n $(n+3)$-tuples.*

PROOF. Suppose this is not the case. Then P contains at least three $(n+3)$-tuples which are congruently contained in E_n. Now since any three $(n+3)$-tuples of P have $n+1$ points in common, we may assume the labelling so that $p_1, p_2, \cdots, p_{n+1}, p_{n+2}, p_{n+3}$; $p_1, p_2, \cdots, p_{n+1}, p_{n+2}, p_{n+4}$; $p_1, p_2, \cdots, p_{n+1}, p_{n+3}, p_{n+4}$ are euclidean $(n+3)$-tuples. Then the determinant $D(p_1, p_2, \cdots, p_{n+3}, p_{n+4})$ of the set P has rank less than or equal to $n+2$, since the principal minors $D(p_1, p_2, \cdots, p_{n+1}, p_{n+i})$ vanish for $i=2, 3, 4$ (every $n+2$ points of P being euclidean) and

$$\begin{aligned} D(p_1, \cdots, p_{n+1}, p_{n+2}, p_{n+3}) &= D(p_1, \cdots, p_{n+1}, p_{n+2}, p_{n+4}) \\ &= D(p_1, \cdots, p_{n+1}, p_{n+3}, p_{n+4}) = 0 \end{aligned}$$

(since the three $(n+3)$-tuples are euclidean).

It follows that each $(n+3)$-tuple of P has a vanishing determinant D and hence is congruently contained in E_n. But this implies that P is euclidean, contrary to the hypothesis.

LEMMA 3.2. *Let P be a pseudo-E_n $(n+3)$-tuple. Then each $n+2$ points of P contains an independent $(n+1)$-tuple.*

PROOF. If this is not the case we may assume the labelling so that $p_1, p_2, \cdots, p_{n+2}$ are $n+2$ points of P with each of its $(n+1)$-tuples dependent. Then $D(p_1, p_2, \cdots, p_{n+2})$ has rank not exceeding $n+1$, for $D(p_1, p_2, \cdots, p_{n+2})$ vanishes and $D(p_1, p_2, \cdots, p_n, p_{n+1}) = D(p_1, p_2, \cdots, p_n, p_{n+2}) = D(p_1, p_2, \cdots, p_n, p_{n+1}, p_{n+2}) = 0$. Thus all first minors of $D(p_1, p_2, \cdots, p_{n+2})$ vanish, and hence $D(p_1, p_2, \cdots, p_{n+3})$ is zero. This is impossible since P is a pseudo-E_n $(n+3)$-tuple.

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[5] We are referring here to the second of the two references in the preceding footnote.

LEMMA 3.3. *Let P and Q be two pseudo-E_n $(n+3)$-tuples with $n+2$ points of P congruent with $n+2$ points of Q. Then P and Q are congruent.*

PROOF. By the preceding lemma the $n+2$ points of P which are congruent to $n+2$ points of Q contain an independent $(n+1)$-tuple. Selecting the labelling so that the $n+2$ points $p_1, p_2, \cdots, p_{n+1}, p_{n+2}$ of P are congruent with the $n+2$ points $q_1, q_2, \cdots, q_{n+1}, q_{n+2}$ of Q, it follows easily that the distances of p_{n+3} from the points of the first set are the same functions of the mutual distances of $p_1, p_2, \cdots, p_{n+2}$ as the distances of q_{n+3} from the points of the congruent set $q_1, q_2, \cdots, q_{n+2}$, and hence the two $(n+3)$-tuples are congruent.

THEOREM 3.1. *If a semimetric space S of more than $n+3$ points has each of its $(n+2)$-tuples congruently contained in E_n, then S is congruently contained in E_n.*

PROOF. Suppose S is not congruently contained in E_n. Then S has at least one $(n+3)$-tuple with this property, and any $(n+4)$-tuple containing this $(n+3)$-tuple is a pseudo-E_n set. Let $P=(p_1, p_2, \cdots, p_{n+4})$ be such an $(n+4)$-tuple. By Lemma 3.1 at most two of the $(n+3)$-tuples of P can be euclidean, say $p_1, p_2, \cdots, p_{n+2}, p_{n+3}$ and $p_1, p_2, \cdots, p_{n+2}, p_{n+4}$.

To show that $p_1, p_2, \cdots, p_{n+2}$ forms an equilateral set (which furnishes the desired contradiction to the assumption that S is not congruently contained in E_n) let p_ip_j, p_rp_s be any pair of its nonzero distances. Then the indices i, j, r, s are selected from the numbers $1, 2, \cdots, n+2$, and so the two $(n+3)$-tuples $p_1, p_2, \cdots, p_{i-1}, p_{i+1}, \cdots, p_{r-1}, p_{r+1}, \cdots, p_{n+2}, p_{n+3}, p_{n+4}, p_i$; $p_1, p_2, \cdots, p_{i-1}, p_{i+1}, \cdots, p_{r-1}, p_{r+1}, \cdots, p_{n+2}, p_{n+3}, p_{n+4}, p_r$ are pseudo-E_n. From Lemma 3.3 we have $p_ip_j=p_rp_j$.

But from the two pseudo-E_n $(n+3)$-tuples $p_1, p_2, \cdots, p_{j-1}, p_{j+1}, \cdots, p_{s-1}, p_{s+1}, \cdots, p_{n+2}, p_{n+3}, p_{n+4}, p_j$; $p_1, p_2, \cdots, p_{j-1}, p_{j+1}, \cdots, p_{s-1}, p_{s+1}, \cdots, p_{n+2}, p_{n+3}, p_{n+4}, p_s$ we have $p_jp_r=p_sp_r$. Hence $p_ip_j=p_rp_s$ and the $n+2$ points $p_1, p_2, \cdots, p_{n+2}$ form an equilateral $(n+2)$-tuple. They are not, then, congruently contained in E_n, contrary to the hypothesis, and the theorem is proved.

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TOPOLOGICAL ANALOG OF THE WEIERSTRASS DOUBLE SERIES THEOREM

G. T. WHYBURN

Since any light interior transformation of a sphere or a Riemann surface into a sphere is topologically equivalent to an analytic transformation and any non-constant analytic transformation is light and interior,[1] it might be expected in view of the Weierstrass Double Series Theorem that the limit of a uniformly convergent sequence of light interior mappings of a sphere into a sphere would be light and interior. However, this is readily seen not to be the case. For if we let r denote $|z|$ and for each $n>0$ we map the complex sphere onto itself by the function $f_n(z)$ defined by

$$\begin{aligned} w &= z && \text{for } r \geqq 2,\\ w &= z/n && \text{for } r \leqq 1,\\ w &= (r-1)z + (2-r)(z/n) && \text{for } 1 < r < 2, \end{aligned}$$

each mapping is topological, whereas the sequence $f_n(z)$ converges uniformly to the mapping $f(z)$:

$$\begin{aligned} w &= z && \text{for } r \geqq 2,\\ w &= 0 && \text{for } r \leqq 1,\\ w &= (r-1)z && \text{for } 1 < r < 2. \end{aligned}$$

The latter mapping is neither light nor interior. (Of course, analytic mappings topologically equivalent to the transformations $f_n(z)$ can be chosen so as to exhibit a wide range of behaviors, since they need only be topological mappings.)

It will be noted in this example, however, that if we factor the limit mapping $w=f(z)$ into the form $f_2f_1(z)$ where f_1 is monotone and f_2 is light,[2] *the transformation* f_2 *is light and interior.* This suggests the possibility, which will indeed be realized in the much more general situation where the mappings operate on an arbitrary locally connected continuum, that the limit of a uniformly convergent sequence of light interior mappings always factors into a monotone transformation followed by a light interior one. In fact, it will be shown that if the

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[1] See S. Stoïlow, *Leçons sur les principes topologiques de la théorie des fonctions analytiques*, Paris, 1938, Gauthier-Villars.

[2] See, for example, the author's *Analytic topology*, Amer. Math. Soc. Colloquium Publications, New York, 1942, chap. 8. Note references therein also to S. Eilenberg.

individual mappings in the sequence admit of such factorization, so also does their limit.

Now it was shown by Wallace[3] that a continuous transformation $f(A)=B$ on a locally connected continuum A factors into this form if and only if it is quasi-monotone, that is, provided the inverse of each continuum K in B having a non-empty interior relative to B consists of a finite number of components each of which maps onto all of K under f or, equivalently, provided that for each connected open set R in B each component of $f^{-1}(R)$ maps onto R under f. Thus our main conclusion asserts in these terms that the class of quasi-monotone mappings of a locally connected continuum A onto another one B is closed in the space B^A of all mappings of A into B. The Weierstrass Double Series Theorem asserts, of course, that the class of analytic transformations is closed. In view of the equivalence theorem of Stoïlow[1] our result is analogous, except that the "constants" (more precisely, "the continua on which the mappings are constant") must first be factored out.

THEOREM. *The limit mapping $f(A)=B$ of a uniformly convergent sequence of quasi-monotone mappings $f_n(A)=B$ on a locally connected continuum A is itself quasi-monotone.*

In view of the fact mentioned above that a continuous mapping $g(A)=B$ on a locally connected continuum A is quasi-monotone if and only if each component of the inverse of an arbitrary region (that is, connected open subset) R in B maps onto R under g, our theorem is a direct consequence of the following more general one.

THEOREM. *Let the sequence of continuous mappings $f_n(A)=B$ on the locally connected continuum[4] A converge uniformly to the mapping $f(A)=B$. If for each $\epsilon>0$ there exists an integer N such that for any $n>N$ it is true that if R is any region in B and Q is any component of $f_n^{-1}(R)$, then*

$$f_n(Q)\cdot\{R - R\cdot V_\epsilon[F(R)]\} \neq 0$$

implies

$$f_n(Q) \supset R - R\cdot V_\epsilon[F(R)],$$

then f is quasi-monotone.

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[3] A. D. Wallace, *Quasi-monotone transformations*, Duke Math. J. vol. 7 (1940) pp. 136–145.

[4] It is understood that a continuum is compact, metric and connected. For any set X and any $\epsilon>0$, $V_\epsilon(X)$ denotes the set of all points x whose distance $\rho(x, X)$ from X is less than ϵ. Also if G is any open set, $F(G)$ denotes its boundary, that is, $\bar{G}-G=F(G)$.

PROOF. To prove f quasi-monotone it suffices, in view of the factorization process, to show that for any $y\in B$ and any neighborhood V of a component X of $f^{-1}(y)$, $f(V)$ contains a neighborhood of y. By local connectedness, there exists a region U in A such that

$$V\supset \overline{U}\supset U\supset X.$$

Further, since X is a component of $f^{-1}(y)$, we may choose U also so that if $F=F(U)$, $f(F)\cdot y=0$. Let R be a region in B containing y and such that $\overline{R}\cdot f(F)=0$. Let

$$\text{(i)}\qquad \begin{aligned} 3\epsilon &= \rho[R, f(F)],\\ F_n &= f_n(F). \end{aligned}$$

By uniform convergence there exists an integer N such that for any $n>N$,

$$\text{(ii)}\qquad \begin{aligned} F_n &\subset V_\epsilon[f(F)],\\ f_n(X) &\subset R. \end{aligned}$$

Now for any $n>N$, let R_n be the component of $B-F_n$ containing R and let Q_n be the component of $f_n^{-1}(R_n)$ containing X. By (ii) we have

$$f_n(Q_n)\cdot R\neq 0.$$

Whence by (i)

$$f_n(Q_n)\cdot\{R_n - R_n\cdot V_\epsilon[F_n]\}\neq 0;$$

and since

$$F_n\supset F(R_n),$$

this gives

$$f_n(Q_n)\cdot\{R_n - R_n\cdot V_\epsilon[F(R_n)]\}\neq 0.$$

By hypothesis this implies that for each $n>N$,

$$f_n(Q_n)\supset R_n - R_n\cdot V_\epsilon[F(R_n)]\supset R.$$

Thus, since $Q_n\subset U$,

$$f_n(\overline{U})\supset f_n(Q_n)\supset R,$$

for every $n>N$.

Now by uniform convergence, $f_n(\overline{U})$ converges to $f(\overline{U})$. Thus we have

$$f(\overline{U}) = \lim f_n(\overline{U})\supset R,$$

so that

$$f(V) \supset f(\overline{U}) \supset R,$$

as was to be proven.

COROLLARY. *If the sequence $f_n(A)=B$ of light interior mappings on a locally connected continuum A converges uniformly to the mapping $f(A)=B$, then f admits of unique factorization into the form $f=f_2f_1$ where f_1 is monotone and f_2 is light and interior.*

The case in which A is a topological sphere is of special interest, not only because of its connection with analytic functions but also because in this case the action of the separate factor mappings can be more completely analyzed. For the monotone factor $f_1(A)=A'$ always yields[5] a cactoid A' and then the light interior factor $f_2(A')=B$ operating on the cactoid A' lends itself readily[6] to analysis.

In conclusion it may be noted that our main conclusion obtained here for monotone mappings is known[7] to hold also for monotone mappings, that is, the limit of a uniformly convergent sequence of monotone mappings of a locally connected continuum A onto another one B is monotone.

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[5] See R. L. Moore, *Concerning upper semi-continuous collections*, Monatshefte für Mathematik und Physik vol. 36 (1929) pp. 81–88.

[6] See footnote 2, chap. 10.

[7] Footnote 2, p. 174.

THE "FUNDAMENTAL THEOREM OF ALGEBRA" FOR QUATERNIONS

SAMUEL EILENBERG AND IVAN NIVEN

We are concerned with polynomials of degree n of the type

$$f(x) = a_0xa_1x \cdots xa_n + \phi(x),$$

where $x, a_0, a_1, \cdots, a_n$ are real quaternions ($a_i \neq 0$ for $i = 0, 1, \cdots, n$) and $\phi(x)$ is a sum of a finite number of similar monomials $b_0xb_1x \cdots xb_k$, where $k < n$.

THEOREM 1. *The equation* $f(x) = 0$ *has at least one solution.*[1]

The 4-dimensional euclidean space R_4 of all quaternions will be made compact by adding the point ∞ to form a 4-dimensional sphere S_4. Setting $f(\infty) = \infty$ we get a mapping

$$f: S_4 \rightarrow S_4.$$

The continuity of f at ∞ follows from the fact that $|f(x)|$ increases without limit as $|x|$ increases without limit, a fact which is obvious from the definition of f. It should be noted that this argument is not valid for polynomials of degree n with more than one term of degree n.

Theorem 1 is then a consequence of the following theorem which asserts that "essentially" the equation $f(x) = 0$ has exactly n solutions.

THEOREM 2. *The mapping* $f: S_4 \rightarrow S_4$ *has the degree* n *(in the sense of Brouwer).*[2]

We define a mapping $g: S_4 \rightarrow S_4$ as follows:

$$g(x) = x^n \quad \text{for} \quad x \in R_4, \qquad g(\infty) = \infty.$$

Theorem 2 is a consequence of the following two lemmas.

LEMMA 1. *The mappings* f *and* g *of* S_4 *into* S_4 *are homotopic.*

LEMMA 2. *The mapping* g *has degree* n.

PROOF OF LEMMA 1. Define for $0 \leqq t \leqq 1$

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[1] This result has been obtained for the special case in which all terms of $f(x)$ have the form ax^k; cf. Ivan Niven, *Equations in quaternions*, Amer. Math. Monthly vol. 48 (1941) pp. 654–661.

[2] Cf. Alexandroff and Hopf, *Topologie*, Berlin, 1935, chap. 12, for the theory of the degree of a mapping.

$$f_t(x) = a_0xa_1x \cdots xa_n + (1 - t)\phi(x) \quad \text{for} \quad x \in R_4,$$
$$f_t(\infty) = \infty.$$

Clearly $f_t(x)$ is continuous in both x and t, and consequently f is homotopic to the mapping

$$f_1(x) = a_0xa_1x \cdots xa_n \quad \text{for} \quad x \in R_4,$$
$$f_1(\infty) = \infty.$$

Now for each $i = 0, 1, \cdots, n$ choose a continuous path $a_{i,t}$ in $R_4 - (0)$, where $1 \leqq t \leqq 2$, with end points $a_{i,1} = a_i$ and $a_{i,2} = 1$. Define for $1 \leqq t \leqq 2$

$$f_t(x) = a_{0,t}xa_{1,t}x \cdots xa_{n,t} \quad \text{for} \quad x \in R_4, \ f_t(\infty) = \infty.$$

Again $f_t(x)$ is continuous in x and t and since $f_2(t) = g(t)$ the conclusion follows.

Proof of Lemma 2. It is known[3] that the equation

$$g(x) = i, \quad \text{that is,} \quad x^n = i, \tag{1}$$

has exactly n roots in quaternions, and hence these must be simply the nth roots of i in complex numbers. We shall compute the Jacobian of the mapping function g for the roots of (1), and show that it is positive at each of these roots. This imples our conclusion.[4]

Any quaternion $x = x_1 + x_2i + x_3j + x_4ij$ can be written in the form $x = k(\cos\theta + \tau\sin\theta)$, where

$$k = (x_1^2 + x_2^2 + x_3^2 + x_4^2)^{1/2},$$
$$\sin\theta = \pm\,(x_2^2 + x_3^2 + x_4^2)^{1/2}/k, \qquad \cos\theta = x_1/k,$$
$$\tau = \pm\,(x_2i + x_3j + x_4ij)/(x_2^2 + x_3^2 + x_4^2)^{1/2}.$$

The last equation is invalid if $x_2^2 + x_3^2 + x_4^2 = 0$; in this case let τ be any unit vector. The sign used in the last two equations is arbitrary; however, in the special case to be considered in which x represents any nth root of i, it will be convenient to let this sign coincide with that of x_2. Now we have

$$x^n = k^n(\cos n\theta + \tau \sin n\theta),$$

and writing $x^n = f_1 + f_2i + f_3j + f_4ij$ we obtain

$$f_1 = f_1(x_1, x_2, x_3, x_4) = k^n \cos n\theta,$$

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[3] Cf. Ivan Niven, *The roots of a quaternion*, Amer. Math. Monthly vol. 49 (1942) pp. 386–388, or Louis Brand, a paper with the same title, same journal and volume, pp. 519, 520.

[4] Cf. Alexandroff and Hopf, p. 477.

$$f_r = f_r(x_1, x_2, x_3, x_4) = k^n x_r \sin n\theta / \pm (x_2^2 + x_3^2 + x_4^2)^{1/2} \quad (r = 2, 3, 4).$$

Computing $\partial f_r/\partial x_s$ $(r, s = 1, 2, 3, 4)$, and using the following relations which hold for all the roots of (1), $k = 1$, $\sin n\theta = 1$, $\cos n\theta = 0$, $\sin \theta = x_2$, $\cos \theta = x_1$, $x_3 = x_4 = 0$, $\tau = i$, we find that the Jacobian of the system is

$$\begin{vmatrix} nx_2 & -nx_1 & 0 & 0 \\ nx_1 & nx_2 & 0 & 0 \\ 0 & 0 & 1/x_2 & 0 \\ 0 & 0 & 0 & 1/x_2 \end{vmatrix}.$$

Since $x_1^2 + x_2^2 = 1$ and $x_2 \neq 0$, this has the positive value n^2/x_2^2. This completes the proof.

The above proofs can be extended to the case where the coefficients of $f(x)$ belong to the Cayley algebra with 8 units, using mappings of an 8-dimensional sphere S_8 into itself. The following two facts should be kept in mind. (1) Due to the non-associativity of the Cayley numbers each monomial of $f(x)$ must be parenthesized in a definite manner, and a monomial of degree k may contain more than $k+1$ constants. (2) The monomial x^n need not be parenthesized.

The proof also applies equally well to the fundamental theorem of algebra in the usual case of complex variables. The resulting proof will use the degree of a mapping of the 2-sphere S_2 into itself. This proof differs slightly from the topological proof of Alexandroff and Hopf[5] which uses the degree of a mapping of S_1 into itself and the theorem of Rouché (in a topological formulation).

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[5] Loc. cit. p. 469. Cf. also Wei-Liang Chow, Math. Ann. vol. 116 (1939) p. 463.

ON ORTHOGONAL LATIN SQUARES

HENRY B. MANN[1]

An m-sided Latin square is an arrangement of the numbers $1, 2, \cdots, m$ into m rows and m columns in such a way that no row and no column contains any number twice. Two Latin squares are said to be orthogonal if when one is superimposed upon the other every ordered pair of numbers occurs once in the resulting square. Various methods have been devised for the construction of sets of orthogonal squares. However no method has as yet been given which would yield all possible sets of orthogonal Latin squares. In constructing orthogonal sets it is of value to have simple criteria which enable us to decide whether a given Latin square can be a member of an orthogonal pair.

A Latin square to which an orthogonal square exists will be called a basis square. In this note we shall derive two simple necessary conditions for a square to be a basis square.

THEOREM 1. *If in the Latin square L of side $4n+2$ the square formed by the first $2n+1$ rows and the first $2n+1$ columns contains fewer than $n+1$ numbers which are different from $1, 2, \cdots, 2n+1$ then L is not a basis square.*

PROOF. Denote by I the square formed by the first $2n+1$ rows and the first $2n+1$ columns, by II the square formed by the first $2n+1$ rows and the last $2n+1$ columns, by IV the square formed by the last $2n+1$ rows and the last $2n+1$ columns. Then if a number occurs a times in I it must occur $2n+1-a$ times in II and $2n+1-(2n+1-a)=a$ times in IV. Hence in I and IV together every number occurs $2a$ times. Assume now that I contains fewer than $n+1$ numbers different from $1, 2, \cdots, 2n+1$ and let L' be a square orthogonal to L. In the square resulting from superimposing L' on L every pair $1, i\ 2, i\ \cdots\ 2n+1, i$ must occur. Hence every number i in L' occurs $2n+1$ times combined with a number of the set $1, 2, \cdots, 2n+1$ in L. But at most $2n$ numbers of the set $1, 2, \cdots, 2n+1$ occur in L outside of I and IV. Hence at least $2n+2$ numbers i of L' occur combined with the numbers $1, 2, \cdots, 2n+1$ in I and IV together an odd number of times. But at most $2n$ numbers of L' occur in I and IV combined with numbers of L which are different from $1, 2, \cdots, 2n+1$.

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Hence at least 2 numbers of L' would occur in I and IV an odd number of times; this is impossible.

COROLLARY 1. *If a* $(4n+2)$*-sided Latin square* L *contains in any subsquare formed by the rows* $i_1, i_2, \cdots, i_{2n+1}$ *and the columns* $j_1, j_2, \cdots, j_{2n+1}$ *fewer than* $n+1$ *numbers different from a given set of* $2n+1$ *numbers* $k_1, k_2, \cdots, k_{2n+1}$, *then* L *is not a basis square.*

Corollary 1 follows from Theorem 1 if we permute the rows, columns and numbers of L in a suitable manner. It may be remarked that using this corollary Tarry's [1][2] proof of the nonexistence of an orthogonal pair of 6-sided Latin squares can be simplified considerably.

We shall say that a Latin square L contains a Latin subsquare of side m if only m different numbers appear in a subsquare of L formed by m rows and m columns. From Corollary 1 we have another corollary:

COROLLARY 2. *If a* $(4n+2)$*-sided Latin square contains a subsquare of side* $2n+1$ *then it is not a basis square.*

Every multiplication table of a group forms a Latin square. A group of order $4n+2$ has a subgroup of order $2n+1$ which generates a Latin subsquare of side $2n+1$. We therefore have the following corollary:

COROLLARY 3. *A multiplication table of a group of order* $4n+2$ *is not a basis square.*

THEOREM 2. *If in the* $(4n+1)$*-sided Latin square* L *the square formed by the first* $2n$ *rows and the first* $2n$ *columns contains fewer than* $n/2$ *numbers different from* $1, 2, \cdots, 2n$, *then* L *is not a basis square.*

PROOF. Denote by I the square formed by the first $2n$ rows and the first $2n$ columns and by II the square formed by the last $2n+1$ rows and the last $2n+1$ columns. By an argument similar to that used in the proof of Theorem 1 it may be shown that if a number occurs a times in I it must occur $a+1$ times in II. Hence it occurs $2a+1$ times in I and II together. I and II together contain numbers different from $1, 2, \cdots, 2n$ at most $2n+1+n-1=3n$ times and outside of I and II numbers of the set $1, 2, \cdots, 2n$ occur at most $n-1$ times. If L' were orthogonal to L it would follow, using the methods of the proof of Theorem 1, that at least two numbers of L' would occur in I and II together an even number of times; this is impossible.

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[2] Numbers in brackets refer to the References listed at the end of the paper.

COROLLARY 4. *If in the* $(4n+1)$*-sided Latin square* L *a subsquare formed by any* $2n$ *rows and any* $2n$ *columns contains fewer than* $n/2$ *numbers different from a given set* $k_1, k_2, \cdots, k_{2n}$ *of* $2n$ *numbers, then* L *is not a basis square.*

COROLLARY 5. *If a* $(4n+1)$*-sided Latin square* L *contains a* $2n$*-sided Latin subsquare, then* L *is not a basis square.*

In the following we shall use some notations and definitions introduced by the author in a previous paper [3].

Let P_i be the permutation which transforms $1, 2, \cdots, m$ into the ith row of the Latin square L. We identify L with its m row permutations and write $L=(P_1, P_2, \cdots, P_m)$. If $L=(P_1, \cdots, P_m)$, $L'=(P_1', P_2', \cdots, P_m')$ then we define $LL'=(P_1P_1', \cdots, P_mP_m')$. LL' is not necessarily a Latin square. From Theorem 1 of [3] it follows that two Latin squares A and B are orthogonal if and only if there exists a Latin square C such that $AC=B$. If P is any permutation and $A=(A_1, \cdots, A_m)$ a Latin square then we put $PA=(PA_1, \cdots, PA_m)$, $AP=(A_1P, \cdots, A_mP)$. PA and AP are also Latin squares. If $AC=B$ then $PAQQ^{-1}CR=PBR$. Hence we have the following lemma:

LEMMA 1. *If* A *is orthogonal to* B *then* PAQ *is orthogonal to* PBR *for any permutations* P, Q, R.

A is said to be equivalent to PAQ.

If A_1 is the identical permutation then A is said to be reduced. Clearly if A is reduced then also $P^{-1}AP$ is reduced.

We can now proceed to prove the following theorem.

THEOREM 3. *Every 5-sided basis square whose first row is* 12345 *is a multiplication table of the cyclic group of order* 5.

PROOF. It follows from Corollary 5 that a 5-sided basis square cannot contain a 2-sided Latin subsquare. Hence every row of a 5-sided basis square must be obtained from every other row by a cyclic permutation of order 5. Let $L=(1, P_2, P_3, P_4, P_5)$ be a basis square. P_2 is a cyclic permutation of order 5. Since in the symmetric group all cyclic permutations of the same order belong to the same class there exists a permutation Q such that $Q^{-1}P_2Q=(12345)$. $Q^{-1}LQ=L'$ is by Lemma 1 a basis square. Its first row is 12345 and its second row 23451. Considering that no 2-sided subsquare may be contained in a 5-sided basis square we obtain as possible third rows of L' the rows 31524, 35214, 34512. The first two of these third rows lead necessarily to Latin squares with 2-sided subsquares, the third leads to a multiplication table of the group of order 5.

We shall say that a Latin square is based on a permutation group G if its rows can be obtained from the row $1, 2, \cdots, m$ by the permutations of G.

THEOREM 4. *Two Latin squares based on two different permutation groups of order* 5 *cannot be orthogonal.*

Let Z and Z' be two such groups. If $P\neq 1$ is in Z then $P^{-1}Z'P\neq Z'$, for otherwise one could obtain a group of degree 5 and order 25. By transforming Z' with the permutations of Z we can therefore obtain 5 different permutation groups all different from Z. But the symmetric group of degree 5 has only 6 subgroups of order 5. Hence every subgroup of order 5 which is different from Z can be obtained by transforming Z' with the permutations of Z. Such a transformation applied to Z itself leaves all elements of Z fixed. Hence if a Latin square based on Z were orthogonal to a Latin square based on Z' it would also be orthogonal to a Latin square based on any other group of order and degree 5.

It is therefore sufficient to show that the Latin square

$$Z = \begin{array}{c} 12345 \\ 23451 \\ 34512 \\ 45123 \\ 51234 \end{array}$$

cannot be orthogonal to any Latin square L whose rows are 12345, 24531, 35214, 43152, 51423. By Lemma 1, Z is also orthogonal to LP_1^{-1} where P_1 is the first row permutation of L. Since the permutations of L form a group, LP_1^{-1} contains the same permutations as L. Thus it can always be achieved that the row 12345 stands over the first row of Z. Then since the pairs 11, 22, 33, 44, 55 all occur in the first row, the row 24531 must stand over the fourth row of Z and the row 35214 over the second row of Z. But then the pair 23 would occur twice. This proves Theorem 4.

In [3] the following definition was given. A biunique mapping S of a group G into itself is said to be a complete mapping if every element of G can be represented in the form XX^S where X is an element of G and X^S the image of X under the mapping S.

It was shown in [3] that two orthogonal Latin squares L_1, L_2 based on the same permutation group define a complete mapping S of the abstract group and that the squares can then be written as $L_1=(P_1, \cdots, P_m)$, $L_2=(P_1P_1^S, \cdots, P_mP_m^S)$. If the mapping $T=1+S$, that is to say the mapping defined by $X^T = XX^S$, is an automorphism then S is said to be derived from the automorphism T. In this case $L_1 = (P_1, \cdots, P_m)$, $L_2 = (P_1^T, \cdots, P_m^T)$.

We shall state this result as a lemma.

LEMMA 2. *If the orthogonal Latin squares L_1 and L_2 are based on the same permutation group G then $L_1 = (P_1, \cdots, P_m)$, $L_2 = (P_1P_1^S, \cdots, P_mP_m^S)$,* where *$P_1, \cdots, P_m$ are the permutations of G, and S is a complete mapping of G.*

In [3] it was shown: If S is a complete mapping of G and $P_1, \cdots, P_m$ is a regular representation of G then the Latin squares $(P_1, \cdots, P_m)$ and $(P_1P_1^S, \cdots, P_mP_m^S)$ are orthogonal.

We shall now define a slightly different procedure by which orthogonal squares may be obtained from complete mappings. It has been shown in [3] that every complete mapping can be transformed into a complete mapping S for which $1^S = 1$. In the following we shall consider only complete mappings which have this property.

Let $G_1 = 1, \cdots, G_m$ be the elements of a group G. Let S be a complete mapping of G. We form two Latin squares L_1 and L_2 in the following manner: In the ith row and kth column of L_1 write l if $G_iG_k = G_l$. In the ith row and kth column of L_2 write l if $G_iG_kG_k^S = G_lG_l^S$. L_1 and L_2 are Latin squares since they are multiplication tables of G. Moreover L_1 is orthogonal to L_2. Otherwise for some i, j, k, l with $(i, j) \neq (k, l)$ we should have

$$G_iG_j = G_kG_l = G_m \tag{1}$$

and

$$G_iG_jG_j^S = G_kG_lG_l^S = G_nG_n^S. \tag{2}$$

(Equations (1) and (2) must hold if the pair m, n occurs in the ith row and jth column and in the kth row and lth column.) But from (1) and (2) follows $G_j^S = G_l^S$.

Since S is a biunique mapping this is possible only if $j = l$. It follows then from (1) that $i = k$ contrary to our assumption.

Now let $L_1 = (P_1, \cdots, P_m)$, $L_2 = (P_1', \cdots, P_m')$. Let L_1 consist of the same permutations as L_2; we shall show that the mapping T defined by $G_i^T = G_iG_i^S$ is an automorphism.

To prove this we have to show that from

$$G_lG_i = G_k \tag{3}$$

follows

$$G_lG_l^SG_iG_i^S = G_kG_k^S. \tag{4}$$

By assumption $P_l=P'_j$ for some j. Hence if $G_lG_\iota=G_k$ we must have

$$G_jG_iG_\iota^S = G_lG_k^S. \tag{5}$$

P_l transforms 1 into l. Hence also P'_j transforms 1 into l and by our definition of L_2 we have

$$G_j11^S = G_lG_l^S. \tag{6}$$

But $11^S=1$. Hence $G_j=G_lG_l^S$ and (4) follows from (5). Hence we have the following theorem.

THEOREM 5. *If a group G admits a complete mapping which is not derived from automorphisms then there exist two orthogonal Latin squares based on two different permutation groups both isomorphic to G.*

COROLLARY 6. *The group of order 5 only admits complete mappings which are derived from automorphisms.*

The corollary is an immediate consequence of Theorems 4 and 5.

A set of $m-1$ orthogonal m-sided squares is said to be a complete set of Latin squares.

In the following we shall abbreviate finite two-dimensional projective geometry by PG_2.

In his beautiful paper [2] R. C. Bose established the connection between PG_2's and complete sets of Latin squares as follows.

Let G be a PG_2 with $m+1$ points on every line. We pick any line L of G and call it the line at infinity. Let $P_0, \cdots, P_m$ be the points of L. Through every one of these points pass m lines different from L. We number these lines and let L_{ij} ($i=0, \cdots, m$; $j=0, \cdots, m-1$) denote the jth line passing through P_i. Every finite point P can then be identified with an $(m+1)$ad of numbers $(I_0, \cdots, I_m)$, where $I_j=k$ if L_{jk} passes through P. It is possible to identify every point with such an $(m+1)$ad of numbers since every pair of points determines exactly one line. A complete set of Latin squares can now be formed in the following way. Write the number k in the ith row and jth column of the square L_r ($r=2, \cdots, m$) if k is the rth number of the $(m+1)$ad whose first number is i and whose second number is j. That is to say, we let the first two numbers be the row and column numbers. From the fact that two lines intersect in one and only one point it follows that $L_2, \cdots, L_m$ are orthogonal Latin squares.

Conversely if a complete set of Latin squares is given, R. C. Bose constructs a PG_2 in the following manner: Every "finite" point of the PG_2 is identified with an $(m+1)$ad of numbers $(I_0, \cdots, I_m)$

where, for $r \geqq 2$, the number I_r is the number in the I_0th row and the I_1st column of the rth Latin square. We form m^2+m lines L_{ik} $(i=0, \cdots, m;\ k=0, \cdots, m-1)$, where L_{ik} is the set of all points whose ith number is k. Thus we obtain $m+1$ sets of parallel lines, each set containing m lines. From the orthogonality of the Latin squares it follows that two nonparallel lines intersect in only one point. We add now one point to each set of parallels and let this additional point lie on every line of the set. The set of added points forms the "line at infinity." Two lines then intersect in one and only one point. From this and the fact that $m+1$ lines pass through every point it follows easily that every pair of points is contained in one and only one line. Thus we obtain a PG_2 with m^2+m+1 points arranged in m^2+m+1 lines.

We shall exemplify R. C. Bose's construction by constructing a finite projective geometry consisting of 13 points from a set of 2 orthogonal 3-sided Latin squares. We start from a 3-sided Graeco-Latin square whose boxes, that is to say our finite points, we shall denote by letters. Thus we have

$$\begin{array}{ccc} 11a & 22b & 33c \\ 23d & 31e & 12f \\ 32g & 13h & 21i. \end{array}$$

The sets of parallel lines are: Rows: *abc, def, ghi*. Columns: *adg, beh, cfi*. Numbers of the first Latin square: *afh, bdi, ceg*. Numbers of the second Latin square: *aei, bfg, cdh*.

Adding the "points at infinity" we obtain the finite geometry

$$\begin{array}{ccccc} abcj & adgk & afhl & aeim & jklm \\ defj & behk & bdil & bfgm & \\ ghij & cfik & cegl & cdhm & . \end{array}$$

Let $0, 1, g_2, \cdots, g_{m-1}$ be the numbers of a Galois field. Form the Latin squares

$$L_j = \begin{array}{cccc} 0 & 1 & \cdots & g_{m-1} \\ g_j & g_{j+1} & \cdots & g_j + g_{m-1} \\ g_j g_2 & g_j g_2 + 1 & \cdots & g_j g_2 + g_{m-1} \\ \cdot & \cdot & \cdots & \cdot \\ g_j g_{m-1} & g_j g_{m-1} + 1 & \cdots & g_j g_{m-1} + g_{m-1} \end{array} \qquad j = 1, 2, \cdots, m-1.$$

R. C. Bose [2] has shown that $L_1, \cdots, L_{m-1}$ form a complete set of Latin squares. We shall say that this set is based on the Galois field $GF(m)$. We shall further say that a PG_2 is equivalent to a complete set S of Latin squares if one can be obtained from the other by Bose's method.

We shall prove the following theorem.

THEOREM 6. *If a PG_2 is equivalent to a complete set of Latin squares based on a Galois field then it is the analytic geometry of this Galois field.*

PROOF. The complete set of Latin squares can by hypothesis be represented by m^2 different $(m+1)$ads of numbers $(I_0, \cdots, I_m)$ of the Galois field, where $I_r = \alpha_r I_0 + I_1$ $(r \geqq 1)$ with $\alpha_1 = 0$ and $\alpha_i \neq \alpha_j$ for $i \neq j$. The lines of the PG_2 consist of the points ($(m+1)$ads) with fixed I_r and of one point at infinity. Let $(1, I_0, I_1)$ be the coordinates (x, y, z) of every finite point and let the coordinates of the infinite point corresponding to the rth set of parallels be $(0, 1, -\alpha_r)$ for $r \geqq 2$, $(0, 1, 0)$ for $r=1$ and $(0, 0, 1)$ for $r=0$. Then all points on a line satisfy an equation $ax+by+cz=0$, namely $-I_r x + \alpha_r y + z = 0$ for $r \geqq 1$ and $-I_0 x + y = 0$ for the lines of the first set. Hence the PG_2 is the analytic geometry of $GF(m)$.

As an application of Theorems 3, 4, 5, and 6 we shall prove the following:

THEOREM 7. *Every PG_2 with 6 points on every line is the analytic geometry of $GF(5)$.*

PROOF. From Theorems 3, 4, and Corollary 6 it follows that every orthogonal pair L_1, L_2 of 5-sided reduced squares is based on the same permutation group. If $L_1 = (1, P_2, \cdots, P_5)$ then, by Lemma 2, $L_2 = (1, P_2 P_2^S, \cdots, P_5 P_5^S)$ where S is a complete mapping of the cyclic group of order 5. The mapping defined by $P_i^T = P_i P_i^S$ is by Corollary 6 an automorphism. Hence $L_2 = (1, P_2^T, \cdots, P_5^T)$ where T is an automorphism. We write the group of order 5 as the additive group of remainders mod 5. Its only automorphisms are those induced by multiplication with the remainders 1, 2, 3, 4. Therefore if P_i is the permutation obtained by adding i to every remainder P_i^T must be the permutation obtained by adding αi to every remainder where α depends only on T. Hence every complete set of 5-sided Latin squares is based on the remainder system mod 5. Since the remainders mod 5 form a Galois field, Theorem 7 follows from Theorem 6.

It is also easy to show that every complete set of 2, 3, and 4-sided Latin squares is based on a Galois field. Hence also the uniqueness of PG_2's with 3, 4, and 5 points on every line can be proved in the same manner.

The uniqueness of finite geometries with less than 6 points on every line was first proved by J. H. M. Wedderburn and O. Veblen [4]. The uniqueness of finite geometries with 6 points on every line was first demonstrated by C. R. MacInnes [5] in a rather laborious tactical enumeration of cases.

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BARD COLLEGE

SOME THEOREMS ON CO-TERMINAL ARCS

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It is the purpose of this note to prove certain properties of sums of simple arcs which have one or both end points in common. The investigation was undertaken to answer a question, that of the validity of Theorem 3 below, raised by Miss Harlan C. Miller. An example is included to show that two of the results obtained are not valid for irreducible continua in general.

THEOREM 1. *If H and K are two distinct arcs from A to B, then each point of $H+K-H\cdot K$ belongs to a simple closed curve lying in the closure of $H+K-H\cdot K$.*

PROOF. Let P be any point of $H+K-H\cdot K=N$, and let S be the component of N which contains it. The set S is an arc segment; let its end points be X and Y. Suppose that no simple closed curve lying in $\overline{N}$ contains P. Then $\overline{N}-S$ contains no continuum containing both X and Y, for if it did it would contain an arc from X to Y, and this arc plus S would be a simple closed curve lying in $\overline{N}$ and contain-

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ing P. There exist, therefore, two disjoint closed sets U, containing X, and V, containing Y, such that $\overline{N}-S=U+V$. The components of $H+K-(U+V)$ are, except for S, subsets of $H\cdot K$ and hence are segments of arcs or segments plus one end point. Let U' be U plus all components of $H+K-(U+V)$ which have no end points in V, and let V' be V plus all components of $H+K-(U+V)$ which have no end points in U. Then U' and V' are closed and disjoint. Hence the components of $H+K-(U'+V')$ are finite in number since each of them has one end point in U' and the other in V'. Since all these segments but S are subsets of both H and K, while S is a subset of only one of these, it follows that the number of segments between U' and V' of one of the arcs H and K is even, while that of the other is odd. This is impossible since both H and K have A and B as end points.

THEOREM 2. *If H, K, and L are three distinct arcs from A to B, then there are two of them neither of which is a subset of the sum of the other two.*

PROOF. Suppose that H is a subset of $K+L$ and K is a subset of $H+L$; then $H+K-H\cdot K=N$ is a subset of L. This implies that $\overline{N}$ is a subset of L. By Theorem 1 there is a simple closed curve in $\overline{N}$. Hence the arc L contains a simple closed curve. It follows from this contradiction that one of the arcs H and K fails to be a subset of the sum of the other two. Suppose H does. Then it may be shown in the same way that one of the arcs K and L fails to be a subset of the sum of the other two, and the theorem is proven.

THEOREM 3. *If H, K, and L are three arcs emanating from a point A and no one of them is a subset of any other, then there is one of them which fails to be a subset of the sum of the other two.*

PROOF. Suppose that each of the arcs H, K, and L is a subset of the sum of the other two. Let the end points, other than A, of H, K, and L be B_H, B_K, and B_L, respectively. No two of the points B_H, B_K, and B_L coincide, for if they did a contradiction could be reached, as it was in the proof of Theorem 2, by the use of Theorem 1. If B_H did not belong to L, some neighborhood with respect to $H+K+L$ of B_H would be a subset of $H\cdot K$, which is impossible unless B_H and B_K coincide. Hence B_H and, similarly, B_K belong to L. Suppose that B_H precedes B_K in the order AB_L on L. Not every point of the interval AB_H of L belongs to H, or if it did H would be a subset of L. Hence there exists a segment S of the interval AB_L of L which contains no point of H. Let X and Y be the end points of S. Denote the intervals of AX and YB_L of L by U and V, respectively. A contradiction may now be reached by an argument very similar to the last part of the proof of Theorem 1.

That Theorems 2 and 3 do not remain true if the word "arcs" in their hypotheses is replaced by the words "irreducible continua" is shown by the following example.

Example. There exist in the plane two points A and B and three continua, each irreducible from A to B, such that each of them is a subset of the sum of the other two.

Construction. Let C be a unit circle and S be C plus its interior. Denote by I_H, I_K, and I_L three mutually exclusive and non-abutting segments of C. For each positive integer n let D_{Hn}, D_{Kn}, and D_{Ln} be simple domains lying in $S-C$ such that if X is H, K, or L and $E_{Xn}=D_{X1}+D_{X2}+\cdots+D_{Xn}$, then (1) $\overline{D}_{X1}\cdot C=\overline{I}_X$, (2) $\overline{D}_{Xn}\cdot C=0$ if $n>1$, (3) $\overline{D}_{Xn}\cdot\overline{D}_{X(n+1)}$ is an arc of length less than $1/n$, (4) $\overline{D}_{Xn}\cdot\overline{D}_{X(n+i)}=0$ if $i>1$, and (5) each point of S is within $1/n$ units of E_{Xn}. Furthermore let D_{Xn} be such that $E_{Hn}\cdot E_{Kn}=E_{Kn}\cdot E_{Ln}=E_{Ln}\cdot E_{Hn}$. Then if $M_X=S-(I_X+D_{X1}+D_{X2}+\cdots)$, it follows that each of the sets M_H, M_K, and M_L is an indecomposable continuum,[1] that each of them is a subset of the sum of the other two, and that no one of them is a subset of any other. It is readily seen that there are two points between which each of these continua is irreducible.

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[1] It will be noted that the construction of the separate continua H, K, and L was made by the "canal" method. See K. Yoneyama, *Theory of continuous set of points*, Tôhoku Math. J. vol. 12 (1917) pp. 60–62.

COMPONENT ORBITS UNDER POINTWISE RECURRENT HOMEOMORPHISMS

MARIANO GARCÍA

1. **Introduction.** The notion of a component orbit is due to G. E. Schweigert,[1] and is closely associated with two theorems established by D. W. Hall and G. E. Schweigert,[2] but so far has been considered only in connection with pointwise periodic homeomorphisms. It is the purpose of this paper to consider an extended concept of this notion and, by applying methods analogous to those used by G. T. Whyburn[3] in obtaining the results of Hall and Schweigert, exhibit certain theorems that contain these results as special cases.

2. **Preliminary definitions and remarks.** Throughout this paper, X will denote a separable metric space. Let $f(X)=X$ be a homeomorphism defined on X and let $x\in X$.

The point x is *recurrent under* f provided that given any neighborhood U of x, there exists a positive integer n such that $f^n(x)\in U$.

The mapping f is *pointwise recurrent* on the set $A\subset X$ provided each point of A is recurrent under f. It should be noted that pointwise recurrence, as defined here, is equivalent to pointwise almost periodicity as used by Ayres, Whyburn, and others.

A non-empty invariant set L whose components can be ordered in a sequence

$$\cdots, A_{-2}, A_{-1}, A_0, A_1, A_2, \cdots$$

such that

$$f(A_i) = A_{i+1}, \qquad i = 0, \pm 1, \pm 2, \cdots,$$

will be termed a *component orbit.*

A component orbit one of whose components is invariant under a positive power of f will be termed a *periodic component orbit.*[4]

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[1] *A note on the limit of orbits*, Bull. Amer. Math. Soc. vol. 46 (1940) pp. 963–969.

[2] *Properties of invariant sets under pointwise periodic homeomorphisms*, Duke Math. J. vol. 4 (1938) pp. 719–724.

[3] *Analytic Topology*, Amer. Math. Soc. Colloquium Publications vol. 28 (1942) pp. 259–261. We shall refer to this book hereafter as ATW.

[4] Schweigert's component orbits become periodic component orbits under this definition.

By the *set-orbit* of a set $A \subset X$ we mean

$$\sum_{-\infty}^{+\infty} f^i(A),$$

and a set-orbit is *non-wandering*[5] if there exists a positive integer N such that $A \cdot f^N(A) \neq 0$.

It follows without difficulty that a component orbit is the set-orbit of a connected set, and conversely, the set-orbit of a connected set is a component orbit. Also, a periodic component orbit is the non-wandering set-orbit of a connected set, and conversely, the non-wandering set-orbit of a connected set is a periodic component orbit. Also, if a component orbit contains a periodic component orbit, it is a periodic component orbit, and in a compact space X, a component orbit is periodic if it is closed. Finally we remark that the orbit of a point x

$$\sum_{-\infty}^{+\infty} f^i(x)$$

is a component orbit, and if x is periodic, its orbit is a periodic component orbit.

3. **The main results.** We now prove the principal theorems.

THEOREM I. *Let $f(X)=X$ be a homeomorphism on a compact space X. If $G_1, G_2, \cdots$ is a sequence of component orbits whose limit inferior contains a periodic component orbit Q, and if both f and f^{-1} are pointwise recurrent on* $\lim\sup G_i - Q$, *then* $\lim\sup G_i$ *is a periodic component orbit.*

PROOF. Let Q_0 be a component of Q and let K_0 be the component of $L = \lim\sup G_i$ which contains Q_0. Since L is invariant, the image of K_0 under any power of f is a component of L. Let $f(K_0)=K_1, f(K_1)=K_2, \cdots, f(K_{r-1})=K_r, f(K_r)=K_0$, so that

$$K = \sum_{0}^{r} K_i$$

is a periodic component orbit which contains Q. We shall show $K=L$.

Suppose this is not true. Then we can find[6] an open set V containing K, whose boundary $F(V)$ does not intersect L and such that L is not contained wholly in V.

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[5] In this connection see G. D. Birkhoff and P. A. Smith, *Structure analysis of surface transformations*, J. Math. Pures Appl. vol. 7 (1928) p. 350.

[6] ATW, Theorem (9.3), p. 15. The proof of Theorem I follows closely the proof of Theorem (6.5), p. 259.

Select a point x_i from each G_i such that $x_i \to x \in Q_0$. Infinitely many G_i intersect $X-\bar{V}$ and thus we may assume that for each i there exists a point $y_i \in G_i$ such that $y_i \in X-\bar{V}$. Let C_{x_i} be the component of G_i containing x_i. Since $x \in \liminf C_{x_i}$, $\limsup C_{x_i}$ is connected[7] and hence is contained in K_0. Also,[8] for almost all i, and therefore we may assume for all i, $C_{x_i} \subset V$. Each C_{x_i} contains a point z_i of the orbit of y_i. A subsequence of $\{z_i\}$, which we may suppose is the whole sequence, converges to a point $z \in K_0$.

There exists a subsequence $\{z_{i_j}\}$ of $\{z_i\}$ such that for each i_j, y_{i_j} is a positive power of z_{i_j} or for each i_j, y_{i_j} is a negative power of z_{i_j}. We shall assume that this sequence is $\{z_i\}$ and that y_i is a negative power of z_i for each i, since the proof for the case where y_i is a positive power of z_i for each i is perfectly analogous.

Now, let b_i be the first point in the sequence

$$z_i,\ f^{-1}(z_i),\ f^{-2}(z_i),\ \cdots$$

which is in $X-V$. A subsequence of $\{b_i\}$, which again we may suppose is the whole sequence, converges to a point $b \in L$. Furthermore, $b \notin \bar{V}$, since $L \cdot F(V)=0$.

For each i, $f(b_i)$ is in V and consequently

$$f(b_i) \to f(b) \in V \cdot (L-K),$$

since $L-K$ is invariant.

Now, there exists an integer N_1 such that

$$f(b_i) \neq z_i \qquad \text{for} \quad i > N_1.$$

This is true because otherwise we should have

$$f(b_i) \to z$$

and therefore

$$b_i \to f^{-1}(z) \neq b.$$

Thus the points $f(b_i)$, $i>N_1$, are negative powers of z_i which are in V, and therefore the points $f^2(b_i)$, $i>N_1$, are in V. Also,

$$f^2(b_i) \to f^2(b) \in V \cdot (L-K).$$

But again, there exists an integer $N_2>N_1$ such that

$$f^2(b_i) \neq z_i \qquad \text{for} \quad i > N_2.$$

The points $f^3(b_i)$, $i>N_2$, are then in V and

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[7] ATW, Theorem (9.1), pp. 14–15.

[8] ATW, Theorem (7.2), p. 12.

$$f^{s}(b_i) \to f^{s}(b) \in V\cdot(L-K).$$

But we can find an integer $N_3 > N_2$ such that

$$f^{3}(b_i) \neq z_i \qquad \text{for} \quad i > N_3,$$

and continuing in this manner we obtain

$$f^{n}(b) \in V, \qquad n = 1, 2, 3, \cdots .$$

This contradicts the recurrence of the point b under f, since $b \notin \overline{V}$ and therefore we can find a neighborhood R of b such that

$$f^{n}(b)\cdot R = 0, \qquad n = 1, 2, 3, \cdots .$$

Had we considered a sequence $\{z_i\}$ for which y_i is a positive power of z_i for each i, we would have obtained a contradiction to the recurrence under f^{-1} of some point of lim sup $G_i - Q$.

THEOREM II. *Let $f(X) = X$ be a homeomorphism on a compact space X. If $G_1, G_2, \cdots$ is a sequence of component orbits whose limit inferior contains a periodic component orbit Q and if f is pointwise recurrent on* lim sup $G_i + \sum_{i=1}^{+\infty} G_i$, *then* lim sup G_i *is a periodic component orbit.*

PROOF. This follows from the proof of the preceding theorem. Let O_i be the orbit determined by y_i and z_i in the proof of Theorem I. There exists a neighborhood U_i of z_i, of diameter less than $1/i$, such that $U_i \subset V$ and, due to the recurrence of z_i under f, corresponding to U_i we can find a point $r_i \in U_i$ such that r_i is a positive power of y_i. Let b_i be the first point in the sequence

$$r_i,\ f^{-1}(r_i),\ f^{-2}(r_i),\ \cdots$$

which is in $X - V$. Since $r_i \to z$, we can proceed as in the proof of Theorem I to show that the point b, to which we may assume $\{b_i\}$ converges, is not recurrent under f.

From the previous proofs we immediately have the following corollary.

COROLLARY 1. *Under the hypotheses of Theorem* I *or of Theorem* II, *the number of components of* lim sup G_i *can not exceed the number of components of Q. Thus, if* lim inf G_i *contains an invariant connected set (in particular, if it contains a fixed point), then* lim sup G_i *is connected.*

COROLLARY 2. *Let $f(X) = X$ be a homeomorphism on a compact space X. Let G be a component orbit in X. If f is pointwise recurrent on $\overline{G}$ and $\overline{G}$ contains a periodic component orbit Q, then $\overline{G}$ is a periodic component orbit and the number of its components does not exceed the number of components of Q.*

PROOF. This is a direct consequence of Theorem II and Corollary 1, for we need only take $G_i = G$ for all i. Then $\lim G_i = \overline{G}$ and the conclusion follows.

COROLLARY 3. *Any pointwise periodic continuous mapping g of a compact totally disconnected space X into itself has equicontinuous powers.*[9]

PROOF. It is a well known fact[10] that under the above hypotheses g is a homeomorphism of X onto itself. Let us show that the orbit decomposition given by g is continuous.[11]

Let (O_n) be a sequence of orbits of X such that $x \in \liminf (O_n)$. Then, letting O_x be the orbit of x,

$$O_x \subset \liminf (O_n) \subset \limsup (O_n).$$

Now, by Theorem I, $\limsup (O_n)$ is a periodic component orbit and, since X is totally disconnected, $\limsup (O_n)$ is an orbit.

Thus, $\lim (O_n) = O_x$ and therefore the orbit decomposition is continuous, from which it follows that f has equicontinuous powers.[12]

THEOREM III. *Let $f(X) = X$ be a homeomorphism on a compact space X. If $L = L_1 + L_2$ is a separation of a non-empty, closed, invariant set L in X such that f is pointwise recurrent on L and, for $i = 1$ or for $i = 2$, L_i has the property that the closure of the orbit of each of its points is a periodic component orbit,*[13] *then there exists a positive integer N such that*

$$f^N(L_i) = L_i, \qquad i = 1, 2.$$

PROOF. It is no restriction to assume that L_1 has the indicated property: $x \in L_1$ implies that the closure of the orbit of x is a periodic component orbit.

Since f is pointwise recurrent[14] on L, if the theorem is false, for each n there exists an $x_n \in L_1$ such that $f^{n!}(x_n) \in L_2$.

Choose a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ converging to a point $x \in L_1$. Let O be the orbit of x. By hypothesis,

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[9] For the meaning of the terms involved in this corollary, see ATW, pp. 34, 239, 252.

[10] ATW, Theorem (1.1) and Corollary (1.11), p. 240.

[11] ATW, pp. 129–130.

[12] ATW, Theorem (6.4), p. 257. See also Corollary (6.42), p. 258.

[13] We could substitute "contains a periodic component orbit" for the phrase "is a component orbit," but in virtue of Corollary 2 this does not matter.

[14] In connection with the phrase that follows, see ATW, Theorem (4.12), p. 247, keeping in mind that "pointwise almost periodic," as used there, is "pointwise recurrent" in our notation.

$$\bar{O} = A_0 + A_1 + \cdots + A_k,$$

where A_0 is a component of $\bar{O}$ and

$$f(A_i) = A_{i+1}, \qquad i = 0, 1, \cdots, k-1,$$

and $f(A_k) = A_0$. We may also assume that $A_r \neq A_s$ for $r \neq s$, r and s being any integers, $0 \leqq r,\ s \leqq k$.

Then $x \in A_j$, where j is an integer such that $0 \leqq j \leqq k$.

Define $g(y) = f^k(y)$, $y \in L$. By a result of W. H. Gottschalk,[15] g is pointwise recurrent on L.

For each n_i, let G_{n_i} be the orbit under g of x_{n_i}. Now,

$$x \in \liminf G_{n_i}.$$

Furthermore, since

$$\sum_{i=-\infty}^{+\infty} f^{ki}(x) \subset \liminf G_{n_i},$$

$$A_j = \overline{\sum_{i=-\infty}^{+\infty} f^{ki}(x)} \subset \liminf G_{n_i}.$$

But now $g(L) = L$ is a homeomorphism on the compact set L and $\{G_{n_i}\}$ is a sequence of orbits whose limit inferior contains the connected set A_j. Also, A_j is invariant under g and g is pointwise recurrent on L. Thus, by Theorem II, lim sup G_{n_i} is connected and consequently

$$\limsup G_{n_i} \subset L_1,$$

since $x \in L_1$.

But for each i,

$$f^{n_i l}(x_{n_i}) \in L_2$$

and for $n_i > k$,

$$f^{n_i l}(x_{n_i}) = g^{n_i l/k}(x_{n_i}).$$

Thus it follows that $G_{n_i} \cdot L_2 \neq 0$ for $n_i > k$. This gives

$$L_2 \cdot \limsup G_{n_i} \neq 0,$$

which contradicts lim sup $G_{n_i} \subset L_1$.

4. **Conclusion.** The main results that Hall and Schweigert obtained relative to component orbits can be stated in the following way:[16]

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[15] *Powers of homeomorphisms with almost periodic properties*, Bull. Amer. Math. Soc. vol. 50 (1944) Corollary 1, p. 223. This result is very useful here.

[16] See footnote 3.

THEOREM 1. *Let $f(X)=X$ be a pointwise periodic homeomorphism on a compact space X. Then if $G_1, G_2, \cdots$ is a convergent sequence of periodic component orbits in X, the limit set L is a periodic component orbit.*

THEOREM 2. *Let X be compact and suppose $f(X)=X$ is pointwise periodic. Any non-empty disconnected invariant closed set L in X has the property that for every separation*

$$L = L_1 + L_2$$

there exists an integer N such that

$$f^N(L_i) = L_i, \qquad i = 1, 2.$$

Theorem 1 follows immediately from either of Theorems I and II of this paper, for since f is pointwise periodic on X, f and f^{-1} are pointwise recurrent on X. Also, since X is compact, any set which is the limit of a convergent sequence of component orbits is non-empty, and therefore contains a periodic orbit, which clearly is a periodic component orbit. Thus, the hypotheses of Theorems I and II are satisfied and hence $\lim G_i$ is a periodic component orbit. In this connection it may be remarked that a homeomorphism $f(X)=X$ is pointwise periodic if and only if every component orbit in X is a periodic component orbit, where X is not necessarily compact.

That Theorem I is considerably more general than Theorem 1 is evident in more than one sense. Referring to the notation employed in Theorem I, examples satisfying the hypotheses of the theorem can be constructed for which f is not pointwise recurrent on Q and such that f is not pointwise periodic on $\limsup G_i - Q$.

Theorem 2, which incidentally may be used to prove Corollary 3 of this paper, is a direct consequence of Theorem III. For if f is pointwise periodic on X, the closure of any orbit in X is a periodic component orbit; namely, the orbit itself.

Here again Theorem III is of more generality than Theorem 2. Using the notation of Theorem III, examples fulfilling the conditions of the theorem can be given for which there exist points in L_2, say, the closures of whose orbits are not even component orbits, and such that f is not pointwise periodic on L_1 nor on L_2.

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GROUPS WITHOUT PROPER ISOMORPHIC QUOTIENT GROUPS

REINHOLD BAER

If f is a homomorphism of the group G, and if g is an isomorphism of the image group G^f, then fg is a homomorphism of G too; and this homomorphism is an isomorphism if, and only if, f is an isomorphism. Consequently the following three properties of the group G imply each other.

(1) *Homomorphisms of G upon isomorphic groups are isomorphisms.*

(2) *Homomorphisms of G upon itself are isomorphisms.*

(3) *If N is a normal subgroup of G such that G and G/N are isomorphic, then $N=1$.*

Groups meeting these requirements (1) to (3) shall be termed Q-groups. A direct product of an infinity of isomorphic groups different from 1 is certainly not a Q-group. On the other hand it is readily seen that groups satisfying the ascending chain condition for normal subgroups are Q-groups. Deeper is the fact that the group G is a Q-group if it belongs to one of the following three classes of groups.

(a) Free groups on a finite number of generators.[1]

(b) Groups of finite dimensional matrices with coefficients from a field, which are generated by a finite number of elements.[2]

(c) Free products of a finite number of abelian groups each of which is generated by a finite number of elements.[3]

All these groups are generated by a finite number of elements. But we are able to show that this latter condition is neither necessary nor sufficient for a group to be a Q-group. The complexity of the situation is increased by the fact that neither subgroups nor quotient groups of Q-groups need be Q-groups. Thus it becomes desirable to obtain general criteria for a group to be a Q-group, and this is the main object of the present note.

The subgroup S of the group G shall be termed *completely characteristic, if $S=S^f$ for every homomorphism f of G which satisfies $G=G^f$.* Examples of completely characteristic subgroups are the commutator subgroup of G and its generalizations. If G happens to be a Q-group, then every characteristic subgroup of G is completely characteristic.

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[1] Levi [1]; Magnus [2]. The numbers in brackets refer to the bibliography at the end of the paper.

[2] Malcew [1].

[3] Fouxe-Rabinowitch [1].

THEOREM 1. *The group G is a Q-group if there exists a well ordered ascending chain of completely characteristic subgroups S_v with the following properties.*

(i) $S_0 = 1$; $S_k = G$ *for some* k.

(ii) S_{v+1}/S_v *is a* Q-*group for every* $v < k$.

(iii) *If* v *is a limit ordinal, then every element in* S_v *is contained in some* S_u *with* $u < v$.

PROOF. Suppose that f is a homomorphism of G satisfying $G = G^f$, and that N is the kernel[4] of this homomorphism f. We are going to prove by complete (transfinite) induction that the crosscut[5] of N and S_v is 1. Since this assertion is certainly true for $v = 0$, we may assume its validity for every $u < v$.

Case 1. $v = u + 1$ is not a limit ordinal.

Then we infer from our induction hypothesis that $N \cap S_u = 1$. We deduce furthermore $S_v = S_v^f$ and $S_u = S_u^f$ from the fact that the subgroups S are completely characteristic subgroups of G and that $G = G^f$. Thus f induces a homomorphism of S_v/S_u such that $S_v/S_u = (S_v/S_u)^f$. But S_v/S_u is by (ii) a Q-group and f consequently induces an isomorphism in S_v/S_u. If x is an element in the crosscut of N and S_v, then $(S_u x)^f = S_u$. Hence $S_u x = S_u$ and x is in S_u. But the crosscut of S_u and N is 1, proving $x = 1$. Thus $N \cap S_v = 1$.

Case 2. v is a limit ordinal.

If x belongs to $S_v \cap N$, then x belongs by (iii) to $S_u \cap N$ for $u < v$. Hence $x = 1$ may be inferred from the induction hypothesis, showing again that $N \cap S_v = 1$.

Thus it follows in particular that $N = N \cap G = N \cap S_k = 1$, proving that f is an isomorphism, as we desired to show.

The subgroup S of the group G has been termed *strictly characteristic,*[6] *if* $S^f \leqq S$ *for every homomorphism* f *of* G *satisfying* $G^f = G$. The members $Z_v(G)$ of the *ascending central chain* of the group G are strictly characteristic subgroups of G. To prove this it suffices to recall their inductive definition.

(i) $Z_0(G) = 1$.

(ii) $Z_{v+1}(G)/Z_v(G)$ is the center[7] of $G/Z_v(G)$.

(iii) If v is a limit ordinal, then $Z_v(G)$ is the (set theoretical) join of the subgroups $Z_u(G)$ for $u < v$.

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[4] The kernel of the homomorphism f consists of the elements in G which are mapped upon 1 by f.

[5] The crosscut of the sets A and B will always be denoted by $A \cap B$.

[6] For a discussion of this and related concepts see Baer [2].

[7] The center of the group G consists of all the elements z in G which satisfy $zx = xz$ for every x in G.

Let us note that the group G is termed *nilpotent*[8] if there exists an ordinal z such that $G=Z_z(G)$.

The members of the ascending central series, though strictly characteristic, need not be completely characteristic. This will be apparent from the following example,[9] which shows at the same time the impossibility of substituting "strictly characteristic" for "completely characteristic" in Theorem 1.

Let B be an abelian group of type p^∞. Then B contains one and only one subgroup B_i which is a cyclic group of order p^i, and every element in B is in at least one B_i. Denote by G the group obtained by adjoining to B an element g subject to the relation

$$g^{-1}xg = x^{1+p}$$

for x in B. It is readily seen that $B_i=Z_i(G)$ for finite i, that $B=Z_\omega(G)$ and that $G=Z_{\omega+1}(G)$. Thus G is nilpotent and the quotient groups of successive terms in the ascending central series are cyclic groups (and therefore Q-groups). But G is not a Q-group, since G and G/B_1 are clearly isomorphic groups.

COROLLARY. *If G is a nilpotent group and if every $Z_{v+1}(G)/Z_v(G)$ is a Q-group, then the following condition is necessary and sufficient for G to be a Q-group.*

Every $Z_v(G)$ is a completely characteristic subgroup of G.

The sufficiency of this condition is an immediate consequence of Theorem 1; its necessity may be derived from the fact that every characteristic subgroup of a Q-group is completely characteristic. The impossibility of omitting this condition from the statement of this corollary is a consequence of the considerations immediately preceding this corollary.

LEMMA. *If S is a strictly characteristic subgroup of the group G, and if G/S is a Q-group, then S is completely characteristic.*

PROOF. If f is an endomorphism of G such that $G=G^f$, then $S^f\leq S$ and an endomorphism of G/S upon itself is defined by mapping the coset X in G/S upon the coset SX^f in G/S. Since G/S is a Q-group, it follows that this endomorphism of G/S is an automorphism. From this fact we conclude that the element x in G belongs to S if, and only if, x^f belongs to S. But to every element y in S there exists an element z in G such that $z^f=y$; and from the preceding remark we infer now that z belongs to S too, proving $S=S^f$, as we desired to show.

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[8] See Baer [1] for a discussion of this concept of nilpotency.

[9] This is essentially a restatement of Baer [1, Example 2.5, p. 406].

The next criterion is an almost exact dual to Theorem 1 with the important difference that we need only consider strictly characteristic subgroups.

THEOREM 2. *The group G is a Q-group if there exists a well ordered descending chain of strictly characteristic subgroups S_v with the following properties.*

(i) *$S_0 = G$; $S_k = 1$ for some ordinal k.*

(ii) *S_v/S_{v+1} is a Q-group for every $v < k$.*

(iii) *If v is a limit ordinal, then S_v is the crosscut of the subgroups S_u for $u < v$.*

PROOF. If the homomorphism f of the group G satisfies $G = G^f$, then we are going to prove the following assertion by complete (transfinite) induction with respect to v.

(v) *$S_v = S_v^f$ and f induces an isomorphism of G/S_v.*

Since this assertion (u) certainly holds true for $u = 0$, we may assume its validity for every $u < v$, in order to prove (v). We distinguish two cases.

Case 1. $v = u + 1$ is not a limit ordinal.

Then we infer from the induction hypothesis that $S_u = S_u^f$ and that f induces an isomorphism in G/S_u. Since S_v is strictly characteristic, we have $S_v^f \leqq S_v$. If X is a coset in S_u/S_v, then mapping X upon $S_v X^f$ effects a homomorphism of S_u/S_v upon itself, that is, S_u/S_v is the image of S_u/S_v under this homomorphism. But S_u/S_v is by (ii) a Q-group, and thus f induces a proper automorphism of S_u/S_v. Suppose now that X is a coset of G/S_v which is mapped upon 1 by the homomorphism f of G/S_v. Then $X^f \leqq S_v$ and therefore $S_u X^f = S_u$. But f induces an isomorphism in G/S_u so that $X \leqq S_u$ and X is a coset of S_u/S_v. But then $X = 1$, since f induces an isomorphism in S_u/S_v, and we have shown that f induces a proper automorphism of G/S_v. If x is an element in S_v, then there exists an element y in G such that $x = y^f$. The coset $S_v y$ is mapped by f into part of S_v. Since f induces an isomorphism in G/S_v, as we have shown just now, it follows that $S_v y = S_v$, and this proves $S_v = S_v^f$, completing the proof of the case $(v) = (u+1)$.

Case 2. v is a limit ordinal.

If X is a coset of G/S_v such that $X^f \leqq S_v$, then $X^f \leqq S_u$ for $u < v$. From the induction hypothesis we infer that f induces an isomorphism in G/S_u for $u < v$. Since f maps the coset $S_u X$ of G/S_u upon the identity of G/S_u, it follows that $X \leqq S_u$ for $u < v$, and we infer from (iii) that $X \leqq S_v$. Thus f induces a proper automorphism of G/S_v; the fact that $S_v = S_v^f$ is verified as in Case 1.

Thus we have verified (v) for every $v \leqq k$. From condition (i) and proposition (k) we infer finally that f is an automorphism of G, as we desired to show.

To enunciate two important consequences of this theorem we need the *descending central series* of G, whose members ${}^{v}G$ are defined inductively as follows:

(i) ${}^{0}G=G$.

(ii) ${}^{v+1}G$ is generated by all the commutators $x^{-1}y^{-1}xy$ for x in G and y in ${}^{v}G$.

(iii) If v is a limit ordinal, then ${}^{v}G$ is the crosscut of all the ${}^{u}G$ for $u<v$.

It is readily verified that ${}^{v}G^{f} \leqq {}^{v}G$ for every endomorphism[10] f of G; that is, the subgroups ${}^{v}G$ are fully invariant,[11] and they are both completely and strictly characteristic subgroups of G. This descending central chain certainly meets the requirement (iii) of Theorem 2, and thus we obtain the following criterion.

COROLLARY 1. *The group G is a Q-group, if there exists an ordinal k such that ${}^{k}G=1$, and if the* (*abelian*) *groups ${}^{v}G/{}^{v+1}G$ are all Q-groups.*

An abelian group is certainly a *Q-group* if it is generated by a finite number of elements. It is well known furthermore that ${}^{i}G/{}^{i+1}G$, for finite i, is generated by a finite number of elements if G is generated by a finite number of elements. Thus the following fact is a special case of Corollary 1.

COROLLARY 2.[12] *The group G is a Q-group if it is generated by a finite number of elements, and if ${}^{\omega}G=1$.*

If F is a free group, then Magnus[13] has shown that ${}^{\omega}F=1$, and thus the last criteria are generalizations of the theorem[14] that free groups on a finite number of generators are Q-groups.

In order to apply Theorem 2 to abelian groups we introduce some notations. If A is an abelian group, then denote by $F_p(A)$ the set of all the elements in A whose order is a power of the prime number p. Clearly $F_p(A)$ is a fully invariant subgroup of A. The product $F(A)$ of all the $F_p(A)$ is their direct product and consists of all the elements of finite order whereas $A/F(A)$ contains no elements of finite order except 1.

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[10] Endomorphisms of the group G are homomorphisms of G into G.

[11] This concept is due to Levi [1]; for further discussion see Baer [2].

[12] Fouxe-Rabinowitch [1].

[13] Magnus [1].

[14] See footnote 1.

COROLLARY 3. *If $A/F(A)$ as well as every $F_p(A)$ is a Q-group, then the abelian group A is a Q-group.*

PROOF. Let $S_0=A$, $S_1=F(A)$, $S_i=\prod_{i\leq j}F_{p_j}(A)$ where p_1, $p_2, \cdots$ is some enumeration of the primes. The subgroups S_i are clearly strictly characteristic and their crosscut $S_\infty=1$. Since $S_0/S_1=A/F(A)$ and since S_i/S_{i+1} is, for positive i, isomorphic to $F_{p_i}(A)$, the hypotheses of Theorem 2 are satisfied by this descending chain of subgroups, proving the validity of our present contention.

Let us note that $F_p(A)$ is a Q-group if it is finite, and that $F/F(A)$ is a Q-group if it is of finite rank, that is, if there exists an integer k such that every subgroup of $F/F(A)$ which may be generated by a finite number of elements may also be generated by k elements. Combining Corollaries 1 and 3 and the preceding remarks it is easy to construct any number of examples of Q-groups that cannot be generated by a finite number of elements.

Examples. 1. It is known that the free group on two generators contains a free group on an infinity of generators as a subgroup.[15] The former group is a Q-group, the latter is not. This shows that *not every subgroup of a Q-group need be a Q-group.*

2. The additive group R of the rational numbers cannot be generated by a finite number of elements. But R is a Q-group. For if R is considered modulo a subgroup different from 0, then all the elements in the quotient group are of finite order. If J is the subgroup of the integers in R, and if E is the subgroup of the even integers in R, then $E\leqq J$ and R/J and R/E are isomorphic groups so that R/E is not a Q-group, showing that *quotient groups of Q-groups need not be Q-groups.*

3. We are going to construct an example of a group G with the following properties.

(a) *G is generated by two elements.*

(b) $(G, G'')=1$.

(c) *G is not a Q-group.*

Here we denote by G' the commutator subgroup of G, by G'' the commutator subgroup of G', and by (G, G'') the subgroup generated by all the commutators $x^{-1}y^{-1}xy$ for x in G and y in G''.

This example shows that Corollary 2 of Theorem 2 is in certain respects a "best" result.

To construct the group G we proceed as follows. Let F be a free group on two generators and denote by u, v a free pair of generators of F. Then F/F' is the free abelian group of rank 2. The group F'/F''

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[15] For example, the commutator subgroup.

may be shown to be a free abelian group, a basis of which is given by the elements

$$(i, j) = u^{-i}v^{-j}u^{-1}v^{-1}uv\,v^{j}u^{i}$$

for integral i, j. We put $N=(F, F'')$. Then F''/N is part of the center of F/N; it is a free abelian group, generated by the elements (commutators)

$$(i, j)(i', j')(i, j)^{-1}(i', j')^{-1}.$$

Every subgroup of F''/N is a normal subgroup of F/N, since F''/N is part of the center of F/N. We denote by U the subgroup of F generated by N and the elements $(0, j)(0, j')(0, j)^{-1}(0, j')^{-1}$ for all positive j, j', and we denote by V the subgroup of F generated by N and all the elements $(0, j)(0, j')(0, j)^{-1}(0, j')^{-1}$ for j, j' not negative. It is clear that U and V are normal subgroups of F and that

$$N < V < U < F''.$$

We are going to prove now that $G=F/V$ is the desired group. It is immediately clear that conditions (a) and (b) are satisfied by G. To prove (c) we construct an automorphism of the group F which maps V upon U. There exists one and only one automorphism f of F which maps u upon uv and v upon v. We have

$$F'^{f} = F', \qquad F''^{f} = F'', \qquad N^{f} = N,$$

since these subgroups are characteristic subgroups of F. Furthermore we have

$$(0, 0)^{f} = (u^{-1}v^{-1}uv)^{f} = (uv)^{-1}v^{-1}(uv)v = v^{-1}(0, 0)v = (0, 1),$$

and consequently

$$(0, j)^{f} = (0, j + 1)$$

for integral j. From this last equation one derives the equality $V^{f} = U$, completing the proof.

From the preceding arguments one deduces readily the noteworthy fact that the ascending chain condition is not satisfied by the normal subgroups of a nonabelian free group.

Dualizing the concept of a Q-group we define as an *S-group* a group G meeting the following requirement.

(S) *If the subgroup S of G is isomorphic to G, then $S=G$.*

It is readily seen that every group satisfying the descending chain condition for subgroups is an S-group. An example of an S-group not satisfying this chain condition is the additive group of rational numbers.

To enunciate conveniently a criterion for S-groups we need the following concept. The subgroup C of the group G is said to be an *S-characteristic subgroup of G* if $C^f \leq C$ for every isomorphism f of G into itself. Clearly every fully invariant subgroup is S-characteristic and every S-characteristic subgroup is characteristic. If G happens to be an S-group, then conversely every characteristic subgroup of G is S-characteristic.

THEOREM 3. *The group G is an S-group if there exists a well ordered ascending chain of S-characteristic subgroups $N(v)$ of G with the following properties:*

(i) $N(0)=1$ *and* $N(t)=G$ *for some ordinal* t.

(ii) $N(v+1)/N(v)$ *is an S-group.*

(iii) *If v is a limit ordinal, then every element in $N(v)$ is contained in some $N(u)$ for $u<v$.*

PROOF. Suppose that f is an isomorphism of G upon some subgroup S of G. We are going to prove by complete (transfinite) induction that $N(v)^f=N(v)$ for every v. This assertion is certainly true for $v=0$; and thus we may assume its validity for every $u<v$. If v happens to be a limit ordinal, then our contention is directly derived from (iii) and the induction hypothesis. If, however, $v=u+1$ is not a limit ordinal, then we deduce from $N(u)^f=N(u)$ and $N(v)^f\leq N(v)$ that f induces an isomorphism of the group $N(v)/N(u)$ upon a subgroup of $N(v)/N(u)$. But it follows from (ii) that $N(v)/N(u)$ is an S-group. Hence $(N(v)/N(u))^f=N(v)/N(u)$ or $N(v)=N(v)^fN(u)=N(v)^fN(u)^f =N(v)^f$, as we desired to show. Now $G=N(t)=N(t)^f=G^f$ is a consequence of (i).

We are interested in the relations between Q-groups and S-groups. There are groups that are both, like the finite groups or the additive group of rational numbers. Any direct (or free) product of an infinity of isomorphic groups ($\neq 1$) is neither. The infinite cyclic groups are Q-groups, but not S-groups, and the abelian groups of type p^∞ are S-groups, but not Q-groups. Thus the following theorem seems to be of some interest.

THEOREM 4. *If N is a fully invariant subgroup of the free group F, and if $G=F/N$ is an S-group, then G is a Q-group.*

PROOF. If L is a normal subgroup of G such that G and G/L are isomorphic groups, then denote by M the uniquely determined normal subgroup of F such that $N\leq M$ and $M/N=L$. There exists an isomorphism g of G upon G/L, and hence there exists an endomorphism f of F satisfying $X^g=MX^f$ for every coset X of $F/N=G$. (To prove the existence of f, consider a free set of generators of F and define f on this free set of generators in accordance with our requirement. Then f can easily be seen to be a desired endomorphism.) Then $F=F^fM$ and $F^f\cap M=N^f\leqq N$, since N is a fully invariant subgroup of F. Since M and N are normal subgroups of F, we deduce from Dedekind's law that

$$F^fN \cap M = (F^f \cap M)N = N^fN = N,$$

and from $N\leqq M$ we infer $(F^fN)M=F^fM=F$. The subgroup $(F^fN)/N = V$ of $G=F/N$ has therefore the following properties:

$$V \cap L = 1 \quad \text{and} \quad VL = G.$$

Thus every coset of G/L contains one and only one element of the subgroup V of G, proving that V and G/L are isomorphic. But G/L and G are isomorphic and G is an S-group, proving that $V=G$. This implies finally $L=1$, showing that G is a Q-group.

It is apparent from the proof of this theorem that it remains valid if we substitute for the hypothesis that F be a free group the following assumption:

If M and N are normal subgroups of F, if $N\leqq M$, and if g is an isomorphism of F/N upon F/M, then there exists an endomorphism f of F, satisfying

$$X^g = MX^f$$

for every coset X of F/N.

It has been shown that the properties (1) to (3) may be deduced (a) from the existence of an ascending chain of completely characteristic subgroups, meeting certain requirements (Theorem 1), and (b) from the existence of a descending chain of strictly characteristic subgroups, satisfying certain conditions (Theorem 2). But these properties (a) and (b) are not independent. For the following correspondence between subgroups maps every ascending chain of completely characteristic subgroups upon a descending chain of strictly characteristic subgroups. If S is a subgroup of the group G, then let S^* be *the centralizer of S in G*, which consists exactly of those elements in G commuting with all the elements in S. To substantiate our contention we enumerate a number of properties of this operation.

(1) If $S\leqq T$, then $T^*\leqq S^*$.

(2) If S is the (set theoretical) join of the subgroups S_v, then S^* is the crosscut of the subgroups S_v^*.

These two properties are immediately obvious. It should be noted, however, that neither the dual nor the converse of (2) is true.

(3) If S is a completely characteristic subgroup of G, then S^* is a strictly characteristic subgroup of G.

Proof. If $G=G^f$ is satisfied by the endomorphism f of G, if s is an element in S, and if t is an element in S^*, then there exists an element s' in S such that $s=s'^f$; and we find:

$$st^f = s'^f t^f = (s't)^f = (ts')^f = t^f s'^f = t^f s,$$

proving that t^f belongs to S^* too, or $S^{*f} \leqq S^*$.

Remark. The impossibility of proving that S^* is completely characteristic, if S is completely characteristic, may be shown by the following example: If $S=G$, then S is certainly completely characteristic. But S^* is the center of G; and we have pointed out before that the center is not always completely characteristic.

Note that in the proof of (3) we actually verified the following fact. If S is a subgroup of G, if f is an endomorphism of G, and if $S \leqq S^f$, then $S^{*f} \leqq S^*$. This implies in particular:

(4) S^* is a normal subgroup of G, if S is a normal subgroup of G.

To enunciate the next statement it will be convenient to introduce some notations. If X and Y are subsets of G, then we denote by (X, Y) the subgroup generated by all the commutators $x^{-1}y^{-1}xy$ for x in X and y in Y. For these subgroups P. Hall proved the following important inequality:

(H)[16] If A, B, C are normal subgroups of G, then

$$(A, (B, C)) \leqq (B, (C, A))(C, (A, B)).$$

Now we are able to prove

(5) If $(G, T) \leqq S$ is satisfied by the subgroups S and T of G, then $(S^*, S^*) \leqq T^*$.

Proof. We deduce from (H) that

$$(T, (S^*, S^*)) \leqq (S^*, (S^*, T)) \leqq (S^*, (G, T)) \leqq (S^*, S) = 1, \text{ proving (5).}$$

We define *the derived series* $G^{(v)}$ of the group G inductively as follows.

(i) $G^{(0)}=G$.

(ii) $G^{(v+1)}$ is the commutator subgroup of $G^{(v)}$.

(iii) If v is a limit ordinal, then $G^{(v)}$ is the crosscut of the subgroups $G^{(u)}$ for $u<v$.

If in particular there exists an ordinal s such that $G=G^{(s)}$, then G may be termed *soluble*.[17]

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[16] Hall [1].

[17] A different concept of solubility which is narrower than the present one has been discussed in Baer [1].

THEOREM 5. *Nilpotent groups are soluble.*

PROOF. Put $C_v = Z_v(G)^*$ for every v. Then we have $C_0 = C_1 = G$, since $Z_1(G)$ is the center of G. From $(G, Z_{v+1}(G)) \leqq Z_v(G)$ and property (5) we deduce that

$$(C_v, C_v) \leqq C_{v+1}.$$

If v is a limit ordinal, then $Z_v(G)$ is the join of the $Z_u(G)$ for $u < v$, and we infer from (2) that C_v is the crosscut of the subgroups C_u for $u < v$. Hence it follows by complete (transfinite) induction that

$$G^{(v)} \leqq C_v \text{ for every } v.$$

Since G is nilpotent, there exists an ordinal k such that $G = Z_k(G)$. Clearly C_k is the center of G. But $G^{(k)}$ is part of C_k and C_k is abelian. Hence $G^{(k+1)} = 1$, proving the solubility of G.

Appended to Theorem 1 is a discussion of an example of a nilpotent group G which can be shown to have the property ${}^1G = {}^2G \neq 1$. Thus nilpotency does not imply that the descending central series ends with 1.

On the other hand it may be worth noting that the nonabelian free groups F satisfy ${}^\omega F = 1$, though their center is equal to 1, showing the impossibility of deducing nilpotency from the fact that the descending central chain ends with 1.

We note finally that the situation is altogether different if one is mainly interested in finite chains. For the following theorem is well known:[18] $G = Z_n(G)$ for finite n if, and only if, $1 = {}^mG$ for finite m.

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[18] Hall [1] or Baer [1, footnote on p. 406].

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A SUBSTITUTE FOR THE AXIOM OF CHOICE

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The following result appeared in the 1914 edition of Hausdorff's *Mengenlehre,* p. 140:

(A) *Any partially ordered system contains a maximal simply ordered subsystem.*

This theorem is well known to be equivalent to the axiom of choice (though there does not seem to be a proof of this fact in the literature) and it has been suggested as an alternative for this axiom. The purpose of this note (which is purely methodological) is to propose a simpler but equivalent formulation of (A) as a substitute for the Zermelo axiom. The simplicity lies in the fact that we make no assumptions concerning the relation R which replaces partial order.

Let Q be a set and R an *arbitrary* binary relation on Q. A subset of Q will be termed *R-simple* if for any pair of its elements, a and b, we have either aRb or bRa. The version of (A) we propose is:

(B) *Any R-simple subset of Q is contained in a maximal R-simple subset of Q.*

It is clear that (B) implies (A). Conversely, let Q_0 be an R-simple subset of Q. Let P be the partially ordered (by inclusion) system composed of all R-simple subsets of Q which contain Q_0. Then by (A) there is a maximal simply ordered subsystem P_0 of P. The union of all the sets in P_0 is the desired maximal R-simple subset.

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PROOF OF A THEOREM OF THORIN

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Let

$$A(x, y) = \sum_{j,k=1}^{m,n} a_{jk}x_jy_k$$

be a complex bilinear form, and let $M(\alpha, \beta)$ denote the upper bound of $|A(x, y)|$ for the x's and y's satisfying the condition

$$(\sum |x_j|^{1/\alpha})^\alpha \leqq 1, \qquad (\sum |y_k|^{1/\beta})^\beta \leqq 1.$$

Then $\log M(\alpha, \beta)$ is a convex function of the point (α, β) in the triangle

$$(1) \qquad 0 \leqq \alpha \leqq 1, \qquad 0 \leqq \beta \leqq 1, \qquad \alpha + \beta \geqq 1.$$

The above result is due to M. Riesz.[1] In a recent paper it was shown by Thorin[2] that the theorem is restricted neither to bilinear forms nor to the triangle (1). Thorin's result is as follows.

THEOREM. (i) *Let* $f(z_1, z_2, \cdots, z_r)$ *be an entire function of* r *complex variables* $z_1, z_2, \cdots, z_r$. *Let* K *be a bounded domain* $(v_1, v_2, \cdots, v_r)$ *of the* r*-dimensional Euclidean space, satisfying the conditions* $v_1 \geqq 0$, $v_2 \geqq 0, \cdots, v_r \geqq 0$. *Let* $M(\alpha_1, \alpha_2, \cdots, \alpha_r)$ *denote the upper bound of* $|f(z_1, z_2, \cdots, z_r)|$ *for*

$$|z_1| = v_1^{\alpha_1}, \cdots, |z_r| = v_r^{\alpha_r}, \quad \textit{and} \quad (v_1, v_2, \cdots, v_r) \in K.$$

Then $\log M(\alpha_1, \alpha_2, \cdots, \alpha_r)$ *is a convex function of the point* $(\alpha_1, \alpha_2, \cdots, \alpha_r)$ *in the domain* $0 \leqq \alpha_j < +\infty, j = 1, 2, \cdots, r$.

(ii) *If the points of* K *satisfy a condition*

$$0 < A \leqq v_j \leqq B < +\infty \qquad (j = 1, 2, \cdots, r)$$

then $\log M(\alpha_1, \alpha_2, \cdots, \alpha_r)$ *is convex in the whole space* $-\infty < \alpha_j < +\infty$, $j = 1, 2, \cdots, r$.

In case (ii), vanishing of some of the α's does not require additional discussion. The situation is slightly different in case (i), since $v_j^{\alpha_j}$ has no meaning if both v_j and α_j are zero. The sets $z_1, z_2, \cdots, z_r$

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[1] M. Riesz, *Sur les maxima des formes bilinéaires et sur les fonctionnelles linéaires*, Acta Math. vol. 49 (1926) pp. 465–497.

[2] G. O. Thorin, *An extension of a convexity theorem due to M. Riesz*, Kungl. Fysiografiska Sällskapets i Lund Förhandlinger vol. 8 (1939) no. 14.

for which this occurs have simply to be disregarded in the definition of $M(\alpha_1, \alpha_2, \cdots, \alpha_r)$.

Owing to the importance of the theorem (even in the special case of M. Riesz), any simplification of its proof may be of interest. The proof given below seems to be slightly simpler than the original one of Thorin, and is based on the following elementary principle.[3]

(a) *A necessary and sufficient condition that a real valued function* $\phi(t)$ *be convex in an interval* (a, b) *is that, for any subinterval* (a', b') *and any real number* μ, *the maximum of the function* $\phi(t)+\mu t$ *in* (a', b') *be attained at least at one of the end points* a', b'.

Its proof is immediate. If $\phi(t)$ is convex, so is $\phi_1(t)=\phi(t)+\mu t$. Since no point of the arc $y=\phi_1(t)$, $a'\leqq t\leqq b'$, lies above the chord joining its end points, the maximum of $\phi_1(t)$ in (a', b') is either at $t=a'$ or $t=b'$. Conversely, if $\phi(t)$ is not convex in (a, b), there is an arc $y=\phi(t)$, $a'\leqq t\leqq b'$, which lies partly above the corresponding chord. If $y=-\mu t-\nu$ is the equation of the chord, the sum $\phi(t)+\mu t+\nu$ vanishes for $t=t_1, t_2$, and takes on positive values at some points inside (a', b'). Hence $\phi(t)+\mu t$ does not attain its maximum in (a', b') at either of the end points a', b'.

The following form of the principle will be more convenient for our purposes.

(b) *A necessary and sufficient condition that the logarithm of a non-negative function* $\phi(t)$ *be convex in an interval* (a, b) *is that, for any subinterval* (a', b') *and any real* μ, *the maximum of the function* $\phi(t)e^{\mu t}$ *in* (a', b') *be attained at least at one of the end points* a', b'.

The necessity of the condition follows from (a). So does the sufficiency, if $\phi(t)$ is strictly positive. It is, however, easy to show that if $\phi(t)$ is non-negative, satisfies the condition of (b), and vanishes at a point t_0 of (a, b), then $\phi(t)$ vanishes identically inside (a, b). For if $\phi(t)$ were positive at a point t_1 which lies, for example, to the right of t_0, so that $t_0<t_1<b$, and if $t_1<t_2<b$, then $\phi(t)e^{\mu(t-t_1)}$ is 0 at t_0, is $\phi(t_1)$ at t_1, and is very small at t_2, if μ is negative and large enough. Hence the maximum of $\phi(t)e^{\mu(t-t_1)}$ in (t_0, t_2) is attained at neither end point. This proves (b), if we agree to consider the function which is $-\infty$ everywhere inside (a, b) as convex.

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[3] This principle, both in forms (a) and (b), was first stated by S. Saks and used by him to simplify certain proofs from the theory of functions. See his papers *Sur un théorème de M. Montel*, C. R. Acad. Sci. Paris vol. 187 (1928) pp. 276–277, and *On the properties of convex and subharmonic functions* (in Polish), Mathesis Polska vol. 6 (1930).

We now pass to the proof of the theorem of Thorin, starting with part (ii). Let

$$v_j = e^{\xi_j}, \qquad j = 1, 2, \cdots, r.$$

If $(v_1, v_2, \cdots, v_r) \in K$, the ξ's are bounded. Obviously, we may assume here that K is closed. Hence

$$M(\alpha_1, \alpha_2, \cdots, \alpha_r) = \max \left| f(e^{\alpha_1\xi_1 + i\eta_1}, \cdots, e^{\alpha_r\xi_r + i\eta_r}) \right|$$

for $(e^{\xi_1}, e^{\xi_2}, \cdots, e^{\xi_r}) \in K$ and $\eta_1, \eta_2, \cdots, \eta_r$ arbitrary. Since $|f(e^{\alpha_1\xi_1+i\eta_1}, \cdots)|$ is a continuous function of the variables α_j, ξ_j, η_j, it follows that $M(\alpha_1, \alpha_2, \cdots, \alpha_r)$ is a continuous function of the α's. In order to show that its logarithm is convex on any straight line

$$\alpha_1 = \overset{0}{\alpha}_1 + \lambda_1 t,\ \alpha_2 = \overset{0}{\alpha}_2 + \lambda_2 t, \cdots, \alpha_r = \overset{0}{\alpha}_r + \lambda_r t,$$

it is enough to show that, for any real μ, the product

$$M(\overset{0}{\alpha}_1 + \lambda_1 t, \cdots, \overset{0}{\alpha}_r + \lambda_r t)e^{\mu t}$$

has no proper maximum for any finite t.

Let us assume, contrary to this, that for some μ such a maximum does exist. By a change of variable we may adjust it so that the maximum is attained for $t=0$. Then

$$M(\overset{0}{\alpha}_1, \overset{0}{\alpha}_2, \cdots, \overset{0}{\alpha}_r) \geqq M(\overset{0}{\alpha}_1 + \lambda_1 t, \overset{0}{\alpha}_2 + \lambda_2 t, \cdots, \overset{0}{\alpha}_r + \lambda_r t)e^{\mu t} \tag{2}$$

for small $|t|$, and we have strict inequality for some t's arbitrarily small. We may write

$$M(\overset{0}{\alpha}_1, \cdots, \overset{0}{\alpha}_r) = \left| f(e^{\overset{0}{\alpha}_1\overset{0}{\xi}_1 + i\overset{0}{\eta}_1}, \cdots, e^{\overset{0}{\alpha}_r\overset{0}{\xi}_r + i\overset{0}{\eta}_r}) \right|,$$

where $(e^{\xi_1^0}, e^{\xi_2^0}, \cdots, e^{\xi_r^0})$ is a point of K, and the $\eta_1^0, \eta_2^0, \cdots, \eta_r^0$ are suitably chosen. Thus

$$\begin{aligned} &\left| f(e^{\alpha_1^0\xi_1^0 + i\eta_1^0}, \cdots, e^{\alpha_r^0\xi_r^0 + i\eta_r^0}) \right| \\ &\qquad \geqq \left| f(e^{(\alpha_1^0+\lambda_1 t)\xi_1^0 + i\eta_1}, \cdots, e^{(\alpha_r^0+\lambda_r t)\xi_r^0 + i\eta_r})e^{\mu t} \right| \end{aligned} \tag{3}$$

for all real t's sufficiently small in absolute value, and for all real η's. We assert that (3) holds for all *complex* t's of sufficiently small modulus. This follows immediately, for if we replace t by $t+iv$ in (3), the only effect will be to change the η's. It follows in particular that the modulus of the entire function (in t)

$$f(e^{(\alpha_1^0+\lambda_1 t)\xi_1^0 + i\eta_1^0}, \cdots, e^{(\alpha_r^0+\lambda_r t)\xi_r^0 + i\eta_r^0})e^{\mu t}$$

attains a proper maximum at $t=0$, which contradicts the principle of the maximum for analytic functions. Hence part (ii) of the theorem is proved.

As Thorin himself points out, part (i) of the theorem is an immediate consequence of part (ii). For if K_ϵ is the part of K for which $v_j \geqq \epsilon$, $j=1, 2, \cdots, r$, and if $M_\epsilon(\alpha_1, \alpha_2, \cdots, \alpha_r)$ is the function $M(\alpha_1, \alpha_2, \cdots, \alpha_r)$ corresponding to K_ϵ, then

$$\lim_{\epsilon \to 0} M_\epsilon(\alpha_1, \alpha_2, \cdots, \alpha_r) = M(\alpha_1, \alpha_2, \cdots, \alpha_r).$$

This holds both when all the α's are different from zero and when some of them vanish. Hence $M(\alpha_1, \alpha_2, \cdots, \alpha_r)$ is convex for $\alpha_1 \geqq 0, \alpha_2 \geqq 0, \cdots, \alpha_r \geqq 0$. The restriction $\alpha_j \geqq 0$ is introduced to avoid infinite values for M.

Let $k>0$. The theorem of Thorin may be considered as the limiting case $(k=+\infty)$ of a similar theorem in which the expression $M(\alpha_1, \alpha_2, \cdots, \alpha_r)$ is replaced by

$$M_k(\alpha_1, \alpha_2, \cdots, \alpha_r)$$
$$= \max \int_0^{2\pi} \cdots \int_0^{2\pi} \left| f(e^{\alpha_1 \xi_1 + i\eta_1}, \cdots, e^{\alpha_r \xi_r + i\eta_r}) \right|^k d\eta_1 \cdots d\eta_r$$

and $(e^{\xi_1}, \cdots, e^{\xi_r}) \in K$. The limitations on the α's are the same as before. The proof does not differ from the one given above, if we use the fact that the integral

$$\int_0^{2\pi} \cdots \int_0^{2\pi} \left| f(e^{(\alpha_1^0 + \lambda_1 t)\xi_1^0 + i\eta_1}, \cdots, e^{(\alpha_r^0 + \lambda_r t)\xi_r^0 + i\eta_r} \right|^k d\eta_1 \cdots d\eta_r$$

is a subharmonic function of the (complex) variable t, and so cannot attain a proper maximum. That this integral is a subharmonic function of t follows from the fact that $|f(e^{(\alpha_1^0+\lambda_1 t)\xi_1^0 + i\eta_1}, \cdots)|^k$ is subharmonic in t for every fixed set of $\eta_1, \eta_2, \cdots, \eta_r$.

As Thorin pointed out, the function $f(z_1, z_2, \cdots, z_r)$ of his theorem need not be entire. If it is regular in a domain D, the function $M(\alpha_1, \cdots, \alpha_r)$ is convex in a domain of the variables $\alpha_1, \cdots, \alpha_r$. The same remark applies to M_k.

BROWN UNIVERSITY AND
MOUNT HOLYOKE COLLEGE

THOMAS SCOTT FISKE—IN MEMORIAM

Thomas Scott Fiske died at Poughkeepsie, New York, January 10, 1944. He spent the whole of his active life in New York City and for fifty years was connected with Columbia College as a student and as a teacher. Shortly after his graduation he was instrumental in forming the New York Mathematical Society of which he became secretary and later president. This society was the forerunner of the American Mathematical Society. Later, upon the organization of the College Entrance Examination Board, he was made its secretary and it was under the influence of his initiative and guidance that the Board became an important factor in matters pertaining to college entrance requirements. He also served as dean of Barnard College for a short time.

Through these activities, continued over such a long period of time, he came into contact with many Columbia students, with practically all the active mathematicians of the country together with many of those living in Europe, and with many college and school men in other departments. His striking personality and lovable qualities made a deep impression upon these groups, as is evidenced by the action taken by the Society at its semicentennial celebration in 1938 as well as by the action of the College Entrance Examination Board upon the occasion of his retirement from the management of its affairs.

His colleagues at Columbia who were more closely associated with him will feel his loss perhaps more keenly than others, although it has been fifteen years since he gave up the active guidance of the Department of Mathematics there.

Those who knew Fiske well recognized that he had an unusually clear and incisive mind. He had a firm grasp of those parts of mathematics to which he gave his attention, and if his inclination had led him in that direction he probably would have made his mark as a research man. But he preferred to devote his energies to administrative matters and in this field and in his personal relations he will be remembered with admiration and affection.

W. BENJAMIN FITE

RECURSIVELY ENUMERABLE SETS OF POSITIVE INTEGERS AND THEIR DECISION PROBLEMS

EMIL L. POST

Introduction. Recent developments of symbolic logic have considerable importance for mathematics both with respect to its philosophy and practice. That mathematicians generally are oblivious to the importance of this work of Gödel, Church, Turing, Kleene, Rosser and others as it affects the subject of their own interest is in part due to the forbidding, diverse and alien formalisms in which this work is embodied. Yet, without such formalism, this pioneering work would lose most of its cogency. But apart from the question of importance, these formalisms bring to mathematics a new and precise mathematical concept, that of the general recursive function of Herbrand-Gödel-Kleene, or its proved equivalents in the developments of Church and Turing.[1] It is the purpose of this lecture to demonstrate by example that this concept admits of development into a mathematical theory much as the group concept has been developed into a theory of groups. Moreover, that stripped of its formalism, such a theory admits of an intuitive development which can be followed, if not indeed pursued, by a mathematician, layman though he be in this formal field. It is this intuitive development of a very limited portion of a sub-theory of the hoped for general theory that we present in this lecture. We must emphasize that, with a few exceptions explicitly so noted, we have obtained formal proofs of all the consequently mathematical theorems here developed informally. Yet the real mathematics involved must lie in the informal development. For in every instance the informal "proof" was first obtained; and once gotten, transforming it into the formal proof turned out to be a routine chore.[2]

We shall not here reproduce the formal definition of *recursive function of positive integers*. A simple example of such a function is an

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[1] For "general recursive function" see [9] ([8] a prerequisite), [12] and [11]; for Church's "λ-defineability," [1] and [6]; for Turing's "computability," [24] and the writer's related [18]. To this may be added the writer's method of "canonical systems and normal sets" [19]. See pp. 39–42 and bibliography of [6] for a survey of the literature and further references. Numbers in brackets refer to the bibliography at the end of the paper.

[2] Our present formal proofs, while complete, will require drastic systematization and condensation prior to publication.

arbitrary polynomial $P(x_1, x_2, \cdots, x_n)$, with say non-negative integral coefficients, and not identically zero. If the x's are assigned arbitrary positive integral values expressed, for example, in the arabic notation, the algorithms for addition and multiplication in that notation enable us to calculate the corresponding positive integral value of the polynomial. That is, $P(x_1, x_2, \cdots, x_n)$ is an *effectively calculable function of positive integers.* The importance of the technical concept recursive function derives from the overwhelming evidence that it is coextensive with the intuitive concept effectively calculable function.[3]

A set of positive integers is said to be *recursively enumerable* if there is a recursive function $f(x)$ of one positive integral variable whose values, for positive integral values of x, constitute the given set. The sequence $f(1), f(2), f(3), \cdots$ is then said to be a *recursive enumeration* of the set. The corresponding intuitive concept is that of an *effectively enumerable* set of positive integers. To prepare us in part for our intuitive approach, consider the following three examples of recursively enumerable sets of positive integers.

$$\begin{array}{ll}
\text{(a):} & 1^2, 2^2, 3^2, \cdots . \\
\text{(b):} & 1, 2, 2^{1+2}, 2^{1+2+2^{1+2}}, \cdots . \\
\text{(c):} & 1^2, 2^2, 3^2, \cdots \\
 & 1^3, 2^3, 3^3, \cdots \\
 & 1^4, 2^4, 3^4, \cdots \\
 & \cdot \quad \cdot \quad \cdot \quad \cdots \\
 & \cdot \quad \cdot \quad \cdot \quad \cdots \\
 & \cdot \quad \cdot \quad \cdot \quad \cdots .
\end{array}$$

In the first example, the set is given by a recursive enumeration thereof via the recursive function x^2. In the second example, the set is generated in a linear sequence, each new element being effectively obtained from the elements previously generated, in this case by raising 2 to the power the sum of the preceding elements. The set is effectively enumerable, since the nth element of the sequence can be found, given n, by regenerating the sequence through its first n elements. In the third example, we rather imagine the positive integers $1, 2, 3, \cdots$ generated in their natural order, and, as each positive integer n is generated, a corresponding process set up which generates $n^2, n^3, n^4, \cdots$, all these to be in the set. Actually, the standard method for proving that an enumerable set of enumerable sets is enumerable yields an effective enumeration of the set.

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[3] See Kleene [13, footnote 2]. In the present paper, "recursive function" means "general recursive function."

Several more examples would have to be given to convey the writer's concept of a *generated set*, in the present instance of positive integers. Suffice it to say that each element of the set is at some time written down, and earmarked as belonging to the set, as a result of predetermined effective processes. It is understood that once an element is placed in the set, it stays there. The writer elsewhere has referred to a generalization which may be restated *every generated set of positive integers is recursively enumerable*.[4] For comparison purposes this may be resolved into the two statements: every generated set is effectively enumerable, every effectively enumerable set of positive integers is recursively enumerable. The first of these statements is applicable to generated sets of arbitrary symbolic expressions; their converses are immediately seen to be true. We shall find the above concept and generalization very useful in our intuitive development. But while we shall frequently say, explicitly or implicitly, "set so and so of positive integers is a generated, and hence recursively enumerable set," as far as the present enterprise is concerned that is merely to mean "the set has intuitively been shown to be a generated set; it can indeed be proved to be recursively enumerable." Likewise for other identifications of informal concepts with corresponding mathematically defined formal concepts.

At a few points in our informal development we have to lean upon the formal development. The latter is actually yet another formalism, due to the writer [19] but proved completely equivalent to that of general recursive function. It will suffice to give the equivalent of "recursively enumerable set of positive integers" in this development.

A positive integer n is represented in the most primitive fashion by a succession $11 \cdots 1$ of n strokes. For working purposes, we introduce the letter b, and consider "strings" of 1's and b's such as $11b1bb1$. An operation on such strings such as "$b1bP$ *produces* $P1bb1$" we term a normal operation. This particular normal operation is applicable only to strings starting with $b1b$, and the derived string is then obtained from the given string by first removing the initial $b1b$, and then tacking on $1bb1$ at the end. Thus $b1bb$ becomes $b1bb1$. "gP *produces* Pg'" is the form of an arbitrary normal operation. A system in normal form, or normal system, is given by an initial string A of 1's and b's, and a finite set of normal operations "g_iP *produces* Pg_i'," $i=1, 2, \cdots, \mu$. The derived strings of the system are A and all strings obtainable from A be repeated applications of the μ normal operations. Each normal system uniquely defines a set, possibly null, of positive integers, namely the integers represented by those derived strings which are strings of 1's only. It can then be proved that every recursively enumerable set of positive integers is the set of positive integers defined by some normal system, and conversely.[5] *We here, as below, arbitrarily extend the concept recursively enumerable set to include the null set.*

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[4] See [19, p. 201 and footnote 18]. In this connection note Kleene's use of the word "Thesis" in [14, p. 60]. We still feel that, ultimately, "Law" will best describe the situation [18].

By the *basis* B of a normal system, and of the recursively enumerable set of positive integers it defines, we mean the string of letters and symbols here represented by

$$A;\ g_1P \textit{ produces } Pg_1',\ \cdots,\ g_\mu P \textit{ produces } Pg_\mu'.$$

When meaningfully interpreted, B determines the normal system, and recursively enumerable set of positive integers, in question. Each basis is but a finite sequence of the symbols 1, b, P, the comma, semicolon and the letters of the word "*produces*." The set of bases is therefore enumerably infinite, and can indeed be effectively generated in a sequence of distinct elements

$$O: \quad B_1, B_2, B_3, \cdots.$$

Since each B_i defines a unique recursively enumerable set of positive integers and each such set is defined by at least one B_i, O is also an ordering of all recursively enumerable sets of positive integers, though each set will indeed recur an infinite number of times in O. We may then say, in classical terms, that whereas there are $2^{\aleph_0}$ arbitrary sets of positive integers, there are but $\aleph_0$ recursively enumerable sets.

By the *decision problem* of a given set of positive integers we mean the problem of effectively determining for an arbitrarily given positive integer whether it is, or is not, in the set. While, in a certain sense, the theory of recursively enumerable sets of positive integers is potentially as wide as the theory of general recursive functions, the decision problems for such sets constitute a very special class of decision problems. Nevertheless they are important, as is shown by the following special and general examples.

One of the problems posed by Hilbert in his Paris address of 1900 [10, problem 10] is the problem of determining for an arbitrary diophantine equation with rational integral coefficients whether it has, or has not, a solution in rational integers. If the variables in a diophantine equation be chosen from a given enumerably infinite set of variables, it is clear that the set of diophantine equations is enumerably infinite. Indeed they can be effectively put into one-one correspondence with the set of positive integers. Since for any one diophantine equation, and assignment of rational integral values to its variables, it can be effectively determined whether or no the equation is satisfied by those values, the set of diophantine equations having rational integral solutions can be generated. The corresponding integers under the above one-one correspondence can then also be generated, and, indeed, constitute a recursively enumerable set of positive integers.[6] And under that correspondence, Hilbert's problem is transformed into the decision problem of that recursively enumerable set.

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[5] We have thus restricted the normal operations and normal systems of [19] because of the following result. If in the initial string and in the normal operations of a normal system with primitive letters 1, a_1', $\cdots$, a_μ', each a_i', $i=1, \cdots, \mu'$, is replaced by $b1 \cdots 1b$ with i 1's, a normal system with primitive letters 1, b results, defining the same set of strings on 1 only as the original normal system.

The assertions of an arbitrary symbolic logic[7] constitute a generated set A of what may be called symbol-complexes or formulas. We assume that A is a subset of an infinite generated set E of symbol-complexes, which in one case may be the set of meaningful enunciations of the logic, in another the set of all symbol-complexes of a given mode of symbolization. The decision problem of the logic, more precisely its deducibility problem [3], is then the problem of determining of an arbitrary member of E whether it is, or is not, in A. Granting that every generated set is effectively enumerable, the members of E can be effectively set in one-one correspondence with the set of positive integers. The positive integers corresponding to the members of A then constitute a generated, and hence, under our generalization, a recursively enumerable set of positive integers. And under that correspondence the decision problem of the symbolic logic is transformed into the decision problem of this recursively enumerable set of positive integers.

Closely related to the technical concept recursively enumerable set of positive integers is that of a *recursive* set of positive integers. This is a set for which there is a recursive function $f(x)$ such that $f(x)$ is say 2 when x is a positive integer in the set, 1 when x is a positive integer not in the set. We may also make this the definition of the decision problem of the set being *recursively solvable*. For 2 and 1 may be regarded as the two possible truth-values, true, false, of the proposition "positive integer x is in the set," and the definition of recursive set is equivalent to this truth-value being recursively calculable for all positive integers x. If then recursive function is coextensive with effective calculability, recursive solvability is coextensive with solvability in the intuitive sense. In particular, the decision problem of a recursively enumerable set would be solvable or unsolvable according as the set is, or is not, recursive. More generally than in our two illustrations, through the more precise mechanism of Gödel representations [8], a wide variety of decision and other problems are transformed into problems about positive integers; and whether those problems are, or are not, solvable in the intuitive sense would be equivalent to their being, or not being, recursively solvable in the precise technical sense.

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[6] In view of [17] we inadvertantly carried through our formal verification with "rational integral solution" replaced by "positive integral solution."

[7] See Church [5, p. 225] for our omitting the qualifying "finitary."

Gödel's classic theorem on the incompleteness and extendibility of symbolic logics [8] in all but wording led him to the recursive unsolvability of a generalization of the above problem of Hilbert [8, 9, 22]. Church explicitly formulated the concept of recursive unsolvability, and arrived at the unsolvability of a number of problems; certainly he proved them recursively unsolvable [1–4]. The above problem of Hilbert begs for an unsolvability proof (see [17]). Like the classic unsolvability proofs, these proofs are of unsolvability by means of given instruments. What is new is that in the present case these instruments, in effect, seem to be the only instruments at man's disposal.

Related to the question of solvability or unsolvability of problems is that of the reducibility or non-reducibility of one problem to another. Thus, if problem P_1 has been reduced to problem P_2, a solution of P_2 immediately yields a solution of P_1, while if P_1 is proved to be unsolvable, P_2 must also be unsolvable. For unsolvable problems the concept of reducibility leads to the concept of *degree of unsolvability*, two unsolvable problems being of the same degree of unsolvability if each is reducible to the other, one of lower degree of unsolvability than another if it is reducible to the other, but that other is not reducible to it, of incomparable degrees of unsolvability if neither is reducible to the other. A primary problem in the theory of recursively enumerable sets is the problem of determining the degrees of unsolvability of the unsolvable decision problems thereof. We shall early see that for such problems there is certainly a highest degree of unsolvability. Our whole development largely centers on the single question of whether there is, among these problems, a lower degree of unsolvability than that, or whether they are all of the same degree of unsolvability. Now in his paper on *ordinal logics* [26, section 4], Turing presents as a side issue a formulation which can immediately be restated as the general formulation of the "recursive reducibility" of one problem to another, and proves a result which immediately generalizes to the result that for any "recursively given" unsolvable problem there is another of higher degree of unsolvability.[8] While his theorem does not help us in our search for that lower degree of unsolvability, his formulation makes our problem precise. It remains a problem at the end of this paper. But on the way we do obtain a number of special results, and towards the end obtain some idea of the difficulties of the general problem.

1. Recursive versus recursively enumerable sets.

The relationship between these two concepts is revealed by the following

THEOREM. *A set of positive integers is recursive when and only when both it and its complement with respect to the set of all positive integers are recursively enumerable.*[9]

For simplicity, we assume both the set S and its complement $\bar{S}$ to be infinite. If, then, S is recursive, there is an effective method for telling of any positive integer n whether it is, or is not, in S. Generate the positive integers $1, 2, 3, \cdots$ in their natural order, and, as a positive integer is generated, test its being or not being in S. Each time a positive integer is thus found to be in S, write it down as belonging to S. Thus, an effective process is set up for effectively enumerating the elements of S. Hence, S is recursively enumerable. Likewise $\bar{S}$ can be shown to be recursively enumerable.

Conversely, let both S and $\bar{S}$ be recursively enumerable, and let $n_1, n_2, n_3, \cdots$ be a recursive enumeration of S; $m_1, m_2, m_3, \cdots$, of $\bar{S}$. Given a positive integer n, generate in order $n_1, m_1, n_2, m_2, n_3, m_3$, and so on, comparing each with n. Since n must be either in S or in $\bar{S}$, in a finite number of steps we shall thus come across an n_i or m_j identical with n, and accordingly discover n to be in S, or $\bar{S}$. An effective method is thus set up for determining of any positive integer n whether it is, or is not, in S. Hence, S is recursive.

COROLLARY. *The decision problem of a recursively enumerable set is recursively solvable when and only when its complement is recursively enumerable.*

For then and only then is the recursively enumerable set recursive.

It is readily proved that the logical sum and logical product of two recursively enumerable sets are recursively enumerable, the complement of a recursive set, and the logical sum, and hence logical product, of two recursive sets are recursive.

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[8] Both our generalization of his formulation and of his theorem have been carried through, rather hastily, by the formalism of [19], without, as yet, an actual equivalence proof. It may be that Tarski's Theorem 9.1 [23] can be transformed into a like absolute theorem.

[9] The only portion of this theorem we can find in the literature is Rosser's Corollary II [20, p. 88].

Clearly, *any finite set of positive integers is recursive.* For if $n_1, n_2, \cdots, n_\nu$ are the integers in question, we can test n being, or not being, in the set by directly comparing it with $n_1, n_2, \cdots; n_\nu$.[10] Likewise for a set whose complement is finite. For arbitrary infinite sets we have the following result of Kleene [12]. *An infinite set of positive integers is recursive when and only when it admits of a recursive enumeration without repetitions in order of magnitude.* Indeed, if $n_1, n_2, n_3, \cdots$ is a recursive enumeration of S without repetitions in order of magnitude, all n_i's beyond the nth must exceed n. Hence we can test n being, or not being, in S by generating the first n members of the given recursive enumeration of S, and seeing whether n is, or is not, one of them. Conversely, if infinite S is recursive, the recursive enumeration thereof we set up in the proof of our first theorem is of the elements of S without repetition, and in order of magnitude.

A direct consequence of the first half of the last result is the following

THEOREM. *Every infinite recursively enumerable set contains an infinite recursive set.*

For, if $n_1, n_2, n_3, \cdots$ is a recursive enumeration of an infinite set S, for each n_i there must be, in this sequence, a later $n_j > n_i$. Hence, generate the elements $n_1, n_2, n_3, \cdots$ in order, and let $m_1 = n_1$, $m_2 = n_{i_2}$, the first n_i greater than n_1, $m_3 = n_{i_3}$, the first n_i beyond n_{i_2} greater than n_{i_2}, and so on. The sequence $m_1, m_2, m_3, \cdots$ is then a recursive enumeration of a subset of S without repetitions in order of magnitude. That subset is therefore infinite, and recursive.

Basic to the entire theory is the following result we must credit to Church, Rosser, Kleene, jointly [1, 20, 12].

THEOREM. *There exists a recursively enumerable set of positive integers which is not recursive.*[11]

By our first theorem this is equivalent to the existence of a recursively enumerable set of positive integers whose complement is not recursively enumerable. Generate in order the distinct bases $B_1, B_2, B_3, \cdots$ of all recursively enumerable sets of positive integers as mentioned in the introduction, and keep track of these bases as the first, second, third, and so on, in this enumeration O. As the nth basis B_n is generated, with $n=1, 2, 3, \cdots$, set going the processes whereby the corresponding recursively enumerable set is generated, and whenever n is thus generated by B_n, place n in a set U. Being a generated set of positive integers, U is recursively enumerable. A positive integer n, then, is, or is not, in U according as it is, or is not, in the nth recursively enumerable set in O considered as an ordering of all recursively enumerable sets. Hence, n is, or is not, in $\bar{U}$, the complement of U, according as it is not, or is, in the nth set in O. We thus see that $\bar{U}$ differs from each recursively enumerable set in the presence or absence of at least one positive integer. Hence $\bar{U}$ is not recursively enumerable.

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[10] The mere existence of a general recursive function defining the finite set is in question. Whether, given some definition of the set, we can actually discover what the members thereof are, is a question for a theory of proof rather than for the present theory of finite processes. For sets of finite sets the situation is otherwise, as seen in §11.

[11] In each of our existence theorems we show how to set up the basis of the set in question—at least, the corresponding formal proof does exactly that.

COROLLARY. *There exists a recursively enumerable set of positive integers whose decision problem is recursively unsolvable.*

Taken singly, finite sets, or sets whose complements are finite, are rather trivial examples of recursive sets. On the other hand, if we define two sets of positive integers to be *abstractly* the same if one can be transformed into the other by a recursive one-one transformation of the set of all positive integers into itself, then all infinite recursive sets with infinite complements are abstractly the same. Our theory being essentially an abstract theory of recursively enumerable sets, our interest therefore centers in recursively enumerable sets that are not recursive. Such sets, as well as their complements, are always infinite. We do not further pursue the question of two sets being abstractly the same, for that is but a special case of each set being one-one reducible to the other (§4).

2. A form of Gödel's theorem. Given any basis B, and positive integer n, the couple (B, n) may be used to represent the proposition, true or false, "n is in the set generated by B." By interlacing the process for generating the distinct bases in the sequence $B_1, B_2, B_3, \cdots$ and the process for generating the positive integers in the sequence $1, 2, 3, \cdots$ by the addition of 1's, we can effectively generate the distinct couples (B, n) in the single infinite sequence

$$O': \quad (B_1, 1), (B_2, 1), (B_1, 2), (B_3, 1), (B_2, 2), (B_1, 3), \cdots.$$

On the one hand, the set of all couples (B, n) is thus a generated set of expressions which we shall call E. On the other hand, O' leads to an effective 1-1 correspondence between the members of E and the set of positive integers, (B, n) corresponding to m if (B, n) is the mth member of O'. We may call m the Gödel representation[12] of (B, n). Given a generated subset of E, the Gödel representations of its members will constitute a generated set of positive integers, and conversely. Thus, in the former case we can generate the members of the subset of E, and, as a couple (B, n) is generated, find its Gödel representation m by regenerating O'. The set of these m's is thus a generated set. Likewise for the converse. If, therefore, we formally define a subset of E to be recursively enumerable if the set of Gödel representations of its members is recursively enumerable,[13] we can conclude that every generated subset of E is recursively enumerable, and, of course, conversely. Similarly for a like formal definition of a recursive subset of E.

While E is just the set of couples (B, n), it may be interpreted as the set of enunciations "n is in the set generated by B." The subset T of E consisting of those couples (B, n) for which n *is* in the set generated by B may then be interpreted as the set of true propositions in E, while $\overline{T}$, the complement of T with respect to E, consists of the false propositions in E.

Actually, T itself can be generated as follows. Generate B_1, B_2, B_3, $\cdots$ in order. As a B is generated, set up the process for generating the set of positive integers determined by B, and, whenever a positive integer n is thus generated, write down the couple (B, n). Each (B, n) for which n is in the set generated by B will thus be written down, and conversely. This generated set of (B, n)'s is then T. We therefore conclude that *T is recursively enumerable.*

Now let F be any recursively enumerable subset of $\overline{T}$. If (B, n) is in F, it is in $\overline{T}$, and hence n is certainly not in the set generated by B. Now generate the members of F, and if (B, n) is thus generated, find the nth member B_n of $O: B_1, B_2, B_3, \cdots$, and if B_n is B, place n in a set of positive integers S_0. Since S_0 is thus a generated set of positive integers, it is recursively enumerable. It will therefore be determined by some basis B. Let this basis be in the νth in O, that is, let the basis be B_ν, and form the couple (B_ν, ν). Now by construction, S_0 consists of those members of F of the form (B_n, n). Suppose that (B_ν, ν) is in F. Then, on the one hand, proposition (B_ν, ν) being false,

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[12] Rather is the Gödel representation in [8] not just an effectively corresponding positive integer, but one which, when expressed according to a specific algorithm, is "formally similar," in the sense of Ducasse [7, p. 51], to the symbolic expression represented.

[13] In our own development [19], "recursively enumerable subset of E" is defined directly as a normal subset of E, or rather of the set of symbolic representations of the members of E.

ν would not be in the set generated by B_ν, that is (1): ν would not be in S_0. But (B_ν, ν) being of the form (B_n, n), (2): ν would be in S_0. Our assumption thus leading to a contradiction, it follows that (B_ν, ν) *is not in* F. But ν can only be in S_0 by (B_ν, ν) being in F. Hence, ν *is not in* S_0. Finally, (B_ν, ν) as proposition says that ν is in S_0. The proposition (B_ν, ν) is therefore false, that is (B_ν, ν) *is in* $\overline{T}$.

For any recursively enumerable subset F of $\overline{T}$ there is then always this couple (B_ν, ν) in $\overline{T}$, but not in F. On the one hand, then, $\overline{T}$ can never be F. Hence, $\overline{T}$ *is not recursively enumerable.* By the definitions of this section, and the first theorem of the last, it follows that T, *while recursively enumerable, is not recursive.* By the decision problem of T we mean the problem of determining for an arbitrarily given member of E whether it is, or is not, in T. But that can be interpreted as the decision problem for the class of recursively enumerable sets of positive integers, that is, the problem of determining for any arbitrarily given recursively enumerable set, that is, arbitrarily given basis B of such a set, and arbitrary positive integer n whether n is, or is not, in the set generated by B. We may therefore say that the *decision problem for the class of all recursively enumerable sets of positive integers is recursively unsolvable*, and hence, in all probability, unsolvable in the intuitive sense.

On the other hand, since (B_ν, ν) of $\overline{T}$ is not in F, T and F together can never exhaust E. Now T, or any recursively enumerable subset T' of T, in conjunction with F may be called a recursively generated logic relative to the class of enunciations E. For the appearance of (B, n) in T' assures us of the truth of the proposition "n is in the set generated by B," while its presence in F would guarantee its falseness. We can then say *that no recursively generated logic relative to E is complete*, since F alone will lead to the (B_ν, ν) which is neither in T' nor in F. That is, (B_ν, ν) is undecidable in this logic. Moreover, if, with a given "basis" for F, the above argument is carried through formally,[14] the recursively enumerable S_0 obtained above will actually be given by a specific basis B which can be constructed by that formal argument. Having found this B, we can then regenerate O:B_1, B_2, B_3, $\cdots$, until B is reached, and thus determine the ν such that $B = B_\nu$. That is, given the basis of F, the (B_ν, ν) in $\overline{T}$ and not in F can actually be found. If then we add this (B_ν, ν) to F, a wider recursively enumerable subset F' of $\overline{T}$ results. We may then say that *every recursively generated logic relative to E can be extended.* Outwardly, these two results, when formally developed, seem to be

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[14] Here, the basis of F may be taken to be the basis of the recursively enumerable set of Gödel representations of the members of F. But see the preceding footnote.

Gödel's theorem in miniature. But in view of the generality of the technical concept general recursive function, they implicitly, in all probability, justify the generalization that every symbolic logic is incomplete and extendible relative to the class of propositions constituting E.[15] The conclusion is unescapable that even for such a fixed, well defined body of mathematical propositions, *mathematical thinking is, and must remain, essentially creative.* To the writer's mind, this conclusion must inevitably result in at least a partial reversal of the entire axiomatic trend of the late nineteenth and early twentieth centuries, with a return to meaning and truth as being of the essence of mathematics.

3. The complete set K; creative sets. Return now to the effective 1-1 correspondence between the set E of distinct (B, n)'s and the set of positive integers obtained via the effective enumeration O' of E. Since T is a recursively enumerable subset of E, the positive integers corresponding to the elements of T constitute a recursively enumerable set of positive integers, K. We shall call K the *complete set.*[16] Since $\overline{T}$ is not recursively enumerable, $\overline{K}$, which consists of the positive integers corresponding to the elements of $\overline{T}$, is not recursively enumerable. Now let B be the basis of a recursively enumerable subset α of $\overline{K}$. The elements of E corresponding to the members of α constitute, then, a recursively enumerable subset F of $\overline{T}$. Find then the (B_ν, ν) of $\overline{T}$ not in F, and, via O', the positive integer n corresponding to (B_ν, ν). This n will then be an element of $\overline{K}$ not in α.

Actually, we have no general method of telling when a basis B defines a recursively enumerable subset of $\overline{K}$. Indeed, the above method will yield a unique positive integer n for any basis B of a recursively enumerable set α of positive integers. However, when α is a subset of $\overline{K}$, n will also be in $\overline{K}$, but not in α.

Furthermore, even the formal proof of this result merely gives an effective method for finding n, given B. But this method itself can be formalized, so that, as a result, n is given as a "recursive function of B." This can mean that a recursive function $f(m)$ can be set up such that $n = f(m)$ where $B = B_m$. We now isolate this property of K by setting up the

DEFINITION. *A creative set C is a recursively enumerable set of positive integers for which there exists a recursive function giving a unique*

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[15] See Kleene's Theorem XIII in [12] for a mathematically stateable theorem approximating the generality of our informal generalization.

[16] "A complete set" might be better. Just how to abstract from K the property of completeness is not, at the moment, clear. By contrast, see "creative set" below.

positive integer n for each basis B of a recursively enumerable set of positive integers α such that whenever α is a subset of $\bar{C}$, n is also in $\bar{C}$, but not in α.

THEOREM. *There exists a creative set; to wit, the complete set K.*

Actually, the class of creative sets is infinite, and very rich indeed as shown by the following easily proved results.[17] If C is a creative set, and E a recursively enumerable set of positive integers, then if E contains $\bar{C}$, CE is creative, if $\bar{C}$ contains E, $C+E$ is creative. Results of §1 enable us actually to construct creative sets according to the first method by using E's which are the complements of recursive subsets of C. Results of the rest of this section lead to constructions using the second method.

It is convenient to talk as if the n in the definition of a creative set were determined by the α thereof instead of by the basis B of α. Clearly *every creative set C is a recursively enumerable set which is not recursive.* For were $\bar{C}$ recursively enumerable, there could be no n in $\bar{C}$ not in the recursively enumerable subset $\bar{C}$ of $\bar{C}$. *The decision problem of each creative set is therefore recursively unsolvable.* On the other hand, the *complement $\bar{C}$ of any creative set C contains an infinite recursively enumerable set.* Recall that every finite set is recursive, and hence recursively enumerable. With, then, α of the definition of creative set as the null set, find the $n=n_1$ of $\bar{C}$ "not in α." With α the unit set having n_1 as sole member, $n=n_2$ will be in $\bar{C}$, and distinct from n_1. With α consisting of n_1 and n_2, $n=n_3$ will be in $\bar{C}$, and distinct from n_1 and n_2, and so on. The set of positive integers n_1, n_2, n_3, $\cdots$ is then an infinite generated, and hence recursively enumerable, subset of $\bar{C}$.

Actually, with this subset of $\bar{C}$ as α, a new element n_ω of $\bar{C}$ is obtained, and so on into the constructive transfinite. But this process is essentially creative. For any mechanical process could only yield n's forming a generated, and hence recursively enumerable, subset α of $\bar{C}$, and hence could be transcended by finding that n of $\bar{C}$ not in α.

4. **One-one reducibility, to K; many-one reducibility.** Let S_1 and S_2 be any two sets of positive integers. One of the simplest ways in which the decision problem of S_1 would be reduced to the decision problem of S_2 would arise if we had an effective method which would determine for each positive integer n a positive integer m such that n is, or is not, in S_1 according as m is, or is not, in S_2. For if we could

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[17] Of course, all sets abstractly the same as a given creative set, in the sense of §1, are creative. Likewise for our later simple and hyper-simple sets.

somehow determine whether m is, or is not, in S_2, we would determine n to be, or not be, in S_1 correspondingly. If "effective method" be replaced by "recursive method," we shall say, briefly, that S_1 is then *many-one reducible* to S_2. If, furthermore, different n's always lead to different m's, we shall say that S_1 is *one-one reducible* to S_2.[18] "Recursive method" here can mean that $m=f(n)$, where $f(n)$ is a recursive function.

THEOREM. *The decision problem of every recursively enumerable set of positive integers is one-one reducible to the decision problem of the complete set K.*

For let B' be a basis of any one recursively enumerable set S'. The effective one-one correspondence between all (B, n)'s and all positive integers yielded by the effective enumeration O' of E, the set of all (B, n)'s, then yields a unique positive integer m for each (B', n), B' fixed, and thus a unique m for each n, different n's yielding different m's. Now n is, or is not, in S' according as (B', n) is in T, or $\overline{T}$, and hence according as m is in K, or $\overline{K}$, whence our result.

Since K itself is recursively enumerable, we may say that for recursively enumerable sets of positive integers with recursively unsolvable decision problems there is a *highest degree of unsolvability relative to one-one reducibility*, namely, that of K. Actually, one-one reducibility is a special case of all the more general types of reducibility later introduced, and, though the proof of this is still in the informal stage, these latter are special cases of general recursive, that is, Turing reducibility. The same result then obtains relative to these special types of reducibility and, more significantly, for reducibility in the general sense.[19]

We have thus far explicitly obtained two recursively enumerable sets with recursively unsolvable decision problems, the U of our first section, and K. We may note that a certain necessary and sufficient condition for the many-one reducibility of K to a recursively enumerable set, the proof of which is still in the informal stage, has as an immediate consequence that K is many-one reducible to U. It would then follow that K and U are of the same degree of unsolvability relative to many-one reducibility.

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[18] The resulting one-to-one correspondence is then between $S_1+\overline{S}_1$ and a subset, recursively enumerable indeed, of $S_2+\overline{S}_2$. Of course, both $S_1+\overline{S}_1$ and $S_2+\overline{S}_2$ constitute the set of all positive integers.

[19] It seems rather obvious that K and the problem of Church [1] are each at least many-one reducible to the other; likewise for the problem of [1] and of [2, 3]. Had we verified this in detail, we would have called this highest degree of unsolvability of decisions problems of recursively enumerable sets the *Church degree of unsolvability*.

5. Simple sets. It is readily proved that the necessary and sufficient condition that every recursive set be one-one reducible to a given recursively enumerable set of positive integers S is that S is infinite, and $\bar{S}$ contains an infinite recursively enumerable set. We are thus led to ask if there exist sets satisfying the following

DEFINITION. *A simple set is a recursively enumerable set of positive integers whose complement, though infinite, contains no infinite recursively enumerable set.*

We now prove the

THEOREM. *There exists a simple set.*

Recall the set T of all couples (B, n) such that positive integer n is in the recursively enumerable set of positive integers determined by basis B. Since T is recursively enumerable, we can set up an effective enumeration

$$O'': \quad (B_{i_1}, n_1), (B_{i_2}, n_2), (B_{i_3}, n_3), \cdots$$

of its members. The subscript of each B is its subscript in the effective enumeration $O: B_1, B_2, B_3, \cdots$ of all distinct B's. Now the complement of a set containing no infinite recursively enumerable set is equivalent to the set itself having an element in common with each infinite recursively enumerable set. Generate then the distinct bases $B_1, B_2, B_3, \cdots$, and as a B_i is generated, regenerate the sequence O'' of (B, n)'s in T, and the first time, if ever, B is B_i, and *n is greater than $2i$*, place n in a set S. The resulting set S is then a generated, and hence recursively enumerable, set of positive integers. We proceed to prove it simple.

If S' is an infinite recursively enumerable set of positive integers, it will be determined by some basis B_i, and will have some element m greater than $2i$. Since (B_i, m), being then in T, will appear in O'', our construction will place m in S, if some earlier (B_i, n) of O'' has not already contributed an element of S' to S. That is, S has an element in common with each infinite recursively enumerable S'. As for $\bar{S}$ being infinite, note that each B_i contributes at most one element to S. The first n B's in O therefore contribute at most n elements to S. Each B_i with $i \geqq n+1$ can only contribute to S an element greater than $2n+2$. Of the first $2n+2$ positive integers, at most n are therefore in S, and hence at least $n+2$ are in the consequently infinite $\bar{S}$.[20]

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[20] $n > i$ can replace $n > 2i$ in the above construction, but the proof will then depend on there being an infinite number of bases defining the null set.

Having one simple set, the method of our succeeding §8 can be modified to yield a rich infinite class of simple sets. Clearly, *every simple set S is a recursively enumerable set that is not recursive.* For were S recursive, $\bar{S}$ would be an infinite recursively enumerable subset of $\bar{S}$. *The decision problem of each simple set is therefore recursively unsolvable.* We thus have obtained two infinite mutually exclusive classes of recursively enumerable sets with recursively unsolvable decision problems, the class of creative sets, and the class of simple sets. They are poles apart in that the complements of creative sets have a creative infinity of infinite recursively enumerable subsets, those of simple sets, not one.

In passing, we may note that every recursively enumerable set of positive integers S with recursively unsolvable decision problem leads to an incompleteness theorem for symbolic logics relative to the class of propositions $n \in S$, n an arbitrary positive integer. Creative sets S are then exactly those recursively enumerable sets of this type each of which admits a universal extendibility theorem as well, simple sets S those for which, given S, each logic can prove the falsity of but a finite number of the infinite set of false propositions $n \in S$.

It is readily seen that no creative set C can be one-one reducible to a simple set S. For under such a reduction, each infinite recursively enumerable subset of $\bar{C}$, proved above to exist, would be transformed into an infinite recursively enumerable subset of $\bar{S}$, contradicting the simplicity of S. Simple sets thus offer themselves as *candidates* for recursively enumerable sets with decision problems of lower degree of unsolvability than that of the complete set K. Even for many-one reducibility the situation is no longer immediately obvious; for an infinite recursively enumerable subset of $\bar{C}$ could thus be transformed into a finite subset of $\bar{S}$, the complement of simple S, without contradiction. However we can actually go much further than that.

6. Reducibility by truth-tables.

If S_1 is many-one reducible to S_2, positive integer n being, or not being, in S_1 may be said to be determined by its correspondent m being, or not being, in S_2 in accordance with the truth-table

(S_2) m	n (S_1)
$+$	$+$
$-$	$-$

Here, the two signs $+$, $-$ under m represent the two possibilities m is in S_2, m is not in S_2, respectively. And by the sign under n in the same horizontal row as the corresponding sign under m the table in the same language tells whether n correspondingly is $(+)$, or is not $(-)$, in S_1. The table then says that when m is in S_2, n is in S_1, when m is not in S_2, n is not in S_1, as required by many-one reducibility. Now there are altogether four ways in which n being, or not being, in S_1 can be made to depend solely on m being, or not being, in S_2, the signs under n being $+, -$ as above; or $+, +$; $-, -$; $-, +$. If then we have an effective method which for each positive integer n will not only determine a unique corresponding positive integer m, but also one of these four "first order" truth-tables, and if in each case the table is such that for the correct statement of membership or non-membership of m in S_2, it gives the correct statement of membership or non-membership of n in S_1, then the decision problem of S_1 will thus be reduced to the decision problem of S_2. For here also, given n, if we could somehow determine whether m is, or is not, in S_2, we could thereby determine which row of the corresponding table correctly describes the membership or non-membership of m in S_2, and from that row correctly determine whether n is, or is not, in S_1.

More generally, let there be an effective method which for each positive integer n determines a finite sequence of positive integers $m_1, m_2, \cdots, m_\nu$, ν as well as the m's depending on n. Let that method correspondingly determine for each n a "νth order" truth-table of the form

$$
\begin{array}{cccccc|cc}
(S_2) & m_1 & m_2 & \cdots & m_\nu & & n & (S_1) \\
\hline
 & + & + & \cdots & + & & - & \\
 & + & + & \cdots & - & & + & \\
 & \cdot & \cdot & \cdots & \cdot & & \cdot & \\
 & \cdot & \cdot & \cdots & \cdot & & \cdot & \\
 & \cdot & \cdot & \cdots & \cdot & & \cdot & \\
 & - & - & \cdots & - & & - &
\end{array}
.
$$

Each horizontal row, to the left of the vertical bar, specifies one of the 2^ν possible ways in which the ν m_i's may, or may not, be in S_2, to the right of the bar correspondingly commits itself to one of the statements n is in S_1, n is not in S_1. If then for each n that row of the corresponding table which gives the correct statements for the m's being or not being in S_2 also gives the correct statement regarding the membership or non-membership of n in S_1, the decision problem of S_1 is again thereby reduced to the decision problem of S_2.

If such a situation obtains with "effective method" replaced by "recursive method," we shall say that S_1 *is reducible to* S_2 *by truth-tables*. "Recursive method" here can mean that a suitable Gödel representation of the couple consisting of the sequence $m_1, m_2, \cdots,$

m_ν and the truth-table of order ν is a recursive function of n. If the orders of the truth-tables arising in such a reduction are bounded, we shall say that S_1 *is reducible to* S_2 *by bounded truth-tables.* Since there are 2^{2^ν} distinct truth-tables of order ν, reducibility by bounded truth-tables is equivalent to reducibility by truth-tables in which but a finite number of distinct tables arise.

7. **Non-reducibility of creative sets to simple sets by bounded truth-tables.** Let us suppose that creative set C is reducible to simple set S by bounded truth-tables. Let $T_1, T_2, \cdots, T_\kappa$ be the finite set of distinct truth-tables entering into such a reduction. That reduction then effectively determines for each positive integer n a finite sequence of positive integers $m_1, m_2, \cdots, m_\nu$, and a unique T_i, $1 \leqq i \leqq \kappa$.

The gist of our reductio-ad-absurdam proof consists in showing that under the assumed reduction we can obtain for each natural number p a sequence of m's at least p of which are in S. We then immediately have our desired contradiction. For in each case $p \leqq \nu$. The finite set of ν's, the orders of the T_i's, being bounded, p cannot then be arbitrarily large as stated.

More precisely we prove by mathematical induction that under the assumed reduction the following would be true. *For each natural number p an effective process Π_p can be set up which will determine for each recursively enumerable subset α of $\bar{C}$ an element n of $\bar{C}$ not in α, and which for the corresponding $m_1, m_2, \cdots, m_\nu$ and T_i yielded by the assumed reduction will correctly designate p of these m's as belonging to S.* The mode of designation may be assumed to be by specifying the sequence of subscripts, $i_1, i_2, \cdots, i_p$, of the m's to be designated, with say $i_1 < i_2 < \cdots < i_p$. With the assumed reduction adjoined to this process, Π_p then determines for each α in question the quadruplet (n, M, T_i, I), M being the sequence of m's, I the sequence of subscripts of the p designated m's.

For $p = 0$, Π_p is immediately given by the creative character of C. For that immediately gives us for each recursively enumerable subset α of $\bar{C}$ a definite element n of $\bar{C}$ not in α. The assumed reduction yields the corresponding M and T_i; and with no members of M *designated* as being in S, I is the null sequence.

Inductively, assume that we have the process Π_p for $p = k$. Let α be any given recursively enumerable subset of $\bar{C}$, and let $(n', M', T_{i'}, I')$ be the corresponding quadruplet yielded by Π_k. Now suppose n is a positive integer for which the assumed reduction yields the same table $T_{i'}$ as it did for n', and a sequence of m's, M, consequently of the same length as M', having the following property. For each undesignated element of M', the correspondingly placed element of M is identical with that of M'; for each element of M' designated as being in S, the corresponding element of M is also in S. Such an n must then be in $\bar{C}$ along with n'. For that row of $T_{i'}$ which correctly tells of the m's of M' whether they are, or are not, in S will also be the correct row for M. And since in the former case that row must say that n' is in $\bar{C}$, in the latter case it will say that n is in $\bar{C}$, and correctly so. We proceed to show how all such n's may be generated.

We first show how to generate all M's obtainable from M' by replacing the designated elements of M' by arbitrary elements of S. For any one such M, the replacing elements, being finite in number, will be among the first N elements, for some positive integer N, of a given recursive enumeration of S. Generate then the positive integers $1, 2, 3, \cdots$, and as a positive integer N is generated, generate the first N elements of the given recursive enumeration of S. For each N place in a set β the at most N^k sequences M that can be obtained from M' by replacing the designated elements of M' by elements chosen from the first N elements of S. The generated set of sequences β then consists of all M's obtainable from M' by replacing the designated elements of M' by arbitrary elements of S.

The n's we wish to generate are then those positive integers for which the assumed reduction yields the table $T_{i'}$ and a sequence of m's, M, such that M is a member of β. Generate then the elements of β. As an element M of β is generated, generate the positive integers $1, 2, 3, \cdots$, and as a positive integer n is generated, find the corresponding sequence of m's and table yielded by the reduction of C to S. If then that sequence of m's is M, and the table is $T_{i'}$, *add n to the given set α*. As seen above, each such n will be in $\bar{C}$. Hence the resulting generated, and hence recursively enumerable, set α' is a subset of $\bar{C}$ containing α. Our reason for thus adding the desired n's to α instead of just forming the class thereof is that the iterative process we are about to set up requires a cumulative effect.

As a result of our hypothesis and construction we thus have a derived process Π'_k which for every recursively enumerable subset α of $\bar{C}$ yields a definite recursively enumerable subset α' of $\bar{C}$ containing α. Starting with α, we may then iterate the process Π'_k to obtain the infinite sequence $A:\alpha_1, \alpha_2, \alpha_3, \cdots$, where $\alpha_1 = \alpha$, $\alpha_{n+1} = (\alpha_n)'$. Each member of A is thus a recursively enumerable subset of $\bar{C}$, and contained in the next member of A. By applying the original process Π_k to the members of A we correspondingly obtain the infinite sequence $\Sigma:\sigma_1, \sigma_2, \sigma_3, \cdots$, where σ_j is the quadruplet $(n^{(j)}, M^{(j)}, T_i^{(j)}, I^{(j)})$ yielded by Π_k for α_j. We then observe the following. If for $j_1 \neq j_2$

the T's of σ_{j_1} and σ_{j_2} are the same, and the I's are the same, then the sequences obtained from the M's by deleting the designated m's cannot be identical. For if they also were identical, then, with say $j_1 < j_2$, $n^{(j_2)}$ would have been assigned to $\alpha^{(j_1+1)}$, whereas it actually is outside of $\alpha^{(j_2)}$ which contains $\alpha^{(j_1+1)}$. Hence, the infinite sequence Σ', obtained from Σ by deleting from each σ_j the integer $n^{(j)}$ and the designated m's of $M^{(j)}$, itself consists of distinct elements.

It follows that *there are an infinite number of distinct undesignated m's appearing in Σ*. Indeed, the distinct $T_i^{(j)}$'s of Σ are at most κ in number. With $T_i^{(j)}$ fixed, the order $\nu^{(j)}$ of $T_i^{(j)}$ is fixed; and since $1 \leqq i_1^{(j)} < i_2^{(j)} < \cdots < i_k^{(j)} \leqq \nu^{(j)}$, the number of distinct I's is finite. Finally, with T and I fixed, were the total number of distinct undesignated m's finite, the number of distinct ways in which those $\nu^{(j)} - k$ undesignated m's could assume values would be finite. Hence Σ' would be finite, not infinite.

Now were each of this infinite set of undesignated m's in $\bar{S}$, we could regenerate the elements of Σ, and as an element σ_j thereof is generated, place all of its undesignated m's in a set γ, and thus obtain an infinite generated, and hence recursively enumerable, subset of $\bar{S}$. As this contradicts the simplicity of S, it follows that *at least one undesignated m arising in Σ is in S*.

We can then find a unique such m, as well as a σ in which it occurs, as follows. With $N = 1, 2, 3, \cdots$, generate the first N elements of the given recursive enumeration of S, and the first N elements of Σ, and test the latter in order to see if any undesignated m is among those first N elements of S. If a particular undesignated m of Σ in S, proved above to exist, is the Lth member of S, and in the Kth member σ_K of Σ, then an affirmative answer to the above test will certainly be obtained for $N = \max(L, K)$. Find then the first N for which an affirmative answer is obtained, and let (n, M, T_i, I) be the first σ to yield the affirmative answer for this N, $m_{i'}$ the first undesignated m of M thus found to be in S. We can then add $m_{i'}$ to the designated m's of M, thus obtaining a quadruplet (n, M, T_i, I_1), where I_1 designates $(k+1)$ of the m's of M as being in S, and where n is certainly a member of $\bar{C}$ not in the originally given α. But the whole process leading up to (n, M, T_i, I_1) is determined by that α. It is therefore the desired process Π_p for $p = k+1$.

Under the assumed reduction of C to S, Π_p would therefore exist for every natural number p. With α say the null set, we would thus obtain for every natural number p a quadruplet (n_p, M_p, T_{i_p}, I_p) such that p of the members of the sequence M_p are in S. Yet the total length of M_p is the order of T_{i_p}, and hence bounded. Hence the

THEOREM. *No creative set is reducible to a simple set by bounded truth-tables.*

We recall that every recursively enumerable set of positive integers is one-one reducible to the creative set K, the complete set. Hence the

COROLLARY. *Every simple set is of lower degree of unsolvability than the complete set K relative to reducibility by bounded truth-tables.*

8. **Counter-example for unbounded truth-tables.** We recall that for the particular simple set S constructed in §5, of the first $2m+2$ positive integers at most m were in S, m being any positive integer. Hence, of the $m+1$ integers $m+2, m+3, \cdots, 2m+2$, at least one is in $\bar{S}$. By setting $m=2^n-1$, with $n=1, 2, 3, \cdots$, we can effectively generate the infinite sequence of mutually exclusive finite sequences

$$\sigma: \qquad (3, 4), (5, 6, 7, 8), \cdots, (2^n+1, 2^n+2, \cdots, 2^{n+1}), \cdots$$

such that each sequence in σ has at least one member thereof in $\bar{S}$. An effective one-one correspondence between the positive integers $1, 2, 3, \cdots$ and the elements of σ is then obtained by making the positive integer n correspond to the sequence $(2^n+1, 2^n+2, \cdots, 2^{n+1})$ constituting the nth element of σ.

Given a creative set C, regenerate the elements of S, placing each in a set S_1. Furthermore, regenerate the elements of C, and as an element n thereof is generated, place all of the positive integers in the nth sequence of σ in S_1. The resulting set S_1 is a generated, and hence recursively enumerable, set of positive integers. Since S_1 contains S, $\bar{S}$ contains $\bar{S}_1$. As S is simple, $\bar{S}$, and hence $\bar{S}_1$, does not have an infinite recursively enumerable subset. Moreover, $\bar{S}_1$ is also infinite. For $\bar{C}$ is infinite. And, for each element of $\bar{C}$, the corresponding sequence in σ has only those of its members that are already in S also in S_1, and hence at least one element in $\bar{S}_1$. Hence, S_1 is simple.

Likewise we see that a positive integer n is in C, or $\bar{C}$, according as all of the integers in the nth sequence of σ are in S_1, or at least one is in $\bar{S}_1$. If then we make correspond to each positive integer n the sequence of 2^n positive integers $(2^n+1, 2^n+2, \cdots, 2^{n+1})$, and the truth-table of order 2^n in which the sign under n is $+$ in that row in which the signs under the 2^n "m's" are all $+$, and in every other row the sign under n is $-$, we have a reduction of C to S_1 by truth-tables. Hence the

THEOREM. *For each creative set C a simple set S can be constructed such that C is reducible to S by unbounded truth-tables.*

COROLLARY. *A simple set S can be constructed which is of the same degree of unsolvability as the complete set K relative to reducibility by truth-tables unrestricted.*

Simple sets as such do not therefore give us the absolutely lower degree of unsolvability than that of K we are seeking.

9. **Hyper-simple sets.** The counter-example of the last section suggests that we seek a set satisfying the following

DEFINITION. *A hyper-simple set H is a recursively enumerable set of positive integers whose complement $\bar{H}$ is infinite, while there is no infinite recursively enumerable set of mutually exclusive finite sequences of positive integers such that each sequence has at least one member thereof in $\bar{H}$.*[21]

In this definition we may use the original Gödel method for representing a finite sequence of positive integers $m_1, m_2, \cdots, m_\nu$ by the single positive integer $2^{m_1} 3^{m_2} \cdots p_\nu^{m_\nu}$, where $2, 3, \cdots, p_\nu$ are the first ν primes in order of magnitude. A set of finite sequences of positive integers is then recursively enumerable if the set of Gödel representations of those sequences is recursively enumerable.

THEOREM. *A hyper-simple set exists.*

Our intuitive argument must again draw upon the formal development to the effect that each recursively enumerable set of finite sequences of positive integers will be determined by a "basis" B^*, and that all such bases can be generated in a single infinite sequence of distinct bases

$$O^*: \quad B_1^*, B_2^*, B_3^*, \cdots .$$

As in §2, generate the elements of O^*, and as an element B^* is generated, set up the process for generating the set of sequences determined by B^*, and as a sequence s is thus generated, write down the couple (B^*, s). The resulting set of couples is then a generated set, and can indeed be effectively ordered in a sequence of distinct couples

$$O_1^*: \quad (B_{i_1}^*, s_1), (B_{i_2}^*, s_2), (B_{i_3}^*, s_3), \cdots .$$

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[21] Mutually exclusive sequences here mean no element of one sequence is an element of another. Curry suggests that "hyper-simple" is linguistically objectionable, and should be replaced by "super-simple." But we would not then know what to use in place of the letter H.

O_1^* then consists of all distinct couples (B^*, s) such that finite sequence s is a member of the recursively enumerable set of finite sequences of positive integers determined by basis B^*.

Now the condition that no infinite recursively enumerable set of mutually exclusive finite sequences of positive integers has the property that each sequence has at least one positive integer thereof in $\bar{H}$ is equivalent to each such set of sequences having at least one sequence all of whose members are in H. Our method of constructing the desired hyper-simple set H will then consist in placing in H for certain B^*'s in O^* all of the positive integers in a sequence in the set of sequences determined by B^*. For purposes of presentation we shall call each such basis B^* a *contributing basis*, while every B^* determining an infinite recursively enumerable set of mutually exclusive sequences will be called a *relevant basis*. Set H, if recursively enumerable, will then be hyper-simple if each relevant basis is a contributing basis, and $\bar{H}$ is infinite.

If B^* is a relevant basis, then among the infinite number of mutually exclusive sequences generated by B^* there must be a sequence each of whose elements exceeds an arbitrarily given positive integer N. For did every sequence generated by B^* have as element one of the integers $1, 2, \cdots, N$, for any $N+1$ of these sequences at least two would have one of these integers in common. We shall then generate H by regenerating sequence O_1^*, and, as an element $(B_{i_n}^*, s_n)$ thereof is generated, we shall place all the elements of s_n in H if $B_{i_n}^*$ has not thus been made a contributing basis earlier in the process, while the elements of s_n are all greater than a certain positive integer N_n, about to be determined; otherwise none. Inductively, assume N_m to have been determined for $1 \leqq m < n$, and thus the entire process up to the time $(B_{i_n}^*, s_n)$ was brought up for consideration. Let $B_{j_1}^*, B_{j_2}^*, \cdots, B_{j_\nu}^*$ be the bases that have thus far contributed to H, and in the order in which they became contributing bases. These bases are then distinct, and hence their subscripts, which give their position in the sequence O^* of all distinct bases, are distinct. Let $k_1, k_2, \cdots, k_\nu$ be the largest integer placed in H by the first contributing basis, by the first two, $\cdots$, by the first ν. The result being cumulative, $k_1 \leqq k_2 \leqq \cdots \leqq k_\nu$. The crux of our construction is to make N_n depend not on the history of all these ν contributions to H, but only on that part of that history up to and including the last contribution, if any, made by a B^* preceding $B_{i_n}^*$ in O^*. Specifically, if $B_{j_\mu}^*$ is the last of the above ν contributing bases preceding $B_{i_n}^*$ in O^*, that is, with $j_\mu < i_n$, N_n is to be one more than the largest integer present in H as a result of all the contributions made up to and including the contribution made by $B^*_{j_\mu}$. That is, $N_n = k_\mu + 1$. Actually, if none of the ν contributing bases precede $B^*_{i_n}$ in O^*, no condition is to be placed on s_n, and all of its elements are placed in H so long as $B^*_{i_n}$ is distinct from the ν contributing bases obtained thus far.

Furthermore, in our induction assume that we have been able to keep a record of the sequence $B^*_{j_1}, B^*_{j_2}, \cdots, B^*_{j_\nu}$, of $k_1, k_2, \cdots, k_\nu$, and also of $j_1, j_2, \cdots, j_\nu$ up to the time $(B^*_{i_n}, s_n)$ was about to be generated. We then generate $(B^*_{i_n}, s_n)$, and by regenerating O^* find the place of $B^*_{i_n}$ in O^* thus determining the subscript i_n. Our criterion for determining whether, or no, the elements of s_n are to be placed in H then becomes effective. In the latter case, the record is unchanged as we generate $(B^*_{i_{n+1}}, s_{n+1})$. In the former, $B^*_{i_n}$ is written into the record as $B^*_{j_{\nu+1}}$, i_n as $j_{\nu+1}$ while we can write in for $k_{\nu+1}$ the maximum of k_ν and the largest integer in s_n. The entire process is thus effective at each stage, and H is thus a generated, and hence recursively enumerable, set of positive integers. We proceed to prove it hyper-simple.

Let B^* be any relevant basis. Of the finite number of bases preceding B^* in O^*, but a finite number can be contributing bases. Let $B^*_{j_\mu}$ be the last of these contributing bases, if any, appearing in the sequence $B^*_{j_1}, B^*_{j_2}, B^*_{j_3}, \cdots$ of distinct contributing bases determined by the above generation of H. There will then be a sequence s generated by B^* each of whose elements is greater than $k_\mu + 1$. When then (B^*, s), a definite element of O^*_1, is generated in the course of generating H, B^* will contribute each element of s to H unless it became a contributing basis earlier in the process. Hence, every relevant basis is a contributing basis.

It also follows, or is easily seen directly, that the number of contributing bases is infinite. Consider then the infinite sequence of contributing bases $B^*_{j_1}, B^*_{j_2}, B^*_{j_3}, \cdots$, the corresponding infinite sequence of subscripts $j_1, j_2, j_3, \cdots$, and the associated infinite sequence $k_1, k_2, k_3, \cdots$. Since the contributing bases are distinct, so are their subscripts. Hence, for each j_m, among the infinite set of j's following j_m there is a unique least j, $j_{m'}$. Consider then the resulting infinite sequence $j_{\lambda_1}, j_{\lambda_2}, j_{\lambda_3}, \cdots$, where j_{λ_1} is the least j in the whole infinite sequence of j's, while $\lambda_2 = (\lambda_1)'$, $\lambda_3 = (\lambda_2)'$, $\cdots$. Now k_{λ_n} is the largest integer contributed to H through the contributing basis with subscript j_{λ_n}. Since j_{λ_n} is the smallest j following $j_{\lambda_{n-1}}$ it is less than all succeeding j's. Hence B^* with subscript j_{λ_n} precedes in O^* all bases following that B^* in the above infinite sequence of contributing bases. Hence, each element added to H by contributing bases thus following B^* with subscript j_{λ_n} must exceed $k_{\lambda_n} + 1$. It follows, on the one hand, that for each positive integer n, $k_{\lambda_n}+1$ is in $\bar{H}$. On the other hand, $k_{\lambda_{n+1}}$ itself exceeds $k_{\lambda_n}+1$ so that $k_{\lambda_{n+1}}+1>k_{\lambda_n}+1$. These members of $\bar{H}$ therefore constitute an infinite subset of the consequently infinite $\bar{H}$. Hence, H is hyper-simple.

Clearly, *every hyper-simple set H is simple.* For an infinite recursively enumerable subset of $\bar{H}$, as set of unit sequences, would contradict H being hyper-simple. Our construction of §6, in view of §8, gives us, however, *a simple set which is not hyper-simple.* Hyper-simple sets thus constitute a third class of recursively enumerable sets with recursively unsolvable decision problems—a class which is a proper subclass of the class of simple sets.

10. Non-reducibility of creative sets to hyper-simple sets by truth-tables unrestricted.

Let creative set C be reducible by truth-tables to a recursively enumerable set of positive integers H. The given reduction will again determine for each positive integer n a finite sequence of positive integers $m_1, m_2, \cdots, m_\nu$, and a truth-table T of order ν such that that row of the table which correctly tells of the m's whether they are, or are not, in H will correctly tell of n whether it is, or is not, in C. Of course ν and T as well as the m's depend on n, and the set of distinct T's now entering into our reduction may be infinite, and hence the set of distinct ν's unbounded.

Let $l_1, l_2, \cdots, l_\mu$ be any given finite sequence of distinct positive integers. A particular hypothesis on the l's being, or not being, in H may then be symbolized by a sequence of μ signs, each $+$ or $-$, such as $+- \cdots +$, such that the ith sign is $+$, or $-$, according as the hypothesis says that l_i is in H, or $\bar{H}$, respectively. We shall speak of such a sequence of signs as a *truth-assignment* for the l's, the ith sign in that sequence as the *sign of l_i* in that truth-assignment. Of the 2^μ possible truth-assignments for the l's, constituting a set V_1, one and only one correctly tells of each l_i whether it is, or is not, in H. Every set V of truth-assignments for the l's is then a subset of V_1, and will be called a *possible set* of truth-assignments if it includes this *correct* truth-assignment.

Let then V be any given possible set of truth-assignments for the l's. Let n be a positive integer with corresponding $m_1, m_2, \cdots, m_\nu$, T yielded by the given reduction of C to H such that *each m not an l is in H.* The correct row of table T must then have the following two properties. First, the sign under each m not an l must be $+$. Second, the signs under those m's which are l's must be the same as the signs of those integers in some one and the same truth-assignment for the l's in V, in fact, as in the correct truth assignment for the l's. Any row of T having these two properties, given the l's, m's and V, will be called a ***relevant row*** of T. Since for our n the correct row of T is thus a relevant row, it follows that n will surely be in $\overline{C}$ if for each relevant row of T the sign under n is $-$.

Generate then the positive integers $1, 2, 3, \cdots$, and as a positive integer N is generated, generate the first N members of a given recursive enumeration of H, and for each n, with $1 \leqq n \leqq N$, find the corresponding $m_1, m_2, \cdots, m_\nu, T$ yielded by the given reduction of C to H. Of those m's, if any, which are not l's, see if each is one of those first N members of H. If they all are, see if for each relevant row of T the sign under n is $-$. If that also is true, place n in a set α_V. Since each such n must be in $\overline{C}$, as seen above, α_V is a subset of $\overline{C}$. And being a generated set, α_V is therefore a recursively enumerable subset of $\overline{C}$.

C being creative, we can therefore find a definite positive integer n' in $\overline{C}$ but not in α_V, and, by the given reduction, the corresponding $m_1', m_2', \cdots, m_{\nu'}', T'$. Let $p_1, p_2, \cdots, p_\lambda$ be those m''s, if any, which are not l's. Now suppose that each p is in H. Then for at least one relevant row of T' the sign under n' must be $+$. For otherwise, if p_i is say the k_ith element in the given recursive enumeration of H, n' would have been placed in α_V in the above generation thereof for $N = \max(k_1, k_2, \cdots, k_\lambda, n')$. Since n' is in $\overline{C}$, such a relevant row cannot be the correct row. But, with each p in H, the signs in that row under m's that are not l's are correctly $+$. Hence the sign under at least one m' that is an l must be incorrect. But, by our definition of a relevant row, the signs under all such m''s are the same as the signs of those integers in at least one truth-assignment in V. Such a truth-assignment in V cannot therefore be the correct truth-assignment for the l's, and hence may be deleted from V. Perform this deletion for all such truth-assignments in V, and for all such relevant rows of T', to obtain the set of truth-assignments V'. Under our hypothesis that each p is in H, V' will then be a proper subset of V, and yet a possible set of truth-assignments for the l's.

Actually, let V be any given set of truth-assignments for the l's, possible or not. Each step of the above construction can then still be carried out, though the constructed entities need not now have all the properties they otherwise possess.[22] In particular, the set of integers, possibly null, $p_1, p_2, \cdots, p_\lambda$ can be found, all different from any l. Likewise, whether the p's are, or are not, all in H, the subset V' of V can be found. What we can say is that if V is a possible set, and if furthermore each p is in H, then V' is a proper subset of V, and itself is also a possible set of truth-assignments for the l's.

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[22] Recall that in the definition of creative set, §3, each B determines an n, whether the α determined by B is, or is not, a subset of $\overline{C}$.

For the given sequence of l's, start then with $V=V_1$, the possible set of all 2^μ truth-assignments for the l's, obtain the corresponding p's, $p_1', p_2', \cdots, p_{\lambda'}'$ and corresponding V', $V_2=(V_1)'$. With $V=V_2$, likewise find $p_1'', p_2'', \cdots, p_{\lambda''}''$, and $V_3=(V_2)'$, and so on. Now each V_{i+1} is a subset of V_i, while V_1 is but a finite set of 2^μ members. Hence in at most 2^μ steps we shall come across a V_κ such that either $V_{\kappa+1}$ is identical with V_κ, or is null. But if all the p''s, p'''s, $\cdots$, $p^{(\kappa)}$'s were in H, V_1 being a possible set, $V_2, \cdots, V_\kappa$ as well as $V_{\kappa+1}$ would all be possible sets, each a proper subset of the preceding. $V_{\kappa+1}$ could not then either be identical with V_κ, or null. It follows that at least one of the $p_i^{(j)}$'s with $1 \leqq j \leqq \kappa$ is in $\bar{H}$. Each $p_i^{(j)}$ is an integer that is not one of the l's. If then we take this finite set of $p_i^{(j)}$'s and arrange them in a sequence of distinct elements in say order of magnitude, we obtain for our arbitrarily given sequence of distinct positive integers $l_1, l_2, \cdots, l_\mu$ a sequence of distinct positive integers $k_1, k_2, \cdots, k_\nu$ having no element in common with the former sequence, and having at least one element in $\bar{H}$.

Starting with the null sequence as the sequence of l's, we can thus find the sequence of k's, $(k_1', k_2', \cdots, k_{\nu'}')$ of distinct positive integers at least one of which is in $\bar{H}$. Inductively, let us have thus generated the sequences $(k_1', k_2', \cdots, k_{\nu'}'), \cdots, (k_1^{(\mu)}, k_2^{(\mu)}, \cdots, k_{\nu^{(\mu)}}^{(\mu)})$, mutually exclusive, of distinct positive integers, each having at least one element in $\bar{H}$. With the single sequence $k_1', \cdots, k_{\nu^{(\mu)}}^{(\mu)}$ as the sequence of l's, we can find the corresponding sequence of k's, $(k_1^{(\mu+1)}, k_2^{(\mu+1)}, \cdots, k_{\nu^{(\mu+1)}}^{(\mu+1)})$ of distinct positive integers with no element in common with any of the preceding sequences, and having at least one element in $\bar{H}$.

With creative C reducible to recursively enumerable H by truth-tables we can thus obtain an infinite generated, and hence recursively enumerable, set of mutually exclusive finite sequences of positive integers each having an element in $\bar{H}$. The set H is therefore not hyper-simple. Hence the

THEOREM. *No creative set is reducible to a hyper-simple set by truth-tables.*

COROLLARY. *Every hyper-simple set is of lower degree of unsolvability than the complete set K relative to reducibility by truth-tables.*

Despite this result, the brief discussion of Turing reducibility, still in the informal stage, entered into in the next section makes it dubious that hyper-simple sets as such will give us the desired absolutely lower degree of unsolvability than that of K. But, in the absence of a counter-example, they remain candidates for this position.

11. General (Turing) reducibility. The process envisaged in our concept of a generated set may be said to be *polygenic*. In a *monogenic* process act succeeds act in one time sequence. The intuitive picture is that of a machine grinding out act after act (Turing [24]) or a set of rules directing act after act (Post [18]). The actual formulations are in terms of "atomic acts," the first leading to a development proved by Turing [25] equivalent to those arising from general recursive function or λ-definability, and hence of the same degree of generality. In our intuitive discussion the acts may be "molecular."

An effective solution of the decision problem for a recursively enumerable set S_1 of positive integers may therefore be thought of as a machine, or set of rules, which, given any positive integer n, will set up a monogenic process terminating in the correct answer, "yes" or "no," to the question "is n in S_1." Now suppose instead, says Turing [26] in effect, this situation obtains with the following modification. That at certain times the otherwise machine determined process raises the question is a certain positive integer in a given recursively enumerable set S_2 of positive integers, and that the machine is so constructed that were the correct answer to this question supplied on every occasion that arises, the process would automatically continue to its eventual correct conclusion.[23] We could then say that the machine effectively reduces the decision problem of S_1 to the decision problem of S_2. Intuitively this should correspond to the most general concept of the reducibility of S_1 to S_2. For the very concept of the decision problem of S_2 merely involves the answering for an arbitrarily given single positive integer m of the question is m in S_2; and in finite time but a finite number of such questions can be asked. A corresponding formulation of "Turing reducibility" should then be the same degree of generality for effective reducibility as say general recursive function is for effective calculability.[24]

We may note that whereas in reducibility by truth-tables the positive integers m of which we ask the questions "is m in S_2" are effectively determined, for given n, by the reducibility process, in Turing reducibility, except for the first such m, the very identity of the m's for which this question is to be asked depends, in general, on the correct answers having been given to these questions for all preceding m's. The mode of this dependence is, however, effective, hence we still have effective reducibility in the intuitive sense.

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[23] Turing picturesquely suggests access to an "oracle" which would supply the correct answer in each case. The "if" of mathematics is however more conducive to the development of a theory.

[24] A reading of McKinsey [16] suggested generalizing the reducibility of a recursively enumerable set S to a recursively enumerable set S' to the reducibility of S to a finite set of recursively enumerable sets $S_1, S_2, \cdots, S_n$. However, no absolute gain in generality is thus achieved, as a single recursively enumerable set S' can be constructed such that reducing S to $(S_1, S_2, \cdots, S_n)$ is equivalent to reducing S to S'. Points of interest, however, do arise.

Let now creative set C be Turing reducible to a recursively enumerable set S of positive integers. We shall talk as if our intuitive discussion has already been formalized. Generate the positive integers $1, 2, 3, \cdots$, and as a positive integer N is generated, for each n with $1 \leqq n \leqq N$ proceed as follows. Set going the reducibility process of C to S for n. Each time a question of the form "is m in S" is met, see if m is among the first N integers in a given recursive enumeration of S. If it is, supply the answer "yes," thus enabling the reducibility process to continue. Finally, if under these circumstances the process terminates in a "no" for the initial question of n being in C, place n in a set α_0. This α_0 is then a recursively enumerable subset of $\bar{C}$ consisting of all members thereof for which the given Turing reduction of C to S leads only to questions of the form "is m in S" whose answer is "yes."

Find then n_0 of $\bar{C}$ not in α_0, and set the reducibility process going for n_0. Now if at any time a wrong answer is supplied to a question "is m in S," we can nevertheless expect our machine for reducing C to S either to effectively pick up the wrong answer and operate on it to give a next step in the process, or to cease operating. Generate then the positive integers $1, 2, 3, \cdots$, and as a positive integer N is generated, generate the first N members of the given recursive enumeration of S, and make the reducibility process for n_0 *effective* though perhaps *incorrect* as follows. Each time a question of the form "is m in S" is reached, see if m is among the first N members of S. If it is, answer the question "yes," and correctly so; if not, answer the question "no," whether that answer be correct or no. If now this *pseudo-reduction* terminates in a "no," place the finite number of m's thus arising in a set β_{n_0}. Note that β_{n_0} consists of all such m's for all such pseudo-reductions for the given n_0. Being a generated set of positive integers, *β_{n_0} is recursively enumerable.*

Now let the correct, though possibly non-effective, reducibility process for n_0 involve the μ questions "is m_i in S," $i=1, 2, \cdots, \mu$. Let $m_{i_1}, m_{i_2}, \cdots, m_{i_\nu}$ be those of these m's actually in S, and let them be the n_1st, n_2nd, $\cdots$, n_νth members of the given recursive enumeration of S. If then $N \geqq M = \max(n_1, n_2, \cdots, n_\nu)$, or $M=1$ if

$\nu = 0$, the corresponding psuedo-reduction for n becomes the correct reduction. For, inductively, if that be so through the time a question "is m in S" is raised, m will be $m_1, m_2, \cdots$, or m_μ, hence will, or will not, be in S according as it is, or is not, one of the first N members of S. The answer is then correctly given by that pseudo-reduction, which therefore continues to be correct through the raising of the next question. Finally, since n_0 is in $\bar{C}$, the correct reduction, now the pseudo-reduction, must terminate with a "no."

It follows that all N's with $N > M$ merely repeat the contribution to β_{n_0} made by $N = M$, that is, of the integers $m_1, m_2, \cdots, m_\mu$. Since but a finite number of m's are contributed by N's with $N < M$, it follows that *β_{n_0} is a finite set.* Finally, were each of the integers $m_1, m_2, \cdots, m_\mu$ in S, n_0 would be in α_0. Hence, *at least one member of β_{n_0} is in $\bar{S}$.*

Formally, we would thus obtain a basis for a finite recursively enumerable set of positive integers at least one of whose members is in $\bar{S}$. Instead of recursively enumerable sets of finite sequences of positive integers, we would thus be led to consider recursively enumerable sets of bases for finite recursively enumerable sets of positive integers. Though, in the last analysis, each sequence in the former case must be generated atom by atom, there will come a time for each sequence when the process will say "this sequence is completed." In the latter case, in general, we cannot have an effective method which, for each basis, will give a point in the ensuing process at which it can say all members of the finite set in question have already been obtained, even though, with the process made monogenic, there always is such a stage in the process.

This suggests, then, that we strengthen the condition of hyper-simplicity still further by replacing "infinitive recursively enumerable set of mutually exclusive finite sequences of positive integers" in the definition of §9 by "infinite recursively enumerable set of bases defining mutually exclusive finite recursively enumerable sets of positive integers." Whether such a "hyper-hyper-simple" set exists, or whether, if it exists, it will lead to a stronger non-reducibility result than that of the last section we do not know.

On the other hand, an equivalent definition of hyper-simple set is obtained if, for example, we replace the quoted phrase by "recursively enumerable set of finite sequences of positive integers having for each positive integer n a member each of whose elements exceeds n." We now can say that with this as the definition of a hyper-simple set, the corresponding extension to a hyper-hyper-simple set cannot be made. For we prove the

THEOREM. *For any recursively enumerable set of positive integers S, with infinite $\bar{S}$, there exists a recursively enumerable set of bases defining finite recursively enumerable sets of positive integers, each set having at least one element in $\bar{S}$, and at least one set having each of its elements greater than an arbitrarily given positive integer n.*

Briefly, with n given, for each positive integer N, and each positive integer m, place all of the integers $n+1, n+2, \cdots, n+m$ in a set α_n if all, or all but the last, are among the first N members of a given recursive enumeration of S. It is readily seen that α_n is a generated, and hence recursively enumerable, set of positive integers. A corresponding basis $B^{(n)}$ can actually be found, and the set of $B^{(n)}$'s, $n=1, 2, 3, \cdots$, being a generated set, is therefore recursively enumerable. Moreover, if ν_n is the smallest integer in the infinite $\bar{S}$ greater than n, α_n will consist of exactly the integers $n+1, n+2, \cdots, \nu_n$, and hence will be finite, with indeed ν_n as the only element in $\bar{S}$, and with each element greater than n.

As a result we are left completely on the fence as to whether there exists a recursively enumerable set of positive integers of absolutely lower degree of unsolvability than the complete set K, or whether, indeed, all recursively enumerable sets of positive integers with recursively unsolvable decision problems are absolutely of the same degree of unsolvability. On the other hand, if this question can be answered, that answer would seem to be not far off, if not in time, then in the number of special results to be gotten on the way.[25]

Such then is the portion of "Recursive theory" we have thus far developed. In fixing our gaze in the one direction of answering the lower degree of unsolvability question, we have left unanswered many questions that stud even the short path we have traversed. Moreover, both our special, and the general Turing, definitions of reducibility are applicable to arbitrary decision problems whose questions in symbolic form are recursively enumerable, and indeed to problems with recursively enumerable set of questions whose answers belong to a recursively enumerable set. Thus, only partly leaving the field of decisions problems of recursively enumerable sets, work of Turing [26] suggests the question is the problem of determining of an arbitrary basis B whether it generates a finite, or infinite, set of positive integers of absolutely higher degree of unsolvability than K. And if so, what is its relationship to that decision problem of absolutely higher degree of unsolvability than K yielded by Turing's theorem.

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[25] This is a matter of practical concern as well as of theoretical interest. For according as the second or first of the above alternatives holds will the method of reducing new decision problems to problems previously proved unsolvable be, or not be, the general method for proving the unsolvability of decision problem either of recursively enumerable sets of positive integers or of problems equivalent thereto.

Actually, the theory of recursive reducibility can be but one chapter in the theory of recursive unsolvability, and that, but one volume of the theory and applications of general recursive functions. Indeed, if general recursive function is the formal equivalent of effective calculability, its formulation may play a rôle in the history of combinatory mathematics second only to that of the formulation of the concept of natural number.

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THE CITY COLLEGE,
NEW YORK CITY.

THE FEBRUARY MEETING IN NEW YORK

The four hundred second meeting of the American Mathematical Society was held at Columbia University on Friday and Saturday, February 25–26. The attendance was approximately two hundred seventy-five including the following two hundred seven members of the Society:

C. R. Adams, E. B. Allen, H. W. Allen, C. B. Allendoerfer, R. L. Anderson, R. G. Archibald, K. J. Arnold, L. A. Aroian, S. P. Avann, Valentin Bargmann, A. E. Basch, Stefan Bergman, Lipman Bers, J. H. Bigelow, Garrett Birkhoff, Gertrude Blanch, Salomon Bochner, A. H. Bowker, C. B. Boyer, H. W. Brinkmann, H. C. Brodie, A. B. Brown, Hobart Bushey, J. H. Bushey, S. S. Chern, Alonzo Church, Harvey Cohn, L. M. Comer, T. F. Cope, Richard Courant, J. E. Crawford, H. B. Curry, M. D. Darkow, J. A. Daum, Jesse Douglas, Arnold Dresden, Nelson Dunford, William H. Durfee, J. E. Eaton, Benjamin Epstein, W. H. Fagerstrom, J. M. Feld, Will Feller, A. D. Fialkow, C. D. Firestone, Tomlinson Fort, R. M. Foster, F. H. Fowler, G. A. Foyle, W. C. G. Fraser, Bernard Friedman, K. O. Friedrichs, D. L. Fuller, R. E. Fullerton, G. N. Garrison, H. P. Geiringer, Abe Gelbart, B. H. Gere, B. P. Gill, Leonard Gillman, P. H. Graham, M. C. Gray, J. W. Green, Lewis Greenwald, C. C. Grove, E. J. Gumbel, Theodore Hailperin, D. W. Hall, P. R. Halmos, M. G. Harrison, J. O. Hassler, L. A. Hazeltine, M. H. Heins, Olaf Helmer, Max Herzberger, M. R. Hestenes, Einar Hille, Abraham Hillman, M. O. Hizon, Banesh Hoffman, A. P. Holló, Harold Hotelling, S. E. Hotelling, E. M. Hull, H. D. Huskey, L. C. Hutchinson, B. M. Ingersoll, R. P. Isaacs, Fritz John, R. A. Johnson, A. W. Jones, Mark Kac, Aida Kalish, Irving Kaplansky, Edward Kasner, Robert Kates, L. S. Kennison, S. A. Kiss, S. C. Kleene, J. R. Kline, Mark Kormes, Arthur Korn, D. M. Krabill, Jack Laderman, Howard Levene, Howard Levi, Simon Lopata, E. R. Lorch, Lee Lorch, A. N. Lowan, C. I. Lubin, N. H. McCoy, A. W. McMillan, Brockway McMillan, L. A. MacColl, H. F. MacNeish, H. B. Mann, A. J. Maria, M. H. Martin, W. T. Martin, A. E. Meder, K. S. Miller, L. W. Miller, Don Mittleman, Deane Montgomery, D. J. Morrow, David Moskovitz, M. E. Munroe, C. A. Nelson, Otto Neugebauer, A. V. Newton, E. N. Nilson, P. B. Norman, C. O. Oakley, L. F. Ollmann, A. F. O'Neill, Oystein Ore, J. C. Oxtoby, L. G. Peck, A. M. Peiser, F. W. Perkins, Hillel Poritsky, E. L. Post, Willy Prager, Walter Prenowitz, M. H. Protter, A. L. Putnam, R. G. Putnam, H. W. Reddick, Mina Rees, Helene Reschovsky, Moses Richardson, R. G. D. Richardson, D. E. Richmond, C. E. Rickart, J. F. Ritt, R. K. Ritt, E. K. Ritter, H. E. Robbins, M. S. Robertson, P. C. Rosenbloom, J. E. Rosenthal, E. A. Saibel, Raphael Salem, H. E. Salzer, Hans Samelson, Arthur Sard, L. J. Savage, S. A. Schelkunoff, C. E. Shannon, Seymour Sherman, Max Shiffman, L. L. Silverman, D. M. Smiley, M. F. Smiley, P. A. Smith, Ernst Snapper, E. R. Stabler, Fritz Steinhardt, Wolfgang Sternberg, G. R. Stibitz, J. J. Stoker, R. R. Stoll, M. H. Stone, W. J. Strange, E. G. Straus, M. M. Sullivan, Fred Supnick, J. D. Tamarkin, E. B. Tolsted, C. C. Torrance, E. M. Torrance, A. W. Tucker, J. W. Tukey, A. R. Turquette, G. L. Walker, R. M. Walter, W. R. Wasow, G. C. Webber, Louis Weisner, Hermann Weyl, A. M. Whelan, P. M. Whitman, John Williamson, W. H. Wise, Jacob Wolfowitz, Antoni Zygmund.

President M. H. Stone presided at the general session Friday evening at which Professor H. W. Emmons gave an address on *Numerical methods of solving partial differential equations.* The address was followed by a general discussion led by Professors Richard Courant and Harold Hotelling.

On Saturday morning there were two sections, one for papers in Analysis and Geometry at which Professor W. T. Martin presided, and one for papers in Algebra, Statistics, and Applied Mathematics at which Professors Oystein Ore and H. B. Curry presided.

At the general session Saturday afternoon Professor E. L. Post gave an address entitled *Recursively enumerable sets of positive integers and their decision problems.* Professor J. D. Tamarkin presided.

Titles and cross references to the abstracts of the papers read follow below. Papers whose abstract numbers are followed by the letter *t* were read by title. Papers numbered 1 to 9 were presented in the section for Analysis and Geometry, papers 10 to 20 in the section for Algebra, Statistics, and Applied Mathematics, papers 21 to 54 were presented by title. Paper 6 was read by Professor Kasner, paper 13 by Mr. Churchman. Mr. Vazsonyi was introduced by Professor J. R. Kline and Professor Wang by Professor J. A. Shohat.

1. A. M. Peiser: *Some applications of Fourier analysis to the study of real roots of algebraic equations.* (Abstract 50-3-65.)

2. B. M. Ingersoll: *On singularities of solutions of linear partial differential equations.* (Abstract 50-3-60.)

3. C. E. Rickart: *Representation of linear transformations on summable functions.* (Abstract 50-3-67.)

4. M. F. Smiley: *An extension of metric distributive lattices with an application in general analysis.* (Abstract 50-3-74.)

5. Jesse Douglas: *Separable transformations with separable inverse.* (Abstract 50-3-58.)

6. Edward Kasner and John DeCicco: *Scale curves in conformal maps.* (Abstract 50-3-95.)

7. S. S. Chern: *A simple intrinsic proof of the Gauss-Bonnet formula for Riemannian polyhedra.* (Abstract 50-3-90.)

8. J. M. Feld: *On a representation in space of groups of circle and turbine transformations in the plane.* (Abstract 50-3-91.)

9. Wolfgang Sternberg: *On difference equations.* (Abstract 50-1-33.)

10. A. W. Jones: *The lattice isomorphisms of certain finite groups.* (Abstract 50-3-53.)

11. P. M. Whitman: *Lattices and equivalence relations.* Preliminary report. (Abstract 50-3-57.)

12. Raphael Salem: *On a remarkable class of algebraic integers. Proof of a conjecture of Vijayaraghavan.* (Abstract 50-3-55.)

13. Benjamin Epstein and C. W. Churchman: *On the statistics of sensitivity data.* (Abstract 50-3-96.)

14. H. E. Robbins: *On the measure of a random set.* (Abstract 50-3-98.)

15. E. J. Gumbel: *The observed return period.* (Abstract 50-3-97.)

16. Stefan Bergman: *On solutions of certain partial differential equations in three variables.* (Abstract 50-3-78.)

17. R. P. Isaacs: *Airfoil theory for flows of variable velocity.* (Abstract 50-3-83.)

18. Andrew Vazsonyi: *On two-dimensional rotational gasflows.* (Abstract 50-3-88.)

19. Max Herzberger: *The diapoint characteristic.* (Abstract 50-3-82.)

20. David Moskovitz: *Numerical solution of Laplace's and Poisson's equations.* (Abstract 50-3-85.)

21. Stefan Bergman: *A Hermitian metric and its property.* (Abstract 50-3-89-*t*.)

22. C. L. Clark: *Arc reversing transformations.* (Abstract 50-3-99-*t*.)

23. Nathaniel Coburn: *A boundary value problem in plane plasticity for the Coulomb yield condition.* (Abstract 50-3-79-*t*.)

24. A. L. Foster and B. A. Bernstein: *Symmetric definition and duality theorem for rings.* (Abstract 50-1-12-*t*.)

25. K. O. Friedrichs: *The identity of weak and strong extensions of differential operators.* (Abstract 50-3-59-*t*.)

26. H. L. Garabedian: *The analogue of Bromwich's theorem for integral transformations.* (Abstract 50-3-80-*t*.)

27. Mariano García: *Component orbits under pointwise recurrent homeomorphisms.* (Abstract 50-3-100-*t*.)

28. W. H. Gottschalk: *Powers of homeomorphisms with almost periodic properties.* (Abstract 50-3-101-*t*.)

29. D. W. Hall: *On rotation groups of plane continuous curves under pointwise periodic homeomorphisms.* (Abstract 50-3-102-*t*.)

30. C. C. Hsiung: *Some invariants of certain pairs of hypersurfaces.* (Abstract 50-3-92-*t*.)

31. Dunham Jackson: *Orthonormal polynomials on loci of the second degree.* (Abstract 50-5-139-*t*.)

32. Mark Kac: *On real zeros of some functions.* (Abstract 50-3-61-*t*.)

33. Edward Kasner and John DeCicco: *Scale curves in cartography.* (Abstract 50-3-94-*t*.)

34. Jakob Levitzki: *A characteristic minimal condition for semi-*

primary rings. (Abstract 50-1-14-*t*.)

35. A. T. Lonseth: *Dirichlet's principle for* $\Delta u = F_u(u; x, y)$. (Abstract 50-3-62-*t*.)

36. Szolem Mandelbrojt: *Some theorems connected with the theory of infinitely differentiable functions.* (Abstract 50-3-63-*t*.)

37. R. v. Mises: *Note on the numerical solution of linear partial differential equations.* (Abstract 50-3-84-*t*.)

38. H. E. Newell: *The solutions of a certain linear matric differential equation.* (Abstract 50-3-64-*t*.)

39. Isaac Opatowski: *Use of special functions in a problem of uniform strength.* (Abstract 50-3-86-*t*.)

40. Oystein Ore: *Galois connexions.* (Abstract 50-3-54-*t*.)

41. Harry Pollard: *Fourier series with coefficients in a Banach space.* (Abstract 50-3-66-*t*.)

42. H. E. Robbins: *A note on the Riemann integral.* (Abstract 50-3-68-*t*.)

43. F. H. Safford: *Analysis of a non-harmonic wave.* (Abstract 50-3-69-*t*.)

44. Raphael Salem: *On a theorem of Bohr and Pál.* (Abstract 50-3-70-*t*.)

45. Raphael Salem: *Sets of uniqueness and sets of multiplicity.* II. (Abstract 50-3-71-*t*.)

46. H. E. Salzer: *Supplementary calculation of coefficients for numerical integration with central differences.* (Abstract 50-1-41-*t*.)

47. H. E. Salzer: *Table of coefficients for inverse interpolation with advancing differences.* (Abstract 50-3-87-*t*.)

48. A. R. Schweitzer: *On functional equations with solutions containing arbitrary functions.* IV. (Abstract 50-3-72-*t*.)

49. A. R. Schweitzer: *On functional equations with solutions containing arbitrary functions.* V. (Abstract 50-3-73-*t*.)

50. W. J. Thron: *Sets of convergence for continued fractions.* Preliminary report. (Abstract 50-1-34-*t*.)

51. F. T. Wang: *On Riesz summability of Fourier series.* IV. (Abstract 50-3-75-*t*.)

52. F. T. Wang: *Some remarks on oscillating series.* (Abstract 50-3-76-*t*.)

53. P. M. Whitman: *Identities in lattices of equivalence relations.* Preliminary report. (Abstract 50-3-56-*t*.)

54. D. V. Widder: *The iterates of the Laplace kernel.* (Abstract 50-3-77-*t*.)

T. R. HOLLCROFT,
Associate Secretary

BOOK REVIEW

Metric methods in Finsler spaces and in the foundations of geometry. By Herbert Busemann. Princeton University Press, 1942. 2+243 pp., 22 fig. $3.00.

Metric notions have entered the foundations of geometry in various ways. In his *Grundlagen*, Hilbert formulates axioms of congruence which imply the availability of a metric, but the idea of distance is not explicitly used. In the projective framework for the classical noneuclidean geometries (for example, Klein), distance is defined in terms of specific configurations which are themselves the product of rather elaborate preparatory study. In the differential approach (for example, Cartan), the theory of coördinate manifolds is taken for granted, distance is first introduced locally, and even after its extension to nonlocal measurements it plays little essential part in the investigations. Others (Lie, Hilbert (Anhang IV to the *Grundlagen*)) based their work on a group of transformations (motions) which provided the means for defining congruence. The advent of metric spaces (Fréchet) raised the possibility of beginning with metric ideas and developing from them the equipment and results of the earlier theories. Menger isolated and studied the properties a metric space must have in order to possess geodesics which behave conveniently. The present monograph takes up at that point and develops a connected theory, occasional portions of which have appeared elsewhere, of metric spaces with geodesics.

The object of study is a finitely compact (hence separable) metric space Σ which is (internally) convex in the sense that, if $X \neq Z$, there is a point Y between X and Z (that is, such that $XY+YZ=XZ$). Define a segment to be a congruent map of a closed real interval, and a geodesic to be a locally congruent map of the real axis. As a decisive local restriction (D), each point of Σ is assumed to have a neighborhood N such that, if A and B are any distinct points of N, the set of points between A and B forms a segment which can be extended appreciably (in fact, outside N), in either direction past A and B, and which is the *unique* segment between any pair of its points of which neither is too far beyond A or B. Without Axiom D, any two points P and Q can be joined by a segment. With Axiom D, this segment turns out to be unique, if P and Q are not too far apart; further, any pair can be connected by a geodesic, and a given segment can be embedded in one and only one geodesic. If there is a point at which

Σ is one-dimensional, Σ is congruent to a euclidean straight line or circle; if Σ is two-dimensional, Σ is a manifold.

Σ is Minkowskian at a point P if and only if there is a neighborhood N of P into which a metric can be introduced, equivalent in an appropriate sense to the original metric, in such a way that under the new metric N is congruent to an open sphere of a linear space which is so normed (cf. Minkowski) that its unit sphere U is a strictly convex surface with the origin as center. The author outlines a proof that a Finsler space Φ is a space Σ which is Minkowskian at each point if Φ is regular (because U is strictly convex) and symmetric (because U has a center). Starting off, conversely, with a space Σ (of chap. 1), he imposes a local condition (Δ) which enables him to introduce a Minkowski metric m at each point where Δ is satisfied. Δ refers to a trio of points which tend to coincidence, and controls the convergence of the segments and of the ratios of the distances determined by these points. The metric m is defined as follows, in a suitably restricted neighborhood N of P: For A in N and $0 \leqq t \leqq 1$, let A_t be that point on the segment $\overline{PA}$ for which $PA_t = tPA$; then $m(A, B) = \lim t^{-1}A_tB_t$ as $t \rightarrow 0+$. The neighborhood N is expanded to a space Σ_P by continuing each segment $\overline{PA}$ past A up to the boundary of N and then on indefinitely through new points created explicitly for this purpose. $m(A, B)$ is extended throughout Σ_P, which is then seen to be a normed (Minkowskian) linear space associated with Σ at P. It follows from a further local condition Δ', if an open set G of Σ is the intersection of Σ_P and Σ_Q, that the local (normal) coordinates provided for G by these spaces are connected by a transformation of class 1. Finally, the metric m is used to define a function $F(x, \lambda)$ of the sort with which Finsler theories commonly set out. Thus Δ and Δ' are conditions sufficient to insure that Σ be a regular symmetric Finsler space. It may happen that m is euclidean in Σ_P; in the important special case in which this happens at every point, the Finsler space reduces to a Riemann space and Σ_P then differs but slightly from the tangent space widely used in differential geometry.

Specializing the space Σ of chap. 1 in quite a different manner, the remainder (two-thirds) of the book is based on the global Axiom E: Any two distinct points are on at most (hence exactly) one geodesic. Each geodesic of such a (straight line, S.L.) space is either open or closed (congruent to a euclidean straight line or circle), and the closed (open) geodesics form an open (closed) set. In the two-dimensional case any two closed geodesics intersect, and the geodesics through a given point are either all closed or all open. Hence an S.L. plane is either closed (all geodesics closed) or open (all geodesics open). By mapping an S.L. plane Σ in a specific way on a projective plane P^2 with an elliptic metric, it is shown that Σ is homeomorphic to E^2 (euclidean plane) or P^2 according as Σ is open or closed. A homeomorphism carrying geodesics of Σ into straight lines is possible (that is, Σ is Arguesian) if and only if Desargues' Theorem holds. (An Arguesian S.L. space of any dimension is either closed or open.) The inverse problem for E^2 seeks conditions on a family F of curves α in E^2 sufficient to insure that a topologically equivalent new metric can be introduced in E^2 in such a way that the curves α become the geodesics of the S.L. plane thus defined. In a very satisfying way, the sufficiency of the following remarkably weak conditions is established: each α is an open simple unbounded Jordan curve, and any two distinct points of E^2 are contained in eactly one curve of F.

Define a sphere to be convex if no geodesic tangent to it contains a point interior to it. Let each sphere of the S.L. space Σ be convex (condition K). This implies, for fixed P, as X moves on a geodesic g, that PX either is constant or has precisely one minimum which is attained just once. It follows that Σ is either open or closed. In the closed case, with dimension not less than 3, Σ is elliptic. In the open case K is equivalent to convexity in the usual sense and to the existence of precisely one (suitably defined) perpendicular to each geodesic from each external point. By detailed argument in which perpendiculars and baselines, parallels, bisectors (loci $PX=QX$) and their limits, and so on, are studied extensively, several theorems are proved establishing conditions which completely characterize Σ in one way or another. Examples: with dimension not less than 3, K, differentiable spheres (suitably defined), and euclidean parallelism, Σ is Minkowskian. In the plane case, with euclidean parallelism, Σ is Minkowskian if the following strengthened form of K holds: for some fixed $\alpha \geqq 1$, if $XY=YZ=XZ/2$, then $2PY^\alpha \leqq PX^\alpha + PZ^\alpha$ for any P. If all bisectors are linear, regardless of dimension, then Σ is euclidean or hyperbolic.

Along with the classical geometries, n-dimensional S.L. spaces Σ can be characterized in terms of their mobility. An examination of involutoric motions and their fixed points reveals that Σ is homogeneous (congruent to a euclidean, hyperbolic, or elliptic space) if it is symmetric about each geodesic. If Σ is closed and symmetric about each point, Σ is elliptic. In E^n a congruence between two triangles can be extended to a motion of E^n; if Σ has the property that there is a number $\delta>0$ such that a congruence between any two triangles, each with diameter less than δ, can be extended to a motion of Σ, then Σ is homogeneous. An example of Fubini is adapted to show that the similar hypothesis for pairs instead of trios of points is insufficient. Suppose that Σ is open, and admits a transitive abelian group G of motions; it is easily seen that G is closed and simply transitive and generates a Minkowskian plane by carrying either of two intersecting geodesics along the other, whence it follows that Σ is Minkowskian. The plane case is studied in detail by means of translations along geodesics. Typical result: An S.L. plane is Minkowskian, hyperbolic, or elliptic if all possible translations exist along two geodesics one of which is not an asymptote to either orientation of the other. If an S.L. plane admits a transitive group of motions, then: either the plane is closed and its metric elliptic; or parallelism is euclidean and the metric is Minkowskian, or parallelism is hyperbolic and the metric is quasi-hyperbolic (admitting all translations along a geodesic and its nonparallel asymptotes). Attention is called, in conclusion, to a number of unsolved problems.

As the cited results suggest, a superficial refresher in non-euclidean geometry is a desirable preliminary. The exposition is clear and persuasive with the exception of one passage (pp. 49–63) which the reviewer found rough in spots. The situation there is inherently complicated, notation, typography, and display are uncoöperative, and the style seems unnecessarily terse. Slight expansion would have rendered this material much more pleasantly accessible.

Reading is enlivened by a stimulating diffusion of misprints. Only the following, which the author has kindly verified, are thought capable of delaying the attentive reader:

Page	*Line*	*For*	*Read*
9	7	4	5
65	$4'$	A, B	A_r, B_r
97	4	$[tf(n)/n-1/m2^n]$	$t[2f(n)/n-1/m2^n]$
167	8	h	π

With these trivial qualifications, the project has been conceived and executed in a thoroughly workmanlike fashion. Loose ends lead either to specified passages in the literature, or to outstanding problems which are clearly indicated with accompanying remarks and references. Although the argument is basically "coördinate-free," appealing picturesquely to the intuition, one senses none of that loss of rigor which is sometimes felt to accompany "synthetic" methods. In presenting a connected account of the author's valuable contributions, this book gives very useful access to an interesting field of study.

In a recent paper (Trans. Amer. Math. Soc. vol. 54 (1943) pp. 171–184) the author establishes important new results.

1. A closed S.L. space of dimension not less than 3 is elliptic if each of its spheres has order 2 (condition K_1, weaker than K). Thus K_1, which was known (p. 135) to be equivalent to K in open spaces, is now seen to be an adequate basis for the main theorem (p. 124, cited above) proved with K in closed spaces.

2. Several conditions, of which a few have been cited, are found in the book under which an S.L. space is necessarily open or closed. Such conditions are now seen to be entirely superfluous, for any S.L. space is either open or closed with each geodesic in the closed case having the same length.

These facts open the way for substantial improvements in the study based in the book on condition K.

F. A. FICKEN

NOTES

Professor Harry Bateman of the California Institute of Technology and Professor W. M. Whyburn of the University of California at Los Angeles have been elected corresponding members of the Academia Nacional de Ciencias Exactas, Fisicas y Naturales de Lima.

Professor Felix Bernstein of New York University has been appointed professor emeritus.

Professor M. H. Stone of Harvard University has been awarded an honorary doctorate by the University of San Marcos.

Mr. J. C. Adams and Mr. G. E. Nicholson have been appointed lecturers at the University of North Carolina.

Associate Professor H. A. Bender of the University of Akron has been appointed to an associate professorship at Rhode Island State College.

Professor W. C. Brenke of the University of Nebraska has retired.

Assistant Professor H. F. Bright of Denison University has been appointed to an assistant professorship at the University of Rochester.

Visiting Assistant Professor Nancy Cole of Kenyon College has been promoted to a visiting associate professorship.

Dr. R. H. Fox of the University of Illinois has been appointed to an assistant professorship at Syracuse University.

Dr. W. H. Gottschalk of the University of Virginia has been appointed to an assistant professorship at the University of Richmond.

Dr. R. W. Hamming of the University of Illinois has been appointed to an assistant professorship at the University of Louisville.

Assistant Professor J. F. Heyda of Denison University has been appointed to a professorship at St. Ambrose College, Davenport, Iowa.

Dean H. M. Hosford of the College of Arts and Sciences, University of Arkansas, has been promoted to the vice presidency of the University.

Dr. D. H. Hyers of the California Institute of Technology has been appointed to an associate professorship at the University of Southern California.

Professor R. L. Jeffery of Acadia University has been appointed to a professorship at Queens University.

Assistant Professor R. N. Johanson of Hamilton College has been appointed to an assistant professorship at Boston University.

Mr. W. S. Krogdahl has been appointed to an adjunct professorship at the University of South Carolina.

Professor C. V. Newsom of the University of New Mexico has been appointed to a professorship at Oberlin College. He will be chairman of the department of mathematics.

Mr. J. D. Novak of MacMurray College has been appointed to an associate professorship at the University of South Carolina.

Assistant Professor J. F. Randolph of Cornell University has been appointed to a professorship at Oberlin College.

Mr. M. M. Resnikoff has been appointed to a professorship at State Teachers College, Minot, North Dakota.

Dr. Henry Scheffé of Princeton University has been appointed to an assistant professorship at Syracuse University.

Professor M. A. Scheier of St. Bonaventure College has been appointed dean of the Division of Natural Sciences.

Dr. James Singer of Brooklyn College has been promoted to an assistant professorship.

Colonel R. H. Somers, U. S. Army, retired, has been appointed to a visiting lecturership at Dartmouth College.

Associate Professor G. A. Williams of Oregon State College has been promoted to a professorship.

Associate Professor A. S. Winsor of the University of North Carolina has been promoted to a professorship.

The following appointments to instructorships are announced: Dartmouth College: Mr. D. B. Kirk; University of North Carolina: Mr. B. M. Drucker, Mr. T. H. Partrick, Mr. W. H. Peacock, Mr. Irwin Stoner, Mr. W. S. Winn; Purdue University: Mr. H. W. Strand; University of Richmond: Mr. Mariano García.

Professor Emeritus Harris Hancock of the University of Cincinnati died March 20, 1944 at the age of seventy-seven years. He had been a member since 1895.

The following fifty-one doctorates, with mathematics or mathematical physics as a major subject, were conferred during 1943 in universities in the United States and Canada; the major subject is mathematics unless otherwise specified. The university, month in which degree was conferred, minor subject (other than mathematics) and the title of the dissertation are given in each case if available.

W. F. Atchison, Illinois, June, minor in physics, *Virtual sets on an algebraic curve as contrasted with Abelian function theory.*

T. A. Bancroft, Iowa State, August, minor in economics, *Tests of significance considered as an aid in statistical methodology.*

Helen P. Beard, Massachusetts Institute of Technology, December, minor in physics, *Equigonal planes and their connection with the theory of functions of two complex variables.*

E. M. Beesley, Brown, June, I: *Concerning total differentiability of functions of class P.* II: *ϕ-Cantorian functions and their convex moduli.*

D. L. Benedict, Wisconsin, May, *Gamma ray energy standards and resonance half-widths.*

B. H. Bissinger, Cornell, October, *Generalizations of continued fractions.*

Angeline J. Brandt, Michigan, October, *The free Lie ring and Lie representations of the full linear group.*

A. J. Coleman, Toronto, November, *A study in relativistic quantum mechanics, based on Eddington's relativity-theorem of protons and electrons.*

B. H. Colvin, Wisconsin, May, *The expansion problem associated with a third order ordinary differential system of highly irregular type.*

C. R. DePrima, New York University, June, *Characteristic and uniqueness theory for linear hyperbolic partial differential equations.*

Roy Dubisch, Chicago, December, *Composition of quadratic forms.*

William H. Durfee, Cornell, October, *Congruence of quadratic forms over valuation rings.*

Jacques Dutka, Columbia, June, *Transversality in higher space.*

Paul Eberhart, Brown, February, *On the summation of derived series of Fourier series and conjugate series.*

H. J. Eckweiler, New York University, June, minor in physics, *Periodic solutions of a class of nonlinear differential equations (the Van der Pol equation).*

W. S. Erickson, Wisconsin, May, *Asymptotic forms of the solutions of the differential equation for the associated Mathieu functions.*

R. W. Gibson, Illinois, June, minor in physics, *Projective geometry with coordinates from a commutative primary ring.*

Theodore Hailperin, Cornell, June, *A set of axioms for logic.*

A. O. Hanson, Wisconsin, May, *Voltage-measuring and control equipment for the electrostatic generator and an absolute measurement of the lithium proton-neutron threshold.*

Gerald Harrison, California Institute of Technology, June, minor in physics, *The lattice structure of algebraic moduls.*

Boyd Harshbarger, George Washington, February, *On the analysis of a certain six by six—four group lattice design.*

Euphemia L. Haynes, Catholic University, June, minors in physics and education, *Determination of sets of independent conditions characterizing certain special cases of symmetric correspondences.*

H. D. Huskey, Ohio State, March, *Contribution to the problem of Geöcze.*

H. S. Kieval, Cincinnati, June, *On certain types of continued fraction developments.*

Horace Komm, Michigan, February, *Concerning the dimension and the lambda dimension of a partial order.*

Janie C. Lapsley, Illinois, June, minor in chemistry, *Systems of bilinear forms.*

Anne L. Lewis, Chicago, September, *Sufficiency proofs for the problem of Bolza in the calculus of variations.*

J. C. R. Li, Iowa State, July, minor in genetics, *Design and statistical analysis of confounded experiments.*

B. J. Lockhart, Illinois, October, minor in astronomy, *Covariant correspondences and covariant sets of points defined by a given correspondence on an algebraic curve.*

Janet MacDonald, Chicago, September, *Conjugate nets in asymptotic parameters.*

Abraham Miller, New York University, June, minor in physics, *The isoperimetric problem in three or more dimensions.*

L. H. Miller, Ohio State, September, *A method for proving the equivalence of certain orthogonal polynomial expansions.*

C. D. Olds, Stanford, April, *On the number of representations of the square of an integer as the sum of an odd number of squares.*

J. F. Paydon, Northwestern, minor in physics, *The continued fraction as a sequence of linear transformations.*

W. H. Pell, Wisconsin, August, *Thermal deflection of anisotropic thin plates.*

George Piranian, Rice, June, *A study of the position and nature of the singularities of functions given by Taylor series.*

R. C. Rand, Maryland, June, minor in physics, *The rectilinear motion of a gas subsequent to an internal explosion.*

R. H. Scanlan, Massachusetts Institute of Technology, May, minor in physics, *On the existence of certain entire functions of zero type.*

O. H. Schmidt, Brown, June, *On the relation between ancient mathematics and spherical astronomy.*

Abraham Seidenberg, Johns Hopkins, June, *Valuation ideals in rings of polynomials in two variables.*

W. de V. B. Spatz, New York University, June, major in physics, *Factors influencing the Plateau characteristics of self-quenching G-M counters.*

Sister Mary P. Steele, Catholic University, June, minors in philosophy and physics, *A geometric interpretation and some applications of the dihedral group* G_6.

C. F. Stephens, Michigan, October, *Non-linear difference equations analytic in a parameter.*

R. R. Stoll, Yale, June, *Representations of completely simple semigroups.*

W. J. Thron, Rice, June, *Convergence regions for continued fractions.*

H. F. Tuan, Princeton, April, *Groups whose orders contain a prime number to the first power.*

C. M. Turner, Wisconsin, May, *Electrostatic generator with multiple electrodes.*

G. C. Vedova, Maryland, June, minor in physics, *Infinite processes in Greek mathematics.*

T. C. G. Wagner, Maryland, June, minor in physics, *The differential geometry associated with a given area metric.*

Marion D. Wetzel, Northwestern, August, minor in statistics, *The analytic theory of positive definite J-fractions.*

Fumio Yagi, Massachusetts Institute of Technology, May, minor in physics, *On Stieltjes integrals: A convergence theorem and an integral equation.*

The following doctorate was conferred in 1942, but was not included in the list in the preceding volume of this Bulletin (vol. 49, pp. 355–360):

Aughtum S. Howard, Kentucky, September, *Linear second order partial differential equations with constant coefficients.*

ABSTRACTS OF PAPERS

SUBMITTED FOR PRESENTATION TO THE SOCIETY

The following papers have been submitted to the Secretary and the Associate Secretaries of the Society for presentation at meetings of the Society. They are numbered serially throughout this volume. Cross references to them in the reports of the meetings will give the number of this volume, the number of this issue, and the serial number of the abstract.

ALGEBRA AND THEORY OF NUMBERS

103. Leon Alaoglu and Paul Erdős: *On highly composite and similar numbers.*

A number n is highly composite if for all $m<n$, $d(m)<d(n)$; superabundant if for all $m<n$, $\sigma(m)/m<\sigma(n)/n$; and highly abundant if for all $m<n$, $\sigma(m)<\sigma(n)$. If q^k is the highest power of the prime q dividing the highly abundant number n, it is found that with at most $c(\epsilon)$ $\log_2 n/\log_3 n$ exceptions, $(1-\epsilon)$ $\log n$ $\log_2 n/q<q^k$ $\log q$ $<(1+\epsilon)$ $\log n$ $\log_2 n$. The superabundants satisfy a stronger inequality with no exceptions, and a similar formula is proved for the highly composite numbers. For these two latter classes the inequalities determine the prime power factors almost uniquely. Highly composite numbers were introduced by Ramanujan. Pillai's D.Sc. thesis may contain related results, but it was never published and is inaccessible. (Received February 14, 1944.)

104. S. P. Avann: *Relations between join-irreducibles and meet-irreducibles in a modular lattice.* I.

Let τ and τ' be the respective orders of the partially ordered sets P of join-irreducible elements and P' of meet-irreducible elements in a finite modular lattice L. It is proved that $\tau=\tau'$, if certain restrictions are made upon the quotient sublattice structure of L. A reasonable conjecture is that $\tau=\tau'$ without restriction, since P and P' are even isomorphic when L is distributive. In a complete set Q_i of projective prime quotients of L the number of minimal quotients, characterized by join-irreducible numerators, is equal to the number of maximal quotients, characterized by meet-irreducible denominators, if and only if $\tau_i=\tau_i'$ for the simple homomorphic image L_i of L corresponding to Q_i. (Received March 29, 1944.)

105. Reinhold Baer: *Groups without proper isomorphic quotient groups.*

This paper contains a discussion of groups G with the following property: If N is a normal subgroup of G, and G and G/N are isomorphic groups, then $N=1$. (Received February 2, 1944.)

106. Garrett Birkhoff: *Subdirect products in universal algebra.*

It is proved that any abstract algebra A (in a very general sense) is a subdirect union of subdirectly irreducible algebras. This theorem contains as special cases various representation theorems due to Kothe, Stone, McCoy and Montgomery, and the author. (Received March 6, 1944.)

107. R. H. Bruck: *Quasigroups with the inverse property*. I. Preliminary report.

If M is an arbitrary multiplicative system with a single-valued multiplication, the left associator of M is the set of elements a such that $a \cdot xy = ax \cdot y$ for all x, y of M. The middle and right associators are similarly defined. For isotopic systems with units the corresponding associators are isomorphic. The three associators coincide for an I. P. loop. If Q is a quasigroup and U, V are one-to-one mappings of Q upon itself, let $Q(U, V)$ be the principal isotope defined by $xoy = x^U \cdot y^V$. An intensive study is made of the following questions: (I) If Q is either an I. P. loop or a totally symmetric quasigroup, when is $Q(U, V)$ an I. P. loop? (II) If Q is a T. S. quasigroup, when is $Q(U, V)$ a T. S. quasigroup? (III) If Q is an I. P. loop, when is $Q(U, V)$ an I. P. quasigroup? The theory of Moufang loops has intimate connections with (I), and the theory of autotopisms of a loop, with (III). (Received February 28, 1944.)

108. R. H. Bruck: *Simple quasigroups*.

This note is mainly concerned with a proof of the following conjecture, due to A. A. Albert: There exist simple loops of every finite order except order four. The proof is by graphical construction of so-called hyperabelian loops of every composite order different from 4. Other constructions are also considered. A concluding section contains the proof of a theorem on arbitrary quasigroups which might be stated thus: Let Q be a quasigroup homomorphic to a quasigroup R, and let Q_0 be a loop isotopic to Q. Then there exists a loop R_0, isotopic to R, such that Q_0 is homomorphic to R_0. (Received February 28, 1944.)

109. W. B. Fite: *The degree of a linear homogeneous group*.

The problem considered in this paper is that of the relation between the degree of a linear homogeneous group and the abstract properties of the group. In the first place the theorem of Frobenius states that the degree of an irreducible group is a divisor of the order of the group; then the generalization of this by Schur states that the order of the group is divisable by the product of the degree and the order of the central. Later the author called attention to the importance of the commutators of the group in this connection by showing that the degree of an irreducible group is not less than the order of any invariant commutator. The present paper starts from this point and shows that the order of certain non-invariant commutators also has a bearing on the degree of the group whether it be reducible or irreducible. Moreover for groups of a certain category the degree is a multiple of the order of the whole commutator subgroup. (Received February 10, 1944.)

110. Alfred L. Foster and B. A. Bernstein: *A dual-symmetric definition of field*.

In another paper the groundwork was laid for a dual-symmetric treatment of commutative rings with unit—a symmetric treatment exhibiting a ring-duality theory which subsumes the duality theory of Boolean algebra. In the present paper this dual-symmetric method is specialized to the case of fields, yielding a definition of field in terms of two groups which play perfectly symmetrical roles. (Received March 24, 1944.)

111. R. D. James: *Power series expansions for doubly periodic functions of the second kind.* Preliminary report.

It is well known that results on the representation of integers as sums of squares can be obtained by comparing coefficients in the expansions in power series and trigonometric series of the Jacobian elliptic functions (E. T. Bell, J. Reine Angew. Math. vol. 163 (1930) pp. 65–70). Trigonometric series expansions for the doubly periodic functions of the second kind $\vartheta_1'\vartheta_a(x+y)/\vartheta_b(x)\vartheta_c(y)$ are also well known (E. T. Bell, Trans. Amer. Math. Soc. vol. 22 (1921) pp. 198–219) but the corresponding double power series expansions do not seem to have been considered. These expansions are obtained in the present paper. The coefficients turn out to be polynomials in the usual ϑ constants ϑ_2, ϑ_3, and an additional one $\vartheta_0''/\vartheta_0$. In a later paper the arithmetical applications will be considered. (Received March 27, 1944.)

112. S. A. Jennings: *A note on chain conditions in nilpotent rings and groups.*

Let R be a group and let Ω be any set of operators of R which contains all inner automorphisms of R. We say that R is *Ω-nilpotent* if there exists in R a strictly decreasing chain of Ω-subgroups: $R=R_1\supset R_2\supset \cdots \supset R_M\supset R_{M+1}=1$, such that, if $a_i\in R_i$, and $\lambda\in\Omega$, then $a_i^{-r}(\lambda a_i)\in R_{i+1}$, $i=1, \cdots, M$, where in general r is any integer, but if λ is an inner automorphism r equals one. For Ω-nilpotent groups we prove that the maximal or minimal condition for Ω-subgroups implies the corresponding condition for all subgroups. Nilpotent groups, and nilpotent rings and algebras, can be considered as special types of Ω-nilpotent groups, and it follows that in any nilpotent group the maximal or minimal condition for normal subgroups implies the corresponding condition for all subgroups, while in any nilpotent ring or algebra the maximal or minimal condition for two-sided ideals implies the corresponding condition for modules. (Received March 24, 1944.)

113. G. K. Kalisch: *Generalized Hilbert spaces over fields with a non-Archimedean valuation.*

Let F denote an algebraically closed field complete with respect to a non-Archimedean valuation. Consider a complete normed vector space S over F which satisfies the following conditions: I. $|a|=0$ if and only if $a=0$, $|fa|=|f|\,|a|$, $|a+b|\leqq\max(|a|, |b|)$ for all $f\in F$, $a\in S$, $b\in S$. II. There is defined in S a symmetric bilinear inner product (a, b) for all $a\in S$, $b\in S$, such that (1) $|(a, b)|\leqq|a|\,|b|$ for all $a\in S$, $b\in S$; (2) if $T\subset S$ is a finite set then for every $a\in S$ linearly independent of T and orthogonal to T there exists $x\in S$ orthogonal to T such that $|(a, x)|=|a|\,|x|$, $|(x, x)|=|x|^2$; (3) if $R\subset S$ is a countably infinite set such that $(r_i, r_j)=S_{ij}$, $|r_i|=1$, then $\lim\,(a, r_i)=0$ for all $a\in S$. It is proved that S is isomorphic with a space of countable sequences $f=(f_i)$ of elements $f_i\in F$ such that $\lim f_i=0$, $|f|=\max(|f_i|)$, $(f, g)=\sum f_ig_i$. Closed manifolds, orthogonal manifolds, and projections are defined and it is proved that S is the direct sum of any closed manifold and its orthogonal complement. The usual theorems regarding projections and closed manifolds are found to be true. These results lead to investigations to be pursued in subsequent papers. (Received April 1, 1944.)

114. Knox Millsaps: *On powers of third and fourth order matrices.*

Explicit formulae for arbitrary powers of square matrices of the third and fourth orders are obtained by using Fantappie's definition for an analytic function of a matrix. The formulae are somewhat complicated and contain Waring functions of the first and second kinds. As a result of the aforementioned complications some approximate expressions are given for large powers. (Received April 1, 1944.)

115. Albert Newhouse: *On finite extending groups.*

In this paper it is shown that the elements of a finite extending group G of an algebra A of order n over a field F have representations in the total matrix algebra of degree n over F which are matrices whose minimum functions of degree less than or equal to n are divisors of some x^m-1. Furthermore it is shown that there exist extending groups for any algebra of order $n>2$ which cannot be regarded as permutation groups for any basis of the algebra (a problem proposed by A. A. Albert, Ann. of Math. vol. 43 (1942) pp. 685–723). (Received February 17, 1944.)

116. M. F. Smiley: *An application of lattice theory to quasigroups.*

It is shown that O. Ore's generalized Jordan-Hölder theorem (*Chains in partially ordered sets*, Bull. Amer. Math. Soc. vol. 49 (1943) pp. 558–566) applies in the theory of loops (A. A. Albert, *Quasigroups.* I, Trans. Amer. Math. Soc. vol. 54 (1943) pp. 507–519) to yield the Jordan-Hölder theorems recently proved by A. A. Albert (*Quasigroups.* II, abstract 50-1-1). The proof is based on the fact that the normal divisors of A. A. Albert are the subloops H of a loop G which commute and associate with the elements of G in the sense that $xH=Hx$, $(xy)H=x(yH)$, $(xH)y=x(Hy)$, $(Hx)y=H(xy)$ for every x, $y\in G$. (Received April 1, 1944.)

117. R. M. Thrall: *On modular representations induced by ordinary representations.*

Let G be a finite group, and R any representation of G by matrices over a field of characteristic zero. After suitable number theoretic preparation one can replace each matrix in R by its residue class, a matrix in a field of prime characteristic. The set, S, of matrices thus obtained is called an *induced representation* of G. Not all modular representations can be induced. In the present note it is proved if any induced representation is split into constituents, each corresponding to a single minimal non-nilpotent two-sided ideal of the modular group ring, Γ_p, of G, then each such constituent is an induced representation. The main step in the proof is a lemma stating that any idempotent in the center of Γ_p is "induced" by an idempotent in the center of the group ring of G formed relative to a suitable p-adic closed field. (Received March 27, 1944.)

118. Bernard Vinograde: *On properties of completely primary rings.*

A cleft ring is defined as the sum of a semisimple ring and a radical. Let L, T, and R stand for composition series of left, two-sided, and right ideals respectively. Let $C=K+N$ be a completely primary (or primitive) cleft ring with minimum condition on left ideals (ML). Then $C=\sum_i^l Ku_i$ and has a matric representation of finite degree with coefficients from K. If K has only a finite number n of isomorphisms into a proper subset of itself and if C has an LT series, then MR is implied. If $n=1$ then C has an LR series and $C=\sum Ku_i=\sum u_iK$ with $u_ik=k^{S_i}u_i$ where S_i is an automorphism. If S_i is inner then C is the K envelope of an algebra over the center of K. When C is an algebra its L and R series have equal length, so an LT series implies an LR series. More generally, for any cleft C with ML and MR, an LT series implies an LR series.

Uncleft rings are extensions of cleft rings, and the study of extensions naturally starts with inseparable fields. (Received April 1, 1944.)

119. John Williamson: *Hadamard's determinant theorem and the sum of four squares.*

A square matrix H of order n is called an Hadamard or an H-matrix (Jacques Hadamard, *Résolution d'une question relative aux déterminants*, Bull. Sci. Math. (2) vol. 17 (1893) Part I, pp. 240–246) if each element of H has the value ± 1 and if $|H|$ has the maximum possible value $n^{n/2}$. In the first part by an adaptation of methods used by R. E. A. C. Paley, *On orthogonal matrices*, Journal of Mathematics and Physics, Massachusetts Institute of Technology, vol. 12 (1933) pp. 311–320, it is shown that: (1) if there exists an H-matrix of order $m>1$, there exists an H-matrix of order $m(p^h+1)$ where p is an odd prime, and (2) there exists an H-matrix of order $N(N-1)$ where $N=2^t k_1 k_2, \cdots, k_r$ and $k_s=p_s^{h_s}+1\equiv 0$ mod 4, p_s an odd prime. In the second part it is shown that an H-matrix of order $4n$ exists if there exist four polynomials $A_i(x)=\sum_{j=0}^{n-1} a_{ij}x^j$, $i=1, 2, 3, 4$, satisfying the following conditions: $a_{ij}=a_{i,n-j}=\pm 1$, $\sum_{i=1}^{4}(A_i(\omega))^2=4n$ for every nth root ω of unity. Such polynomials $A_i(x)$ are determined for specific small values of n and in particular for $n=43$ thus showing the existence of an H-matrix of order 172, a result not previously known. (Received March 22, 1944.)

ANALYSIS

120. R. P. Agnew: *Summability of subsequences.*

If A is a regular (real or complex) matrix method of summability and x_n is a bounded complex sequence, then there exists a subsequence y_n of x_n such that the set L_Y of limit points of the transform Y_n of y_n includes the set L_x of limit points of the sequence x_n. (Received February 2, 1944.)

121. E. F. Beckenbach and Maxwell Reade: *Further results on mean-values and harmonic polynomials.*

In this paper the authors study the relation between the "vertex averages" used by Walsh (J. L. Walsh, Bull. Amer. Math. Soc. vol. 42 (1936) pp. 923–930) and the "peripheral" and "areal averages" used by Beckenbach and Reade (E. F. Beckenbach and Maxwell Reade, Trans. Amer. Math. Soc. vol. 53 (1943) pp. 230–238). From the relation noted it follows that some of the results of Walsh are equivalent to those obtained by Beckenbach and Reade, and moreover, by following the methods outlined by the latter authors, it is possible to extend Walsh's results to more general "vertex averages." (Received March 27, 1944.)

122. E. F. Beckenbach and Maxwell Reade: *Regular solids and harmonic polynomials.*

Suppose D is a domain containing a regular solid V_0 and ϕ is a class of functions $f(x, y, z)$ defined and continuous on D. It is assumed that if V is similar and parallel to V_0 then the value of $F(x, y, z)$ at the center of V is the mean of values of $f(x, y, z)$ at the vertices. The class ϕ is shown to consist of certain harmonic polynomials. For the five regular solids these classes are given in terms of three spherical harmonics and their partial derivatives. The solution of the problem, suggested by J. L. Walsh (Bull. Amer. Math. Soc. vol. 42 (1936) pp. 923–930), of determining the class of functions satisfying the above mean-value property for all regular solids in D similar but not necessarily parallel to a given one, follows from the preceding results. (Received March 20, 1944.)

123. Stefan Bergman: *Certain classes of analytic functions of two real variables and their properties.*

The author derives conditions in order that the operator $T(f)=K(z, \bar{z})f(z)+\int_{c^1}E(z, \bar{z}, t)f[n(z, t)]dt$ transforms analytic functions into complex solutions, u, of $L(u)=u_{z\bar{z}}+au_z+bu_{\bar{z}}+cu=0$, $z=x+iy$, $\bar{z}=x-iy$, x, y real. Here K, E, n are some fixed functions, c^1 is a fixed curve in the complex t plane, and f ranges over the totality of analytic functions of a complex variable which are regular at the origin. Choosing $K=0$, $n=z(1-t^2)/2$, $c^1=[-1\leqq t\leqq 1]$ he shows that to every L the function $E(z, \bar{z}, t)$ can be determined in such a way that $E(z, 0, t)=(1-t^2)^{-1/2}$ and therefore $u(z, 0)=\int_{-1}^{1}f(z(1-t^2)/2)dt/(1-t^2)^{1/2}$. The above choice of K, E, n, c^1 defines a particular operator, P. The author shows that to every real solution U, of $L(U)=0$, there exists a function f such that $U=\text{Re}\,[P(f)]$. The singularities and the value distribution of the complex solutions, $u=P(f)$, of $L(u)=0$ are studied. The sum of the indices of points where $u(z, \bar{z})=b$ ($\bar{z}$ conjugate to z) and which are situated in the circle $x^2+y^2\leqq r^2$ is denoted by $n[r, (u(z, \bar{z})-b)^{-1}]$. Let $u(z, \bar{z})=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}A_{mn}z^m\bar{z}^n$ be an entire function. An upper bound for the growth of $n[r, (u(z, \bar{z})-b)^{-1}]$ considered as function of r, which is independent of b, is given in terms of the A_{n0}, $n=0, 1, 2, \cdots$, for a large class of differential equations, L. (Received March 27, 1944.)

124. Stefan Bergman: *The determination of some properties of a function satisfying a partial differential equation from its series development.*

Let $u=\sum D_{mn}z^m\bar{z}^n$, $D_{nm}=\bar{D}_{mn}$, $z=x+iy$, $\bar{z}=x-iy$, be a solution of the differential equation $L(U)=U_{z\bar{z}}+aU_z+\bar{a}U_{\bar{z}}+cU=0$, c real, where a and c are entire functions of x and y. For certain types of equations $L(U)=0$ the author determines the upper bounds for the growth of $|U|$ in terms of the subsequence $\{D_{m0}\}$, $m=0, 1, 2, \cdots$. Let $Q^{(n)}(z)$, $n=1, 2, \cdots$, be a set of analytic functions of a complex variable z. The point $2a$ is said to be a branch point of the type $\{Q^{(n)}\}$ of an analytic function $f(z)$ if, in the neighborhood of the point $2a$, $f(z)=\sum_{n=0}^{\infty}a_nz^n$ can be represented in the form $f(z)=\pi^{1/2}\sum_{n=0}^{\infty}2^{-2n}\Gamma(2n+1)[\Gamma(n+1)]^{-1}Q^{(n)}(z)I^{1/2+n}[z^{-1/2}(a-z/2)]^{-1}+g(z)$ where $g(z)$ is regular at $z=2a$ and I^k denotes the integral of the kth order. Necessary and sufficient conditions in terms of the a_n are given in order that f has one and only one singularity on the circle of convergence, which singularity is a branch point of the type $\{Q^{(n)}\}$. Generalization of this result to the case where $u(z, \bar{z})$ are certain complex solutions of $L(u)=0$ are given. (Received March 9, 1944.)

125. B. H. Bissinger and Fritz Herzog: *An extension of some previous results on generalized continued fractions.*

In a previous paper (abstract 49-11-253) the authors have extended certain measure theoretical results concerning simple continued fractions to the f-expansion $x=f(a_1+f(a_2+\cdots$, where f belongs to a certain class F_1 of functions. In this paper the following results are obtained: when $f\in F_1$ the set of all x, $0<x<1$, for which $a_n\neq k_n$ for all n (the k_n being a given sequence of positive integers) is of measure zero if and only if $\sum f'(k_n)$ diverges; for almost all x, $0<x<1$, the a_n assume every positive integer infinitely often; moreover, every finite sequence of positive integers occurs in successive places infinitely often in the f-expansion of almost all x, $0<x<1$. In the case $f(t)=1/t$ the above results represent corresponding statements about simple continued fractions. (Received April 1, 1944.)

126. Augusto Bobonis: *A sufficiency theorem for differential systems.*

The purpose of the present paper is to establish necessary and sufficient conditions for linear-differential systems and boundary conditions of the form $y'=(A+\lambda B)y$, $M(\lambda)y(a)+N(\lambda)y(b)=0$ to have an infinite number of characteristic values. These are obtained on the assumptions that the matrix $\|B\|$ has constant rank and the parameter λ enters linearly in both differential equations and boundary conditions. By defining a problem of Lagrange, and the use of the extremizing properties of the characteristic values, Reid (Trans. Amer. Math. Soc. vol. 44 (1938) pp. 508–521) established necessary and sufficient conditions for a system, not involving the linear parameter in the boundary conditions, to have an infinity of characteristic values. These conditions are obtained here by defining a Bolza problem and use of the extremizing properties of the characteristic values of the accessory boundary value problem arising from this problem of Bolza. (Received March 7, 1944.)

127. H. D. Brunk: *Dirichlet series meromorphic in a half-plane.* Preliminary report.

Applications are made of theorems given in another paper in obtaining information concerning the distribution of singular points (in particular, poles) of functions, given by Dirichlet series, which satisfy certain conditions of boundedness type in certain half-planes. For example, one theorem states that if the function $f(s)$ whose Dirichlet series expansion is $\sum a_n e^{-\lambda_n s}$ satisfies a certain condition of a boundedness type in an open half-plane containing its half-plane of absolute convergence, if the sequence $\{\lambda_n\}$ satisfies certain "density" conditions, and if the coefficients $\{a_n\}$ satisfy an essential condition, and if further $f(s)$ has as its only singularities in an open half-plane containing its half-plane of absolute convergence poles of finite maximum order at certain points $s_0+2\pi ki$ (k taking on only integral values) on its axis of absolute convergence, $f(s)$ having at least one pole on this axis, then $f(s)$ is essentially a linear combination of Taylor-Dirichlet series (series of the form $\sum b_n e^{-ns}$). (Received March 1, 1944.)

128. H. D. Brunk: *Theorems of composition for Dirichlet series.* Preliminary report.

Theorems generalizing for Dirichlet series Hadamard's familiar multiplication theorem for Taylor series have been given by Mandelbrojt (Acta Math. vol. 55 (1930) pp. 1–32). Let $f(s)$, $\phi(s)$, and $H(s)=H(f, \phi)$ represent the (uniform) functions whose Dirichlet series expansions are, respectively, $\sum a_n e^{-\lambda_n s}$, $\sum b_n e^{-\lambda_n s}$, and $\sum a_n b_n e^{-\lambda_n s}$. One theorem of Mandelbrojt imposes certain types of boundedness conditions on $f(s)$ and $\phi(s)$, satisfied in certain half-planes, with neighborhoods of their singular points extracted. The result gives a region in which $H(s)$ is holomorphic. The author of the present paper uses Mandelbrojt's method in generalizing this theorem, to give a theorem in which no such conditions are imposed on the functions $f(s)$ and $\phi(s)$. A region in which $f(s)$ is holomorphic is obtained, defined in terms of regions in which $f(s)$ and $\phi(s)$ are bounded. This theorem is then used to give a theorem closely resembling that of Mandelbrojt, and is also used with a theorem of Schottky to define regions in which $H(s)$ is holomorphic in terms of regions where the functions $f(s)$ and $\phi(s)$ fail to take on two values. (Received February 25, 1944.)

129. H. D. Brunk: *Transformations of Dirichlet series*. Preliminary report.

Let $f(s)$ be a function having a Dirichlet series expansion $\sum a_n e^{-\lambda_n s}$, and $F(s)$ the function whose Dirichlet series expansion is $\sum \psi(a_n) e^{-\lambda_n s}$, where $\psi(z)$ is an entire function, $\psi(z) = \sum c_k z^k$. If $f(s)$ has a nonpositive axis of absolute convergence and satisfies a condition of a boundedness type in a certain half-plane, then, under certain auxiliary conditions on the coefficients $\{a_n\}$, the sequence $\{\lambda_n\}$, and the magnitude of the coefficients $\{c_k\}$, the possible singular points of $F(s)$ are given very simply in terms of those of $f(s)$. (Received February 24, 1944.)

130. R. H. Cameron and W. T. Martin: *Transformations of Wiener integrals under a general class of linear transformations.*

In this paper the authors study the behavior of Wiener integrals under transformations of the form $y(t) = T[x](t) = x(t) + \int_0^1 K(t, s)x(s)ds + x_0(t)$. The function $x(t)$ ranges over a Wiener measurable set S in the space C of all continuous functions $x(t)$ on $0 \leqq t \leqq 1$ that vanish at the origin. The authors obtain the formula for changing variables from y to x in the integral $\int_{TS}^{W} F(y) d_W y$ for any functional $F[y]$ which is Wiener integrable over the transform TS of S by T. Finally, the authors show by evaluating $\int_C^W \exp \{-\lambda^2 \int_0^1 x(s)^2 ds\} d_W x = [\operatorname{sech} \lambda]^{1/2}$ that this transformation theory may be used to calculate new integrals. (Received March 30, 1944.)

131. M. M. Day: *Cluster points of subsequences.*

Buck and Pollard (Bull. Amer. Math. Soc. vol. 49 (1943) pp. 924–931) showed that a number x is a cluster point of a sequence of real numbers $\{x_n\}$ if and only if x is a cluster point of almost every subsequence of $\{x_n\}$. If the index system of integers is replaced by a countable partially ordered system S with no terminal elements, the class C of cofinal subsets of S is a natural substitute for the class of infinite subsets of integers used in defining subsequences. It is shown that a measure can be defined in a natural way in C and that for every function g from S into a neighborhood space satisfying the first countability axiom, x is a cluster point of g if and only if x is a cluster point of $g|E$ for almost-every E in C (where $g|E$ is the function equal to g defined only over E. (Received March 29, 1944.)

132. M. M. Day: *On symmetric closed convex curves.*

An elementary proof not using fixed point theorems is given for the following theorem: If C is a closed convex plane curve with a center of symmetry, there is a parallelogram P circumscribed about C such that the center of P is the center of C and moreover the midpoint of each side of P is on the curve C. (Received March 29, 1944.)

133. M. M. Day: *Inner-product spaces.*

Two characterizations are given of inner-product spaces, that is normed spaces which are linear subsets of generalized Hilbert spaces. The first of these is a weakening of the condition of von Neumann-Jordon (Ann. of Math. vol. 36 (1935) pp. 719–723); it holds for either real- or complex-linear spaces: *The normed linear space B is an inner-product space if and only if* $\|b_1 + b_2\|^2 + \|b_1 - b_2\|^2 = 4$ *whenever* $\|b_1\| = \|b_2\| = 1$. The second condition involves both elements b_i of B and β_i of B^* and is suggested by an elementary inequality that holds in all normed spaces: *The normed real-linear space B is an* *inner product space if and only if* $\|\beta_1+\beta_2\|\cdot\|b_1+b_2\|+\|\beta_1-\beta_2\|\cdot\|b_1-b_2\|=4$ *whenever* $\|\beta_i\|=\|b_i\|=\beta_i(b_i)=1$. Whether this holds for complex linear spaces is unknown. Both sufficiency proofs begin with the real two-dimensional case; they use the theorem of the preceding abstract (*On symmetric closed convex curves*) to select an ellipse and then show that this ellipse must be the unit sphere. (Received March 29, 1944.)

134. Herbert Federer: *The Gauss-Green theorem.*

A new definition of the *exterior normal* of a subset of n-space at a point of n-space is given in terms of the Lebesgue densities of the set and its complement on opposite sides of a plane through the point. This concept is used together with the $(n-1)$-dimensional *surface measure* over n-space previously introduced by the author (see *Surface Area*, I and II, abstracts 49-11-259, 260. These papers will appear in the Transactions) to prove the Gauss-Green formula under weak assumptions. In particular, the formula applies to every bounded open set in the plane, whose boundary has finite Carathéodory linear measure. The present results are less complete in the case of dimension $n>2$. As an application of the general theory it is proved that the integral "around" a bounded open set in the plane of a function, which is analytic inside the open set and continuous on its boundary, equals zero, provided the boundary has finite Carathéodory linear measure. (Received March 22, 1944.)

135. J. G. Herriot: *Nörlund summability of multiple Fourier series.*

In a previous paper (Trans. Amer. Math. Soc. vol. 52 (1942) pp. 72–94) the author showed that certain regular Nörlund methods of summability including Cesàro (C, α), $\alpha\geqq 1$ but not $\alpha<1$, applied to the square partial sums s_{nn} of the double Fourier series of an integrable periodic function of two variables, possess the localization property, and that certain other regular Nörlund methods, including (C, α), $\alpha>0$, applied to the s_{nn}, are effective almost everywhere. The same results are now obtained replacing the s_{nn} by the triangular partial sums T_n^0 formed by adding terms the sum of whose indices does not exceed n. For regular Nörlund methods with $p_n\geqq 0$ (hence including (C, α), $\alpha\geqq 0$), these localization results do not admit of extension to Fourier series of dimension exceeding two. The proof is based on a theorem of Banach and Steinhaus on linear functionals. In the author's previous paper it was shown that certain restricted double Nörlund methods, including restricted (C, α, β), $\alpha, \beta\geqq 1$ but not α or $\beta<1$, applied to the double Fourier series, possess the localization property. It is now shown that for Fourier series of dimension exceeding two many restricted multiple Nörlund methods, including restricted $(C, \alpha, \beta, \cdots, \kappa)$, $\alpha, \beta, \cdots, \kappa\geqq 0$, do not possess the localization property. (Received March 15, 1944.)

136. J. D. Hill: *Cesàro summability of sequences of* 0*'s and* 1*'s.*

Let $0.a_1a_2a_3\cdots$ be the dyadic representation of an irrational number x between 0 and 1. Borel showed that $\lim_n n^{-1}\sum_1^n a_k=1/2$ for almost all x. This result asserts that *almost all* sequences of 0's and 1's are summable $(C, 1)$ to the value 1/2. In this paper it is shown that *almost all sequences of* 0*'s and* 1*'s are summable* (C, r) *to the value* 1/2 *for each* $r>1/2$. The proof depends on the general inclusiveness theorem of Mazur, a well known lemma from the general theory of series, and certain properties of the Rademacher functions. (Received April 1, 1944.)

137. J. D. Hill: *Nörlund methods of summability that include the Cesàro methods of all positive orders.*

H. L. Garabedian (Bull. Amer. Math. Soc. vol. 48 (1942) pp. 124–127) has recently given examples of Hausdorff methods of summability which include the Cesàro methods of all positive orders. In this paper the corresponding problem for Nörlund methods $(N;\ p_k)$ is considered. By employing the general inclusiveness theorem of Mazur (Math. Zeit. vol. 28 (1928) pp. 599–611) a necessary and sufficient condition is given for the relation $(N;\ p_k)\supset(C,\ r)$. Finally, it is shown that $(N;\ \cosh k^{1/2})\supset(C, r)$ for every $r>0$. This article will appear shortly in the American Journal of Mathematics. (Received April 1, 1944.)

138. H. K. Hughes: *The asymptotic expansions of entire functions defined by Maclaurin series.*

Asymptotic expansions of the entire function $f(z)=\sum_{n=0}^{\infty}g(n)z^n$ (radius of convergence $=\infty$), valid for large values of $|z|$, are obtained under the asumption that the function $g(w)$, where $w=x+iy$, can be represented asymptotically in the form $g(w)\sim\sigma^w\sum_{n=0}^{\infty}c_n[\Gamma^{-1}(\alpha w+t+n)]$, where σ and α are positive, while the c's and t are any real or complex constants. Let $Z_\mu=(\sigma z e^{2\pi i\mu})^{(1/\alpha)}$, $Y_\mu=\bar{\alpha}' Z_\mu^{1-t}e^{Z_\mu}\sum_{n=0}^{\infty}c_n(Z_\mu)^{-n}$, and $-\pi<\arg z\leqq\pi$. Then it is shown that $f(z)\sim\sum(Y_\mu)-\sum_{n=1}^{\infty}g(-n)z^{-n}$, the first summation being taken over those integral values of μ which satisfy $|\arg z+2\pi\mu|\leqq\pi\alpha/2$. The proof is based on recent theorems of W. B. Ford (*The asymptotic developments of functions defined by MacLaurin series*, University of Michigan Studies, Scientific series, vol. 11, 1936, p. 30) and C. V. Newsom (Amer. J. Math. vol. 60 (1938) pp. 561–572), wherein the function $f(z)$ is expressed in a form containing a definite integral. It consists largely of finding the asymptotic expansions of this integral. (Received February 21, 1944.)

139. Dunham Jackson: *Orthonormal polynomials on loci of the second degree.*

In a paper presented to the Society some years ago (abstract 45-5-192) but not published, the author discussed the boundedness of orthonormal polynomials in two variables on certain elliptic and hyperbolic loci. The treatment was based on theorems of Peebles concerning the preservation of the property of boundedness of orthonormal polynomials or trigonometric sums in a single variable on transition from one weight function to another. The approach is simplified now by the use of Korous's theorem on the bounds of orthonormal polynomials in place of the theorems of Peebles, and the scope of the results is extended in various ways, by the consideration of a greater variety of domains of orthogonality and especially by the use of affine linear transformations in the plane of the variables. (Received February 14, 1944.)

140. Herman Kober: *Approximation by integral functions in the complex domain.*

Let $F(z)$ be analytic and bounded in a sector of opening θ, $0<\theta<2\pi$, and uniformly continuous on the boundary. Then there exist integral functions $g_\alpha(z)$ such that $|g_\alpha(z)|\leqq A\exp[(\alpha|z|)^\rho]$, $\rho=\pi/(2\pi-\theta)$, and $\lim_{\alpha\to\infty}g_\alpha(z)=F(z)$ uniformly in the sector. Here ρ is the least possible order of the approximations and their types can stay bounded if and only if $F(z)$ itself is an integral function of order ρ and finite type. More general domains than sectors can be handled. Sharper results are possible when the sector is a half-plane and here ordinary boundedness may be replaced by boundedness in the mean of order p with corresponding interpretation of convergence. Extensions to unbounded functions are possible but offer greater difficulties. The author also develops the theory of functions holomorphic and bounded in the mean of order p in a strip and proves that such a function may be approximated uniformly in mean of order p by integral functions of order one. (Received March 17, 1944.)

141. L. H. Loomis: *Abstract congruence and the uniqueness of Haar measure.*

The author proves simultaneously—and hence without use of the axiom of choice—the existence and uniqueness of Haar measure in any locally compact metric space satisfying the following weak combinatorial congruence axiom: if S_1 and S_2 are compact spheres with the same radius and if S_1 can be covered by N open spheres of radius x then so can S_2. A Haar measure is a Lebesgue measure in which "congruent sets" (compact spheres with the same radius, in this case) have equal measures. For intuitive simplicity the theorem is stated and proved in the metric case, but the proof is formulated so as to hold without modification in a suitably restricted uniform structure. The theory of Haar measure in locally compact groups is an immediate application. (Received February 8, 1944.)

142. A. N. Lowan and H. E. Salzer: *Coefficients for complex interpolation within a square grid.*

When interpolation by Taylor series up to terms in $(\Delta z)^4$ is adequate for the desired accuracy, the value of $f(z_0+\Delta z)$ may be obtained with the aid of a complex cubic approximation polynomial derived from the general Lagrange-Hermite interpolation formula. If z_{-1}, z_0, z_1, and z_2 denote points in the z-plane at the corners of a square of length h whose sides are parallel to the axes, then $f(z_0+\Delta z)=f(z_0+Ph)$ $=L_{-1}(P)f(z_{-1})+L_0(P)f(z_0)+L_1(P)f(z_1)+L_2(P)f(z_2)$, where the L's are cubic polynomials in $P=p+iq$. The present table facilitates interpolation in the complex plane, whenever $\Delta z=(p+iq)h$ where p and q range from 0 to 1 at intervals of 0.1. A similar table has been prepared for the case when a complex quadratic polynomial suffices to approximate the function and is based on the three points z_{-1}, z_0, and z_1. It should be noted that for purposes of systematic and extensive subtabulation, a method based on the use of real Lagrangian interpolation coefficients is more economical than the method here discussed. (Received March 10, 1944.)

143. G. W. Mackey: *On infinite dimensional linear spaces.*

A general description of the contents of this paper and statements of some of the theorems which it contains will be found in Proc. Nat. Acad. Sci. U.S.A. vol. 29 (1943) pp. 216–221. (Received March 28, 1944.)

144. A. M. Peiser: *On the average sum of the real roots of a random algebraic equation.*

Let $F=\sum_{j=0}^{n} x_j t^j=0$ be an algebraic equation with real coefficients, and let $t_1, \cdots, t_k$ denote the real roots of $F=0$ in $a\leqq t\leqq b$. It is shown that if $\phi(t)$ is continuous, $a\leqq t\leqq b$, then $\sum_{j=1}^{k}\phi(t_j)=\lim_{\epsilon\to 0}(1/2\epsilon)\int_a^b\phi(t)\psi_\epsilon(F)\left|\partial F/\partial t\right|dt$, where $\psi_\epsilon(x)=1$ if $|x|<\epsilon$ and $\psi_\epsilon(x)=0$ otherwise. If $x_0, \cdots, x_n$ are independent standard normally distributed random variables, and if $\phi(t)=|t|$, then the average value (mathematical expectation $=$ m.e.) of the sum of the absolute values of the roots of $F=0$ in $a\leqq t\leqq b$ is given by $S_n(a, b)=$ m.e. $\left\{\sum_{j=1}^{k}|t_j|\right\}=\lim_{\epsilon\to 0}(1/2\epsilon)\int_a^b|t|$ m.e. $\left\{\psi_\epsilon(F)\left|\partial F/\partial t\right|\right\}dt$ $=(1/\pi)\int_a^b|t|A_n(t)dt$, where $A_n(t)$ is a concrete positive function of t. In particular, for $a>1$, $S_n(-a, a)\sim(2/\pi)\log n$. Kac (Bull. Amer. Math. Soc. vol. 49 (1943) pp. 314–320) has shown that the average number of real roots of $F=0$ in any interval containing 1 and -1 is asymptotically equal to $(2/\pi)\log n$. Thus, the present result makes even more apparent the fact already noticed by Kac that the real roots of random equations show a strong tendency to cluster (on the average) around 1 and -1. Finally it is shown that for $a<1$, $\lim_{n\to\infty} S_n(-a, a) = -(1/\pi)\log(1-a^2)$. (Received March 24, 1944.)

145. Maxwell Reade: *On functions of class PL.*

If $f(x, y)$ is continuous in a domain D, then $f(x, y)$ is said to be of class PL in D provided $f(x, y)\geqq 0$ and $\log f(x, y)$ is subharmonic in D. Generalized Laplacean operators are introduced for functions of class PL in order to characterize those functions in terms of the operators. Since $f(x, y)$ of class PL implies that $f(x, y)$ is subharmonic too, there exist generalized mass distributions μ_1 and μ_2 associated with $f(x, y)$ and $\log f(x, y)$ respectively; relations between μ_1 and μ_2 are given. A typical result is the following theorem. If $f(x, y)$ is continuous and of class PL in a domain D, and if μ_1 is the associated generalized mass distribution for $f(x, y)$, then $\lim_{r\to 0}\{[L(f; P_0, r)]^2 - A(f^2; P_0, r)\}/(r^2/4) = f(P_0)D_a\mu_1(P_0) - f_x^2(P_0) - f_y^2(P_0)$ for almost all points P_0 in D. Here $L(f; P_0, r)$ denotes the peripheral mean and $A(f; P_0, r)$ the areal mean of $f(x, y)$ for a circle with center at P_0 and radius r. (Received March 20, 1944.)

146. Maxwell Reade: *Two applications of generalized Laplaceans.*

Generalized Laplaceans are used to lessen some of the differentiability requirements in (a) the Kierst-Saks problem (S. Saks, Acta Univ. Szeged. vol. 5 (1930–1932) pp. 187–193) and (b) the author's extension of the isoperimetric inequality for functions having subharmonic logarithms (Maxwell Reade, Bull. Amer. Math. Soc. vol. 49 (1943) pp. 894–897). (Received March 22, 1944.)

147. L. I. Schoenfeld: *A transformation formula in the theory of partitions.*

Let $f(x) = \prod_{m=1}^{\infty}(1-x^{m^\kappa})^{-1}$ where κ is a positive integer. Then $f(x) = \sum_{m=0}^{\infty} p(m)x^m$, $p(0)=1$, $|x|<1$, so that $f(x)$ is the generating function of $p(m)$, the number of unrestricted partitions of m into κth powers. A shorter and simpler proof of Wright's Theorem 4 (Acta Math. vol. 63 (1934) pp. 143–191) is given; this theorem gives a transformation formula for $f(x)$ which exhibits its behavior in the neighborhood of its singularities at the rational points of the unit circle. The proof given here makes use of a Mellin transform technique of Rademacher (J. Reine Angew. Math. vol. 167 (1931) pp. 312–336). (Received March 16, 1944.)

148. C. F. Stephens: *Nonlinear difference equations analytic in a parameter.*

The author considers the system of difference equations (1) $y_i(x+1) = \sum_j^n b_{ij}(x)y_j(x) + f_i(y_1(x), \cdots, y_n(x); p(x); x)$, where the f_i are analytic in the $y_i(x)$ and $p(x)$, but only continuous in x in the neighborhood of $(x=\infty, y_1(x)=0, \cdots, y_n(x)=0, p(x)=0)$. The parameter $p(x)$ is periodic of period 1, f_i contain no linear terms in $y_i(x)$ and $p(x)$, and $f_i(0, \cdots, 0; 0; x)\equiv 0$. The purpose of the paper is an investigation of those solutions of (1) which are analytic in $p(x)$ and continuous in x. It is shown that such solutions exist. Past methods made the associated system of linear equations $y_i(x+1) = \sum_j^n b_{ij}(x)y_j(x)$ fundamental and cannot in general be applied to the case where $b_{ij}(x)\equiv 0$. The author arrives at generalizations in two ways. (a) The f_i are taken to be continuous in x and not analytic as is usually required. (b) A new method of treating difference equations is established and can be applied when $b_{ij}(x)=0$, and also when $p(x)$ is replaced by 0 in (1). The method makes use of analytic implicit functions and a matrix transformation, and the case where $b_{ij}(x)\neq 0$ is made to depend upon the case $b_{ij}(x)=0$. (Received February 1, 1944.)

149. Otto Szász: *On uniform convergence of trigonometric series.*

This paper contains generalizations of some theorems due to Chaundy and Jolliffe, to Hardy, and to the author. The following are some of the results. The trigonometric series $\sum_1^\infty b_n \sin nt$ is uniformly convergent if $\sum_n^{2n}|b_\nu - b_{\nu+1}| = O(n^{-1})$ and if the sequence nb_n is Abel summable to zero. The power series $\sum_1^\infty c_n z^n$ is uniformly convergent in $|z|\leqq 1$, if $\sum_n^{2n}|c_\nu - c_{\nu+1}| = O(n^{-1})$ and if $\sum c_n$ is Abel summable. The essential part of the proof concerns the point $z=1$, that is $t=0$; a device of the Tauberian type is employed. (Received March 18, 1944.)

150. F. A. Valentine: *Contractions in non-euclidean spaces.*

Let $f(x)$ be a function mapping a set S in a metric space M into a set S' in a metric space M', and suppose a contraction of the type $\|f(x_1), f(x_2)\|' \leqq \|x_1, x_2\|$ holds in S and S'. The existence of an extension of the range of definition of such a function so as to preserve a contraction depends upon M and M'. In this article the author shows *the extension exists when $M=M'$ is the n-dimensional hyperbolic* space. The proof used is applied to a metric space which includes both the hyperbolic and the spherical cases. Hence a unification of results is also obtained. (Received February 21, 1944.)

151. S. E. Warschawski: *On conformal mapping of nearly circular regions.*

Generalizing results of L. Bieberbach (Sitzungsberichte, Berliner Akademie, 1923) and of A. R. Marchenko (Bull. Acad. Sci. U.S.S.R. 1935), the author proves the following theorem: Let R be a simply connected region with the properties: (i) R contains the origin $w=0$ and its boundary lies in the ring $1\leqq |w| \leqq 1+\epsilon$, ϵ being a fixed positive number; (ii) there exists a number $\eta \geqq \epsilon$ such that any two points P_1 and P_2 of R of distance less than ϵ can be connected in R by an arc of *diameter* less than η. If $w=f(z)$ maps the circle $|z|<1$ conformally onto R ($f(0)=0, f'(0)>0$) then, for all $|z|<1$, $|f(z)-z| \leqq B\epsilon \log (1/\epsilon)+4\eta$, where B is an absolute constant. Analogous results for the derivatives of the mapping function, such as the following, are established. Let C be a simple closed curve $\rho=\rho(\phi)$, $0\leqq\phi\leqq 2\pi$ (ρ, ϕ polar coordinates), such that $1\leqq\rho(\phi)\leqq 1+\epsilon$, $|\rho'/\rho|\leqq\epsilon$, and that $|\rho'(\phi_2)/\rho(\phi_2)-\rho'(\phi_1)/\rho(\phi_1)| \leqq \epsilon|\phi_2-\phi_1|$, $0<\epsilon<1$. If $f(z)$ (normalized as above) maps $|z|<1$ onto the interior of R, then, for $|z|\leqq 1$, $(A(1+\epsilon^2)^{1/2})^{-1} \leqq |zf'(z)/f(z)| \leqq A(1+\epsilon^2)^{1/2}$ and $|f'(z)-1| \leqq 5(A\epsilon + A - 1)$, where $A=4^\epsilon e^{\epsilon^2}$. (Received April 1, 1944.)

APPLIED MATHEMATICS

152. Wilfred Kaplan and Max Dresden: *The mechanism of the condensation of gases.*

The criterion previously formulated (see abstract 49-5-158) for the condensation of a gas: namely, that condensation occurs at energy zero, when the topological structure of the energy surface changes, is further explored. It leads to a qualitative picture of the whole process of condensation. The theory has now been compared with experiment. The critical temperatures of seven inert gases are predicted from the known critical volumes, and a satisfactory agreement with observed critical temperatures is found. (Received March 23, 1944.)

153. Morris Marden: *Axisymmetric harmonic vectors.*

Let $\Phi(x, \rho)$ and $\Psi(x, \rho)$, where $\rho^2 = y^2 + z^2$, be respectively the potential and Stoke's stream function in an axisymmetric flow and let i and m be the unit vectors along the x- and ρ-axis of the meridian plane. Then with the harmonic vector $H = \Phi i - \rho^{-1}\Psi m$ may be associated an analytic function $\phi(x+i\rho)$ of a complex variable as follows (cf. S. Bergman, Bull. Amer. Math. Soc. vol. 49 (1943) pp. 163–174): $H = (2\pi)^{-1} \cdot \int_0^{2\pi} \phi(x + i\rho \cos t)(i + ie^{it}m)dt$. This formula is used in the present paper to study the relation between the two-dimensional flow with complex potential $\phi(x+i\rho)$ and the three-dimensional axisymmetric flow with potential $H(x, \rho)$. The $\phi(u)$ corresponding to certain distributions of doublet sources or vortices is found to be the associate of the $H(x, \rho)$ corresponding to a circular source or vortex filament. The $\phi(u)$ corresponding to "Karman streets" of doublet sources or vortices is thus found to be the associate of an $H(x, \rho)$ corresponding to Karman streets of circular source or vortex filaments. Finally, the energy in an axisymmetric force field is expressed operationally in terms of the energy in the associate two-dimensional field. (Received March 25, 1944.)

154. A. D. Michal: *Physical models of some curved differential-geometric metric spaces of infinite dimensions.* II. *Vibrations of elastic beams and other elastic media as studies in geodesics.* Preliminary report.

This is a continuation of a previous paper (abstract 49-11-289). In this paper a special study is made of the infinite dimensional Riemannian geometries determined by conservative vibration systems of beams and other elastic media. (Received April 1, 1944.)

155. Ida Roettinger: *Certain integral transformations, with applications to boundary value problems.*

The transformations $S\{F(x)\} = \int_0^\pi F(x) \sin (n-1/2)xdx$ and $C\{F(x)\} = \int_0^\pi F(x) \cos (n-1/2)xdx$, $n = 1, 2, 3, \cdots$, where $F(x)$ is sectionally continuous on the interval $(0, \pi)$, are studied. Four convolution theorems are proved, analogous to those given by Kniess for the ordinary sin nx and cos nx transformations. (See H. Kniess, Math. Zeit. vol. 44 (1938) pp. 266–292.) By means of one of these theorems the solution of the heat conduction problem $U_t(x, t) - K(t)U_{xx}(x, t) + A(t)U(x, t) + Q(x, t) = 0$, $U_x(0, t) = B(t)$, $U(\pi, t) = C(t)$, $U(x, 0) = F(x)$ is expressed in terms of the solution of the problem $V_t(x, t) = V_{xx}(x, t)$, $V(0, t) = 0$, $V_x(\pi, t) = 0$, $V(x, 0) = 1$. Similar work has been done by Brown. (See H. K. Brown, Journal of Applied Physics vol. 14 (1943) pp. 609–618.) Furthermore the transformations $\Lambda\{F(x)\} = \int_{-1}^1 F(x) \sin \lambda_n(1+x)dx$ and $\overline{\Lambda}\{F(x)\} = \int_{-1}^1 F(x) \sin \lambda_n(1-x)dx$, where λ_n are the positive roots of $\tan 2\lambda = -\lambda/h$, $h > 0$, are studied and applied to the above heat equation with the boundary conditions $U(-1, t) = H(t)$, $U_x(1, t) + hU(1, t) = G(t)$, $U(x, 0) = F(x)$. Three convolution theorems are proved for the Λ-transformation. (Received March 16, 1944.)

156. J. A. Shohat: *Series expansions for the periodic solution of Van der Pol's equation and its frequency for all values of the parameter.*

If the parameter μ in Van der Pol's equation $d^2u/dt^2-\mu(1-u^2)du/dt+u=0$ is small, power series expansions for the periodic solution u (unique, save for time-translation) and its frequency v can be and have been given, say, by Lidstedt's method. In the present paper the author gives (for the first time, he believes) series expansion for u and v, valid for all values of μ—large and small. Numerical computation agrees quite well with known numerical results. (Received March 13, 1944.)

GEOMETRY

157. Reinhold Baer: *The fundamental theorems of elementary geometry. An axiomatic analysis.*

It is the object of this paper to evaluate the logical interdependence of certain fundamental theorems in elementary geometry. The paper deals with the theorems asserting the copunctuality of each of the following triplets of lines: medians, altitudes, perpendicular bisectors, and bisectors of angles; and the theorem stating that the locus of the points of equal distance from two different points is a line. The framework of our discussion is provided by a general affine plane in which we introduce just as many further relations as are needed for stating the investigated theorems. (Received March 22, 1944.)

158. P. O. Bell: *A study of surfaces by means of a system of differential equations of the first order.*

The projective differential geometry of a surface in ordinary space is studied by means of tetrads of surfaces whose corresponding points x_i $(i=0, 1, 2, 3)$ are linearly independent. The general homogeneous coordinates of x_i satisfy a system of equations $\partial x_i/\partial u^\alpha=C_{h,i\alpha}x_h$, $\alpha=1, 2$, summed for $h=0, 1, 2, 3$. With the points x_i as vertices of a local reference tetrahedron an algebraic surface $a_{ij\cdots l}x^ix^j\cdots x^l=0$ is fixed as u^1, u^2 vary independently, if and only if the coefficients $a_{ij\cdots l}$ are proportional to the corresponding components of the covariant derivatives, of the aggregate of these coefficients, with respect to the connection $C_{h,i\alpha}$. Such conditions of immovability form the basis for a general theory of envelopes. Tetrads of surfaces are first investigated. The study of a surface S_0 is then undertaken by specializing the general theory. Auxiliary surfaces S_1, S_2, S_3 covariantly determined with respect to S_0 are selected so that the fundamental differential equations are as simple as possible and exhibit desirable properties of symmetry. Some differential invariants are characterized geometrically. When the asymptotic curves are parametric on one of the surfaces one of these invariants becomes the projective linear element and another becomes Fubini's element of projective arc length. (Received April 1, 1944.)

159. S. S. Chern: *Laplace transforms of a class of higher dimensional varieties in a projective space of n dimensions.*

In a projective space of n dimensions a class of r-dimensional varieties is defined, which form a natural generalization of the surfaces sustaining conjugate nets. These varieties are characterized by the property that the asymptotic net is an $(r-1)$-parameter linear system of cones whose base cones are linear spaces counted twice. (See E. Cartan, Bull. Soc. Math. France vol. 47 (1919) pp. 125–160.) This geometrical characterization is in general equivalent to the following: On the variety there exist r families of curves such that when the tangent r-plane is displaced along a curve of the family its intersection with a neighboring tangent r-plane is the tangent $(r-1)$-plane formed by the tangents of the other $r-1$ curves. Basing on this property one can define on each of these r tangents AA_i, $i=1, \cdots, r$, $r-1$ points A_{ij}, $j\neq i$, having the property that when the s-plane $AA_{i_1}\cdots A_{i_s}$ is displaced along AA_j, $j\neq i_1, \cdots, i_s$, its intersection with a neighboring s-plane is the $(s-1)$-plane $A_{i_1 j}\cdots A_{i_s j}$. The points A_{ij} describe varieties of the same type and are naturally defined as the Laplace transforms of the given variety, there being altogether $r(r-1)$ transforms. Many well known properties of Laplace transforms can be generalized. (Received February 23, 1944.)

160. S. B. Jackson: *Vertices of plane curves.*

A closed curve of class C'', not a circle, has two vertices by the continuity of the curvature. The present paper seeks to characterize geometrically those curves with exactly two vertices. Let a curve be called normalized if it contains no complete circles, and let a simple closed arc of the curve which is never crossed by the curve be called a simple loop. The following facts are established for any normalized curve C with two vertices: (a) C may be divided into two simple arcs; (b) all double points are simple; (c) C contains exactly two simple loops, one containing each vertex; (d) none of the plane regions bounded by C are bounded always in the same sense except those regions bounded by the loops; (e) at any points of tangency the directed tangents coincide. For a curve which is not normalized these results are modified slightly. Two familiar theorems regarding the number of vertices on an oval are generalized to any simple closed curve. The methods employed are entirely elementary, extensive use being made of the invariance of vertices under direct circular transformations. (Received February 4, 1944.)

161. J. E. Wilkins: *The contact of a cubic surface with a ruled surface.*

It is shown that there exist ∞^1 cubic surfaces having contact of order 5 with a nondevelopable ruled surface. If there is any cubic surface having contact of order 6 with a nondevelopable ruled surface, then the surface is itself a cubic surface. In order to obtain these results, there are first derived power series expansions for a nondevelopable ruled surface to terms of the sixth degree. Similar investigations are made for developable surfaces. (Received April 1, 1944.)

STATISTICS AND PROBABILITY

162. C. W. Churchman and Benjamin Epstein: *Estimates of error in parallel experiments.*

It is common in many types of tests to have not only a random error from test to test due to a large number of unallocable causes, but it is also possible to have systematic errors present. It is because of this possibility that one tests not only samples of the unknown, but also control samples. The purpose of using control samples is two-fold—(a) to find out whether or not abnormal experimental conditions exist during the test and (b) to establish tentatively a level for the particular test under consideration. It is shown that a statistic can be found which gives the most efficient estimate of the corrections to be applied to the unknown under test for a variety of experimental conditions. It is further shown that this statistic must be a linear function of the random variables x and y where x corresponds to the values assumed by the unknown under test and y corresponds to the values assumed by the control from test to test. (Received March 23, 1944.)

163. P. R. Halmos: *Randon alms.*

Suppose that a pound of gold dust is distributed at random among a countably infinite set of beggars, so that the first beggar gets a random portion of the gold, the second beggar gets a random portion of the remainder, and so on ad infinitum. The purpose of this work is to calculate the actual and the asymptotic distributions of x_n and S_n (where x_n is the amount received by the nth beggar and $S_n = \sum_{j \leq n} x_j$) and also to study the rate of convergence of the random series $x_1 + x_2 + x_3 + \cdots$, under the assumption that the phrase "random portion," occurring an infinite number of times in the description of the stochastic process, receives the same interpretation each time. The results may be interpreted as properties of a random distribution of a unit mass on the positive integers; they may be used to explain the experimentally observed distribution of sizes of mineral grain particles; and they occur also as distributions of energy in the theory of scattering of neutrons by protons of the same mass. (Received March 24, 1944.)

164. Henry Scheffé and J. W. Tukey: *Contributions to the theory of non-parametric estimation.*

For problems of non-parametric estimation, concerning an unknown cumulative distribution function $F(x)$, three good solutions are available: (i) confidence intervals for the median of F (W. R. Thompson, K. R. Nair), (ii) tolerance limits for F (Wilks), and (iii) confidence limits for F (Wald, Wolfowitz, Kolmogoroff). Heretofore (i) and (ii) have been limited to the case where $F'(x)$ is known to be continuous. By means of a theorem of general application they are extended to the case where F need only be continuous. The only previous result for discontinuous F is that of Kolmogoroff for (iii). The appropriate modifications of (i) and (ii) extending their validity to this case are found. Some uniqueness results limiting the kind of statistics usable in such problems are obtained. Sufficiently complete tables for applying (i) and (iii) have been published, but computations for (ii) have been extremely few and laborious. A simple formula based on the z and χ^2-distributions is found which gives highly accurate approximations in ranges of practical interest. All the results for (ii) also apply to Wald's tolerance intervals for the multivariate case. (Received March 14, 1944.)

TOPOLOGY

165. E. F. Beckenbach and R. H. Bing: *On generalized convex functions.*

Let $F(x; \alpha, \beta)$ be a two-parameter family of real continuous functions defined in an interval (a, b) such that there is a unique member of the family taking on arbitrary values y_1, y_2 at arbitrary distinct points x_1, x_2 of the interval. A real function $f(x)$ is said to be a sub-$F(x; \alpha, \beta)$ function in (a, b) provided at the midpoint x_0 of each subinterval I of (a, b) we have $f(x_0) \leq F(x_0; \alpha, \beta)$ for that $F(x; \alpha, \beta)$ which coincides with $f(x)$ at the endpoints of I. The family $F(x; \alpha, \beta)$ is not necessarily topologically equivalent to the set of non-vertical line segments in the strip; hence the study of sub-$F(x; \alpha, \beta)$ functions is not topologically equivalent to the study of convex functions. It is shown among other things that if a sub-$F(x; \alpha, \beta)$ function $f(x)$ is bounded, or even measurable, in a subinterval of (a, b), then $f(x)$ is continuous. (Received March 31, 1944.)

166. Samuel Eilenberg: *Skew-invariant cohomology groups.* Preliminary report.

Let X be a topological space with a group W acting as a group of homeomorphisms on X, and let G be an abelian group with W as a group of operators. An n-dimensional cochain f on X with coefficients in G is called skew-invariant provided $f(wT)=wf(T)$ for every singular n-simplex T in X and every $w\in W$. If f is skew-invariant, then so is its coboundary δf. Using *skew-invariant* cochains throughout, the skew-invariant cohomology group $I_n(X, G)$ is defined. Let Y be an arcwise connected space, locally connected in dimension 0 and 1, and let the fundamental group $\pi_1(Y)$ of Y act as a group of operators on the coefficient group G. Let $H_n(Y, G)$ denote the nth cohomology group of Y with G as a group of local coefficients as recently defined by Steenrod (Ann. of Math. vol. 44 (1943) pp. 610–627). Let $\tilde{Y}$ be the universal covering of Y. The group $\pi_1(Y)$ acts on $\tilde{Y}$ as a group of homeomorphisms, namely, the covering homeomorphisms, and one may consider the skew-invariant group $I_n(\tilde{Y}, G)$. The main theorem asserts that $H_n(Y, G)\approx I_n(\tilde{Y}, G)$. (Received March 3, 1944.)

167. R. C. James: *Linear functionals and orthogonality in normed linear spaces.*

Of several definitions of orthogonality in normed linear spaces, perhaps the most interesting is: "$x\perp y$ if and only if $\|x+ky\|\geq\|x\|$ for all k." For any elements x and y there exist numbers a and b such that $x\perp(ax+y)$ and $(bx+y)\perp x$. Since this orthogonality is not symmetric, it is not necessary that $a=b$. The uniqueness of this number a for all x and y $(x\neq 0)$ is equivalent to the Gateaux differentiability of the norm at each nonzero point. The uniqueness of b is equivalent to strict normability. If T is a uniformly convex Banach space and L a linear subset not dense in T, there exists an element x orthogonal to L. From this it follows that for every linear functional on T with modulus 1 there exists an element x such that $|f(x)|=\|x\|$. If the norm of T is Gateaux differentiable at all nonzero points, then every linear functional is a Gateaux differential of $\|x\|$ at some point x_0. (Received March 31, 1944.)

168. N. E. Steenrod: *The classification of sphere bundles.*

The problem of Whitney of classifying k-sphere bundles over a complex B as base space is reduced to a familiar problem of topology as follows. Let S be a $(k+l+1)$-sphere $(l=\dim B)$, R its rotation group, S' a fixed k-sphere in S, R' the subgroup of R mapping S' on itself, and M_l^k the space of left cosets of R' in R. To each map g of B in M_l^k is attached the k-sphere bundle $A(g)$ over B which is the subset $\{(b, s)\}$ of $B\times S$ such that s lies in the image of S' under any rotation of $g(b)$. The function $A(g)$ induces a 1-1 correspondence between the equivalence classes of k-sphere bundles over B and the homotopy classes of maps of B in M_l^k. The homotopy groups of M_l^k are computed for dimensions 1 through 6 for all k, l. This leads to a complete solution of the classification problem for B a sphere of dimension less than or equal to 6. (Received April 1, 1944.)

169. G. S. Young: *A generalization of Moore's theorem on simple triods.*

If n is a non-negative integer, by a T_n-set is meant a continuum which is the sum of an n-cell, c, and an arc, a, such that $c \cdot a$ is a point which is an end point of a and an interior point of c. A T_1-set is a simple triod. In this note it is proved that Euclidean n-space does not contain uncountably many mutually exclusive T_{n-1}-sets. For $n=2$, this is a theorem due to Moore (Proc. Nat. Acad. Sci. U.S.A. vol. 14 (1928) pp. 85–88). (Received March 27, 1944.)

170. G. S. Young: *Concerning spaces in which every arc has two sides.*

Let S denote a connected, locally connected, complete metric space satisfying the following axiom: If AB is an arc and D is a domain containing $AB-(A+B)$, then D contains a connected domain which is separated by $AB-(A+B)$ into two connected domains, each having AB in its boundary. In this paper it is shown that if S is locally compact, it is a 2-manifold without boundary, which is closed if S is compact, and that if S is not locally compact, but satisfies certain "flatness" conditions, then it can be imbedded in a 2-manifold. A similar characterization and imbedding theorem is given for 2-manifolds with boundary. Several characterizations of the sphere are also given. (Received March 27, 1944.)

171. G. S. Young: *On continua whose links are non-intersecting.*

In this note, it is shown that if a compact metric continuum is not a simple link of itself and no two of its links intersect, then uncountably many are degenerate; also that the statement obtained by replacing the words "compact metric continuum" by "connected, locally connected, separable Moore space" is true. (Received March 27, 1944.)

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Gleason, J. M. See Daus, P. H.

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TWIN CONVERGENCE REGIONS FOR CONTINUED FRACTIONS

V. F. COWLING, WALTER LEIGHTON, AND W. J. THRON

1. **Introduction.** Some years ago it was proved [2][1] that the continued fraction

$$1+\frac{a_1}{1}+\frac{a_2}{1+}\cdots \tag{1.1}$$

converges if the complex numbers a_n satisfy the following conditions:

$$|a_{2n+1}| \leqq 1/4, \qquad |a_{2n}| \geqq 25/4 \qquad (n = 1, 2, \cdots). \tag{1.2}$$

Later one of the present authors proved [3] that (1.1) converges if

$$\begin{gathered} |1+a_2| \geqq 1+|a_1|, \\ |1+a_n+a_{n+1}| \geqq 1+|a_{n-1}a_n| \quad (n = 2, 3, \cdots). \end{gathered} \tag{1.3}$$

One of the immediate consequences of this theorem is that if

$$\begin{aligned} &|1+a_2| > 1, && |a_2| \geqq (2+m)/(1-m), \\ &|a_{2n+1}| \leqq m < 1, && |a_{2n+2}| \geqq 2+m+m|a_{2n}| \\ & && \qquad (n = 1, 2, 3, \cdots), \end{aligned} \tag{1.4}$$

then (1.1) converges.

Recently Thron [6] has shown that if

$$|a_{2n+1}| \leqq k^2 < 1, \qquad |a_{2n}| \geqq (1+k)^2 + \epsilon \qquad (\epsilon > 0), \tag{1.5}$$

the continued fraction (1.1) converges. For $k^2<1$ this result can be shown to be a "best" result except possibly for the presence of the quantity ϵ.

The present paper is concerned with establishing convergence criteria of this general type. The principal result is given in Theorem 3.1. The method to be used is the following. Denote the nth approximant of (1.1) by A_n/B_n. Conditions on the numbers a_n are determined which imply that the approximants lie in a given region V of the complex plane. A continued fraction the elements of which are functions of the complex variable z and which reduces to (1.1) for $z=1$ is then introduced. When the given conditions on the numbers a_n are satisfied the approximants of this continued fraction are shown to

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[1] Numbers in brackets refer to the Bibliography at the end of the paper.

form a normal family for z in a region D of the complex plane which includes the point $z=1$. From the known convergence of this family in a sub-region of D the (uniform) convergence of the family in every closed region interior to D may be inferred. The convergence of (1.1) is then trivial.

2. **Two lemmas.** It will be convenient to refer to an open connected set as an *open region* and to use the term *region* to mean an open region plus part or all its boundary. If S is a set of complex numbers and a is a fixed complex number, we shall designate by $a+S$ and by $S+a$ the set obtained by adding to each element of S the number a. Geometrically it is clear that $a+S$ is a translation of the set S by means of the vector a. Similarly aS will denote the set obtained from S by multiplying each element of S by the number a. Geometrically this constitutes a stretching of the set S followed by a rotation of the resulting set through an angle of arg a. By the notation $D[aS]$ is meant the point set intersection of all sets aS as a assumes all values in a given set A.

The following lemma is an extension of a result due to Scott and Wall [4].

LEMMA 2.1. *Let D_1, D_2, V_1, V_2 be any four sets of points in the complex plane which have the following properties*:

(a) $1+D_1 \subset V_1$,
(b) $1+D_2 \subset V_2$,
(c) $1+a/v \in V_1$, *if* $a \in D_1$, *and* $v \in V_2$,
(d) $1+a/v \in V_2$, *if* $a \in D_2$, *and* $v \in V_1$.

If the numbers $a_{2n-1} \in D_1$ and $a_{2n} \in D_2$ $(n=1, 2, \cdots)$, the values of the approximants of the continued fraction (1.1) *are numbers which are contained in V_1, and the values of the approximants of*

$$1+\frac{a_2}{1}+\frac{a_3}{1}+\cdots \tag{2.1}$$

belong to V_2.

The proof of this lemma is by an easy induction.

We proceed with the proof of the following result which is a consequence of Lemma 2.1.

LEMMA 2.2. *Let V_1 and V_2 be any two sets of numbers in the complex plane such that the sets*

$$E_1 = D[v_2(V_1 - 1)],$$
$$E_2 = D[v_1(V_2 - 1)]$$

have the property that

$$1+E_1\subset V_1, \qquad 1+E_2\subset V_2.$$

If the numbers $a_{2n-1}\in E_1$ *and* $a_{2n}\in E_2$ $(n=1, 2, \cdots)$, *the values of the approximants of* (1.1) *belong to* V_1 *and the values of the approximants of* (2.1) *belong to* V_2.

To prove this lemma it will be sufficient to show that conditions (c) and (d) of Lemma 2.1 are satisfied when we have identified the sets D_i of that lemma with the present sets E_i, for $i=1, 2$, respectively. We shall prove that condition (c) holds. The proof for (d) will then follow mutatis mutandis. To that end let a be any point belonging to E_1 and v_2 be any point belonging to V_2 and observe that the definition of E_1 insures the existence of a point $v'_1\in V_1$ such that $a=v_2(v_1'-1)$. It follows that $1+a/v_2=v_1'\in V_1$.

3. **The principal theorem.** We proceed with the proof of the following result.

THEOREM 3.1. *Let k be any real number greater than* 1 *and let* ϵ *and* ϵ_1 *be positive numbers less than k. If the elements* $a_n=r_ne^{i\theta_n}$ *are such that*

$$r_{2n}\geqq 2[k+\epsilon_1-\cos\theta_{2n}], \qquad r_{2n-1}\leqq k-\epsilon \qquad (n=1, 2, \cdots),$$

the continued fraction (1.1) *converges.*

The first step in the demonstration of this theorem is the proof of the following lemma.

LEMMA 3.1. *Let k be any real number greater than* 1. *If the elements* $a_n=r_ne^{i\theta_n}$ *are such that*

$$r_{2n-1}\leqq k, \qquad r_{2n}\geqq 2(k-\cos\theta_{2n}) \quad (0\leqq\theta_{2n}\leqq 2\pi;\ n=1, 2, \cdots),$$

the values of the approximants of (1.1) *belong to* V_1 *and the values of the approximants of* (2.1) *belong to* V_2.

Let the sets V_1 and V_2 of Lemma 2.2 be respectively the sets $|z-1|\leqq k$ and $|z|\geqq 1$. It will follow that (i) the set $E_1=D[v_2(V_1-1)]$ is the set $|z|\leqq k$ and that (ii) $E_2=D[v_1(V_2-1)]$ is the set of points $z=re^{i\theta}$ for which $r\geqq 2(k-\cos\theta)$.

To prove (i) observe that V_1-1 is the set of points z for which $|z|\leqq k$ and that the set $v_2'(V_1-1)\subset v_2(V_1-1)$ whenever v_2 and v_2' are elements of V_2 such that $\arg v_2'=\arg v_2$ and $|v_2'|<|v_2|$. For in these circumstances the set $v_2(V_1-1)$ is the circular region defined by $|z|\leqq k|v_2|$ while the set $v_2'(V_1-1)$ is the region $|z|\leqq k|v_2'|=k$. Thus the set E_1 is the set $|z| \leqq k$. We note that

$$1 + E_1 \subset V_1. \tag{3.1}$$

The proof of (ii) is somewhat more complicated. We observe that V_2-1 is the set of complex numbers z for which $|z+1| \geqq 1$. The set $0(V-1)$ is not well-defined. From continuity considerations we shall define it to be the set exterior to the null circle $|z| = 0$. Consider then the set of products $v_1(V_2-1)$ where $v_1 \in V_1$ and $v_1 \neq 0$. If v_1 and v_1' are any two such elements of V_1 having the properties $\arg v_1 = \arg v_1'$ and $|v_1'| > |v_1|$, then $v_1'(V_2-1) \subset v_1(V_2-1)$. This is true since multiplication of V_2-1 by v_1' subjects the set to the same rotation as multiplication by v_1. Since however $|v_1'| > |v_1|$, the "exterior" set which is $v_1(V_2-1)$ contains the "exterior" set $v_1'(V_2-1)$. Thus $D[v_1'(V_2'-1)]$ is the set $D[v_1(V_2-1)] = E_2$ for v_1' on the boundary of V_1. We can now prove that E_2 is the set of numbers $z = re^{i\theta}$ for which

$$r \geqq 2(k - \cos\theta).$$

To determine the boundary of E_2 it is clearly sufficient to consider only those products $v_1'(v_2'-1)$ where v_1' and v_2' range over the boundaries of V_1 and V_2 respectively. To that end let $r(\alpha)e^{i\alpha}$ and $\rho(\beta)e^{i\beta}$ be arbitrary points of the boundaries of V_1 and V_2-1, respectively. It is clear that

$$r(\alpha) = \cos\alpha + (k^2 - \sin^2\alpha)^{1/2} \qquad (0 \leqq \alpha \leqq 2\pi),$$
$$\rho(\beta) = -2\cos\beta \qquad (\pi/2 \leqq \beta \leqq 3\pi/2).$$

We shall show first that

$$r(\alpha)\rho(\beta) \leqq 2[k - \cos(\alpha+\beta)] \qquad (\text{all } \alpha, \beta). \tag{3.2}$$

It will follow then that the boundary of E_2 is not exterior to the limaçon $r = 2(k - \cos\theta)$. The proof of (3.2) is easy. It is clearly sufficient to prove that

$$-\cos\beta(k^2 - \sin^2\alpha)^{1/2} \leqq k + \sin\alpha\sin\beta.$$

Since $\cos\beta < 0$ and $k > 1$, it is then sufficient to prove that

$$\cos^2\beta(k^2 - \sin^2\alpha) \leqq (k + \sin\alpha\sin\beta)^2.$$

But this is equivalent to

$$0 \leqq (k\sin\beta + \sin\alpha)^2. \tag{3.3}$$

Thus (3.2) is true. However, we observe that inequalities (3.2) and (3.3) are equivalent. With this in mind we can establish that corresponding to each angle $\alpha+\beta=\theta$ there exists an essentially unique choice of α and of β such that the equality sign in (3.2) and in (3.3) holds. It will follow that the limaçon $r=2(k-\cos\theta)$ is the boundary of E_2. The angles α and β referred to are determined by the following conditions:

$$\sin\alpha = -k\sin\theta/\Delta, \qquad \cos\alpha = (1-k\cos\theta)/\Delta,$$
$$\sin\beta = \sin\theta/\Delta, \qquad \cos\beta = (\cos\theta - k)/\Delta,$$

where $\Delta=(1-2k\cos\theta+k^2)^{1/2}$.

It is clear that these equations determine α and β to within an additive integral multiple of 2π. It is easy to see that (3.3) and hence (3.2) reduces to an equality for this choice of α and β. It is not difficult to prove that there is no other choice of α and β having the desired properties. However, since the uniqueness is irrelevant to our present discussion, its proof is omitted.

The proof of the lemma is completed by noting that

$$1+E_2\subset V_2 \tag{3.4}$$

and recalling (3.1).

To complete the proof of Theorem 3.1 we introduce the continued fraction

$$1+\frac{a_1z}{1}+\frac{a_2/z}{1}+\frac{a_3z}{1}+\frac{a_4/z}{1}+\cdots, \tag{3.5}$$

where $z=re^{i\theta}$ is a complex variable. This continued fraction reduces to (1.1) when $z=1$. We shall suppose that the coefficients a_n satisfy the hypotheses of Theorem 3.1. It is clear then that for $|z|\leqq 1+\delta_1$, where $\delta_1=\epsilon/(k-\epsilon)$,

$$|a_{2n-1}z|\leqq k. \tag{3.6}$$

Further, since the numbers a_{2n} lie outside the limaçon $r=2(k+\epsilon_1-\cos\theta)$, positive numbers η and δ_2 can be chosen so small that for $|\theta|\leqq\eta$ and $r\leqq 1+\delta_2$ we have $a_{2n}/z=Re^{i\phi}$ satisfying the conditions

$$R_{2n}\geqq 2(k-\cos\phi_{2n}).$$

Let δ be the smaller of δ_1 and δ_2. In the open region D defined by the inequalities

$$D:\quad |\theta|<\eta, \qquad 0<|z|<1+\delta \tag{3.7}$$

the approximants of the continued fraction (3.5) are a family of rational functions of z. By (3.6) and (3.7) Lemma 3.1 asserts that the values of these approximants lie in the bounded region V_1. Thus this family is a normal family for z in D.

We shall now show that this normal family converges in an open subset of D. Denote by h the smaller of the two numbers $1/4(k-\epsilon)$ and $8(k+\epsilon_1-1)/25$. Then for $|z|\leqq h$, the elements of (3.5) have the property that

$$|a_{2n-1}z|\leqq 1/4,\qquad |a_{2n}/z|\geqq 25/4\qquad (n=1,2,\cdots).$$

Thus by the theorem referred to concerning inequalities (1.2), the continued fraction (3.5) converges for $|z|\leqq h$ and hence in the intersection of this circular region with the region D. By Montel's [1] generalization of the Stieltjes-Vitali theorem the sequence of approximants of (3.5) converges (uniformly) in every closed region interior to D. Thus, in particular, this family converges for $z=1$.

The proof of the theorem is complete.

The following corollary is an immediate consequence of the theorem. The notation is that of the theorem.

COROLLARY 3.1. *If*

$$r_{2n}\geqq 2(k+\epsilon_1+1),\qquad r_{2n-1}\leqq k-\epsilon\qquad (n=1,2,\cdots),$$

the continued fraction (1.1) *converges.*

It is a consequence of Theorem 3.1 that if the hypotheses of this theorem hold, the continued fraction (2.1) converges at least in the wider sense, and converges to a finite value if and only if the continued fraction (1.1) converges to a value not equal to 1.

The following result is thus established. The notation is that of the theorem.

COROLLARY 3.2. *If the elements* $a_n=r_ne^{i\theta_n}$ *are such that*

$$r_{2n-1}\geqq 2(k+\epsilon_1-\cos\theta_{2n-1}),\qquad r_{2n}\leqq k-\epsilon\qquad (n=1,2,\cdots),$$

the continued fraction (1.1) *converges at least in the wider sense.*

We proceed with the proof of the following result.

COROLLARY 3.3. *If the elements* a_{2n} *lie in the circle* $|z|\leqq 1$ *and if the quantities* a_{2n+1} *are outside of and bounded away from the cardioid* $r=2(1-\cos\theta)$ *the continued fraction* (1.1) *converges.*

To prove this one observes that one can choose numbers $k>1$ and $\epsilon_1>0$ such that the numbers a_{2n+1} lie outside the limaçon $r=2(k+\epsilon_1-\cos\theta)$. A positive number ϵ can be chosen so that $|a_{2n}|\leqq k-\epsilon$. The conditions of Theorem 3.1 are then fulfilled.

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THE RICE INSTITUTE

EQUICONVERGENCE THEOREMS FOR ORTHONORMAL POLYNOMIALS

G. E. ALBERT AND L. H. MILLER

1. Introduction. If $\rho(x)$ is a non-negative function integrable on the finite interval (a, b) and positive on a set of positive measure, there exists a unique set of polynomials $\{p_n(\rho, x)\}$ of degree n, $p_n(\rho, x) = c_n x^n + \cdots$, $c_n > 0$, which are orthonormal on (a, b) relative to the weight function $\rho(x)$; that is

$$\int_a^b \rho(x) p_n(\rho, x) p_m(\rho, x) dx = \begin{cases} 0 & \text{if} \quad n \neq m, \\ 1 & \text{if} \quad m = n. \end{cases}$$

In the sequel the abbreviation ONP will be used to denote such a set of polynomials.

Given any function $f(x)$ for which the integral $\int_a^b \rho(x) f(x) dx$ exists, the set of ONP associated with $\rho(x)$ may be used to construct the formal expansion

$$(1) \qquad f(x) \sim \sum_{n=0}^{\infty} a_n p_n(\rho, x), \qquad a_n = \int_a^b \rho(t) f(t) p_n(\rho, t) dt.$$

Much attention has been given to the convergence properties of such series for particular choices of $\rho(x)$. In the succeeding pages known results on this problem are extended by means of what seems to be a new type of proof.

Let $\{p_n(\rho_1, x)\}$ and $\{p_n(\rho_2, x)\}$ be two sets of ONP with different weight functions $\rho_1(x)$ and $\rho_2(x)$ and let

$$(2) \qquad s_n(f; \rho_i, x) = \sum_{k=0}^{n} a_{ki} p_k(\rho_i, x), \qquad i = 1, 2,$$

denote the partial sums of nth degree for the two corresponding expansions (1) associated with $f(x)$. In §§5 and 6 below certain sufficient conditions will be established for the validity of the relation

$$(3) \qquad \lim_{n \to \infty} \{s_n(f; \rho_1, x) - s_n(f; \rho_2, x)\} = 0;$$

that is, conditions under which the two series mentioned are *equiconvergent*. It should be stated that the emphasis of the paper is upon method rather than upon specific conditions. The discerning reader

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will easily see arrangements for the weight functions which yield variations of the conditions cited.

The materials from the theory of ONP that are needed in the proofs to follow are quite elementary and are collected for reference in §2. Two lemmas are the subject matter of §3; from them the remainder of the paper follows essentially as a series of corollaries.

Some interesting relations between bounds for related systems of ONP are indicated in §4. These are obtained as by-products of the method of proof for the equiconvergence theorems.

2. **Notations and formulas.** The symbol (a, b) will denote the closed interval $a \leqq x \leqq b$. It will be assumed without further mention that any weight function and any function $f(x)$ for which a series of type (1) is used will satisfy the conditions set forth for such functions in the introduction. The symbol $L^2(\rho; a, b)$ will denote the class of all measurable functions $f(x)$ for which the integral $\int_a^b \rho(x)[f(x)]^2 dx$ exists.

If $f(x)$ and $g(x)$ are in $L^2(\rho; a, b)$, it is well known that

$$\int_a^b \rho(x)[f(x)]^2 dx = \sum_{k=0}^{\infty} a_k^2, \tag{4}$$

$$\int_a^b \rho(x)f(x)g(x)dx = \sum_{k=0}^{\infty} a_k b_k, \qquad b_k = \int_a^b \rho(t)g(t)p_k(\rho, t)dt, \tag{5}$$

and

$$\int_a^b \rho(x)[f(x) - s_n(f; \rho, x)]^2 dx \leqq \int_a^b \rho(x)[f(x) - \pi_n(x)]^2 dx \tag{6}$$

for an arbitrary polynomial $\pi_n(x)$ of degree n.

Also

$$s_n(f; \rho, x) = \int_a^b \rho(t)f(t)K_n(\rho; x, t)dt \tag{7}$$

where

$$\begin{aligned} K_n(\rho; x, t) &= \sum_{k=0}^{n} p_k(\rho, x)p_k(\rho, t) \\ &= \frac{c_n}{c_{n+1}} \, \frac{p_{n+1}(\rho, t)p_n(\rho, x) - p_{n+1}(\rho, x)p_n(\rho, t)}{t - x}. \end{aligned} \tag{8}$$

In (8), c_n is the leading coefficient of $p_n(\rho, x)$ and one has

$$0 < c_n/c_{n+1} \leqq C = \text{max of } |a| \text{ and } |b|. \tag{9}$$

The reader interested in proofs of the above formulas should consult Jackson [1][1] or Szegö [4].

3. **Two inequalities.** The remaining sections of the paper will be devoted to comparisons of the two sets of ONP, $\{p_n(\rho_1, x)\}$ and $\{p_n(\rho_2, x)\}$, for which the weight functions will be assumed to have the forms

$$\rho_1(x) = \rho(x)[\pi_m(x)]^2 \quad \text{and} \quad \rho_2(x) = \rho(x)\pi_q(x)\sigma(x) \tag{10}$$

or special cases thereof. The following blanket hypotheses will be made concerning the individual factors involved in (10).

(a) *The functions* $\pi_m(x)$ *and* $\pi_q(x)$ *are polynomials of degrees* m *and* q *respectively and* $\rho(x)$, $\pi_q(x)$ *and* $\sigma(x)$ *are non-negative on* (a, b).

(b) *The polynomial* $\pi_q(x)$ *and the function* $\sigma(x)$ *are such that the product* $\pi_q(x)\sigma(x)$ *is measurable and bounded on* (a, b) *by a constant* M_1.

(c) *The polynomial* $\pi_m(x)$ *and the function* $\sigma(x)$ *are such that the quotient* $\pi_m(x)/\sigma(x)$ *is measurable and bounded on* (a, b) *by a constant* M_2.

LEMMA 1. *Let* (a), (b) *and* (c) *be satisfied. If, at a point* ξ *in* (a, b), *there exist two positive numbers* $\epsilon(\xi)$ *and* $\gamma(\xi)$ *such that when* x *is in the common part of the intervals* (a, b) *and* $(\xi-\epsilon, \xi+\epsilon)$ *the condition*

$$\left| \frac{\pi_m(\xi)}{\sigma(\xi)} - \frac{\pi_m(x)}{\sigma(x)} \right| < \gamma(\xi) \cdot |x - \xi| \tag{11}$$

holds, then at the point ξ

$$\begin{aligned} \left\{ \int_a^b \rho(t)\pi_q(t)\sigma(t) \left[\frac{\{\pi_m(t)\}^2}{\sigma(t)} K_{n+q}(\rho_1; \xi, t) - \pi_q(\xi)K_k(\rho_2; \xi, t) \right]^2 dt \right\}^{1/2} & \\ \leq M_1^{1/2} M_2 \left\{ \sum_{j=k-m+1}^{n+q} [p_j(\rho_1, \xi)]^2 \right\}^{1/2} & \\ + M_1^{1/2} C \left\{ \left(\frac{4M_2^2}{\epsilon^2} + \gamma^2 \right) \sum_{j=k-m}^{k-m+1} [p_j(\rho_1, \xi)]^2 \right\}^{1/2} & \end{aligned} \tag{12}$$

for all k *and* n *such that* $m \leqq k \leqq n$. *The constant* C *is the bound indicated in* (9).

PROOF. Since $\pi_q(\xi)p_j(\rho_2, \xi)$ is a polynomial of degree $j+q$ in ξ, one has

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[1] The numbers in brackets refer to the bibliography.

$$\begin{aligned}\pi_q(\xi)p_j(\rho_2, \xi) &= \int_a^b \rho_1(t)\pi_q(t)p_j(\rho_2, t)K_{n+q}(\rho_1; \xi, t)dt \\ &= \int_a^b \rho_2(t)\left[\frac{\{\pi_m(t)\}^2}{\sigma(t)} K_{n+q}(\rho_1; \xi, t)\right]p_j(\rho_2, t)dt\end{aligned} \tag{13}$$

for all $j \leqq n$. This implies

$$\pi_q(\xi)K_k(\rho_2; \xi, t) \equiv s_k\left(\left[\frac{\pi_m^2}{\sigma} K_{n+q}(\rho_1)\right]; \rho_2, t\right).$$

In view of (6), the left member of (12) is dominated by

$$\begin{aligned}&\left\{\int_a^b \rho_2(t)\left[\frac{\{\pi_m(t)\}^2}{\sigma(t)} K_{n+q}(\rho_1; \xi, t) - \frac{\pi_m(\xi)\pi_m(t)}{\sigma(\xi)} K_{k-m}(\rho_1; \xi, t)\right]^2 dt\right\}^{1/2} \\ &\leqq \left\{M_1M_2^2\int_a^b \rho_1(t)\left[\sum_{j=k-m+1}^{n+q} p_j(\rho_1, \xi)p_j(\rho_1, t)\right]^2 dt\right\}^{1/2} \\ &\quad + \left\{M_1\int_a^b \rho_1(t)\left[\frac{\pi_m(t)}{\sigma(t)} - \frac{\pi_m(\xi)}{\sigma(\xi)}\right]^2[K_{k-m}(\rho_1; \xi, t)]^2dt\right\}^{1/2}.\end{aligned} \tag{14}$$

The first term in (14) is clearly bounded by the first term in (12) on the right. In the second term of (14) the range of integration must be broken into the three intervals $(a, \xi-\epsilon)$, $(\xi-\epsilon, \xi+\epsilon)$ and $(\xi+\epsilon, b)$. Using (8), (9), condition (c), and the inequality $|\xi-x| \geqq \epsilon(\xi)$, the contribution to the integral in that term from the first and third subintervals is found to be dominated by

$$\begin{aligned}\frac{4M_1M_2^2C^2}{\epsilon^2}\int_a^b \rho_1(t)\,[p_{k-m+1}(\rho_1, \xi)p_{k-m}(\rho_1, t) - p_{k-m+1}(\rho_1, t)p_{k-m}(\rho_1, \xi)]^2dt \\ \leqq \frac{4M_1M_2^2C^2}{\epsilon^2}\,[\{p_{k-m}(\rho_1, \xi)\}^2 + \{p_{k-m+1}(\rho_1, \xi)\}^2].\end{aligned}$$

Use of (8), (9) and (11) shows that the contribution to the integral under consideration from $(\xi-\epsilon, \xi+\epsilon)$ is dominated by

$$M_1C^2\gamma^2[\{p_{k-m}(\rho_1, \xi)\}^2 + \{p_{k-m+1}(\rho_1, \xi)\}^2].$$

Combining all these results one has (12).

The following special case of Lemma 1 is of special importance in the applications.

LEMMA 2. *Let the hypotheses* (a) *and* (b) *be satisfied. If the quotient* $\pi_m(x)/\sigma(x)$ *satisfies the Lipschitz condition*

$$\left|\frac{\pi_m(x)}{\sigma(x)} - \frac{\pi_m(t)}{\sigma(t)}\right| < \lambda\left|x - t\right| \tag{15}$$

for all x and t in (a, b), then, for every point ξ in (a, b),

$$\begin{aligned}\left\{\int_a^b \rho_2(t)\left[\frac{\{\pi_m(t)\}^2}{\sigma(t)} K_{n+q}(\rho_1; \xi, t) - \pi_q(\xi)K_k(\rho_2; \xi, t)\right]^2 dt\right\}^{1/2} \\ \leqq M_1^{1/2}M_2\left\{\sum_{j=k-m+1}^{n+q} [p_j(\rho_1, \xi)]^2\right\}^{1/2} \\ + \lambda M_1^{1/2}C\left\{\sum_{j=k-m}^{k-m+1} [p_j(\rho_1, \xi)]^2\right\}^{1/2}\end{aligned} \tag{16}$$

for all k and n such that $m \leqq k \leqq n$. The constant C is the bound indicated in (9).

Proof. The hypothesis (c) is automatically satisfied. The proof proceeds like that for Lemma 1 except that (8), (9) and the Lipschitz condition are to be applied directly to the second term in (14) to obtain the second term in (16).

4. **Bounds for ONP.** Lemmas 1 and 2 are powerful tools for establishing bounds for ONP. Since the most interesting bounds are uniform in x on some subset of (a, b), attention will be focused upon the application of Lemma 2; extensions of the results which are obtainable from Lemma 1 are left to the reader.

Theorem 1. *Suppose the hypotheses* (a) *and* (b) *and the Lipschitz condition* (15) *are satisfied. If at a point ξ in (a, b) for which $\pi_q(\xi)$ is different from zero there is a number $H_1(\xi)$ such that $|p_n(\rho_1, \xi)| \leqq H_1$ for all n, then there exists a number $H_2(\xi)$ such that $|p_n(\rho_2, \xi)| \leqq H_2$ for all n. Moreover, if for every point of some closed subset of (a, b) on which $\pi_q(x)$ is different from zero the bound H_1 exists and is independent of x, then H_2 exists and is independent of x on that set.*

Proof. Obviously

$$0 = \int_a^b \rho_2(t)p_n(\rho_2, t)\pi_q(\xi)K_{n-1}(\rho_2; \xi, t)dt.$$

Combining this with (13) for $j = n$, one has

$$\begin{aligned}&\pi_q(\xi)p_n(\rho_2, \xi) \\ &= \int_a^b \rho_2(t)p_n(\rho_2, t)\left[\frac{\{\pi_m(t)\}^2}{\sigma(t)} K_{n+q}(\rho_1; \xi, t) - \pi_q(\xi)K_{n-1}(\rho_2; \xi, t)\right]dt.\end{aligned}$$

THEOREM 2. *If $\rho(x)$ and $\sigma(x)$ are non-negative and $\sigma(x)$ is bounded and measurable on (a, b) and a polynomial $\pi_m(x)$ of degree m can be found such that the quotient $\pi_m(x)/\sigma(x)$ satisfies a Lipschitz condition, then* (17) *holds, for all $f(x)$ in $L^2(\rho;\, a, b)$, at any point ξ in (a, b) at which $\pi_m(\xi)$ is different from zero and $|p_n(\rho, \xi)| \leqq H(\xi)$ where $H(\xi)$ is a number independent of n. Moreover,* (17) *holds uniformly on any closed subset of (a, b) on which $\pi_m(x)$ is different from zero and the number H is independent of x.*

PROOF. To demonstrate the inequality (18) set

$$I_n(\xi) = \int_a^b \rho(t)[K_n(\rho;\, \xi, t) - \sigma(t)K_n(\rho\sigma;\, \xi, t)]^2 dt.$$

Add and subtract $[\pi_m(t)]^2K_n(\rho\pi_m^2;\, \xi, t)$ inside the bracket in the integrand. Here $K_n(\rho\pi_m^2;\, \xi, t)$ denotes the kernel polynomial (8) for the ONP of weight function $\rho(x)[\pi_m(x)]^2$. Applying Minkowski's inequality and a little obvious manipulation, one has

$$\begin{aligned}[I_n(\xi)]^{1/2} &\\ \leqq \Bigg[M_1 \int_a^b \rho(t)\sigma(t)\bigg[&\frac{\{\pi_m(t)\}^2}{\sigma(t)} K_n(\rho\pi_m^2;\, \xi, t) - K_n(\rho\sigma;\, \xi, t)\bigg]^2 dt\Bigg]^{1/2} \\ &+ \left[\int_a^b \rho(t)[\{\pi_m(t)\}^2 K_n(\rho\pi_m^2;\, \xi, t) - K_n(\rho;\, \xi, t)]^2 dt\right]^{1/2},\end{aligned}$$

where M_1 is the bound on $\sigma(x)$. To the first term on the right apply Lemma 2 with $\pi_q(x) \equiv 1$, $k = n$; to the second term apply Lemma 2 with $\pi_q(x) \equiv \sigma(x) \equiv 1$ and $k = n$. The result is

$$\begin{aligned}[I_n(\xi)]^{1/2} \leqq (M_1M_2 + M_3)&\left[\sum_{j=n-m+1}^{n} \{p_j(\rho\pi_m^2,\, \xi)\}^2\right]^{1/2} \\ &+ \lambda C(M_1 + 1)\left[\sum_{j=n-m}^{n-m+1} \{p_j(\rho\pi_m^2,\, \xi)\}^2\right]^{1/2}\end{aligned}$$

for all $n \geqq m$. The constants M_2 and M_3 are bounds on π_m/σ and π_m. By the remark following Theorem 1 the system $\{p_n(\rho\pi_m^2,\, \xi)\}$ is bounded as to n at any point ξ such that $\pi_m(\xi)$ is different from zero and at which the system $\{p_n(\rho,\, \xi)\}$ is so bounded. This proves the inequality (18) under the stated hypotheses. The uniformity clause is evident.

The following more general theorem is obtained by applying Lemma 1 to the integral $I_n(\xi)$.

THEOREM 3. *Let $\rho(x)$ and $\sigma(x)$ be non-negative and $\sigma(x)$ be bounded and measurable on (a, b). Let ξ be a point in (a, b). If there is a polynomial $\pi_m(x)$ of degree m such that $\pi_m(\xi)$ is different from zero and $\pi_m(x)/\sigma(x)$ satisfies the condition* (11) *of Lemma* 1 *and if there is a number $H(\xi)$ independent of n such that $|p_n(\rho, \xi)| \leqq H(\xi)$ for all n, then* (17) *holds for all $f(x)$ in $L^2(\rho; a, b)$.*

Theorems 2 and 3 furnish no information about equiconvergence at a point where the auxiliary polynomial $\pi_m(x)$ vanishes. It is not difficult to construct examples which show that such equiconvergence *cannot be expected* for all functions in $L^2(\rho; a, b)$.

6. **Applications.** Let (a, b) be $(-1, 1)$ and let $\rho(x)$ and $\sigma(x)$ have the forms

$$\rho(x) = (1 - x^2)^{-1/2} \quad \text{and} \quad \sigma(x) = \tau(x)\prod_{k=1}^{h} | x - x_k |^{\lambda_k}$$

where $-1 \leqq x_1 < x_2 < \cdots < x_h \leqq 1$; $\lambda_k \geqq 0$, $k = 1, 2, \cdots, h$, and $0 < A \leqq \tau(x) \leqq B$. Theorem 2 applies if $\tau(x)$ satisfies a Lipschitz condition on $(-1, 1)$ and Theorem 3 applies at any point ξ in $(-1, 1)$ distinct from all the points x_k, $k = 1, 2, \cdots, h$, provided that there is a pair of numbers $\lambda(\xi)$ and $\epsilon(\xi, \lambda)$ such that

$$| \tau(x) - \tau(\xi) | < \lambda | x - \xi |$$

for all x in $(-1, 1)$ such that $|x - \xi| < \epsilon$. In either case the polynomial $\pi_m(x)$ is chosen to be $\prod_{k=1}^{h} (x - x_k)^{m_k}$, where m_k is the smallest integer such that $m_k - \lambda_k \geqq 1$. These results compare the convergence of the series (1) for the ONP of weight function $\rho(x)\sigma(x)$ with the convergence of the Fourier cosine series on the open intervals $x_i < x < x_{i+1}$, $i = 1, 2, \cdots, h$. Either case includes the Jacobi polynomials of weight function $(1 - x)^\alpha(1 + x)^\beta$ where $\alpha, \beta > -1/2$. The cases where α or β or both are less than $-1/2$ are also easily handled. For example, if $\alpha = \beta = -3/4$, set $\rho(x) = (1 - x^2)^{-3/4}$ and $\sigma(x) = (1 - x^2)^{1/4}$ and choose $\pi_m(x)$ appropriately.

The case above using Theorem 3 essentially includes a theorem of Szegö [4; Theorem 13.1.2, p. 307]. The case using Theorem 2 is comparable to results of Peebles [2]. The methods of the present paper are certainly more elementary and more direct than those used by either of the above authors.

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OHIO STATE UNIVERSITY

MATRIX PRODUCTS OF MATRIX POWERS

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1. **Introduction.** Let m n-by-n matrices, A_k, of complex constants, a_{ijk} $(i, j=1, 2, \cdots, n; k=1, 2, \cdots, m)$, be given. We shall denote by $\mathcal{L}$ the set of all matrices,

$$A(t) = \sum_{i=1}^{m} \rho_i(t) A_i,$$

where $\rho_i(t)$ $(i=1, 2, \cdots, m)$ are arbitrary, non-negative, summable functions of the real variable t on the interval T, $a \leqq t \leqq b$. We shall call $\mathfrak{J}$, $\mathcal{S}$, or $\mathcal{X}$ the subsets of $\mathcal{L}$ obtained by restricting the functions $\rho_i(t)$ to polynomial functions, step functions, or step functions which are all zero except one. Since, in each case, the elements of $A(t)$ are summable functions of t on T, it follows that, on T, there exists a unique, absolutely continuous matrix solution,[1] $Y(t)$, of the linear, matrix differential equation and initial condition:

$$dY(t)/dt = Y(t)A(t), \qquad Y(a) = E, \tag{1.1}$$

where E is the n-by-n unit matrix. We shall denote by λ, ι, σ or ξ the set of matrices, $Y(t)$, which are particular values of solutions of (1.1), where $A(t)$ is an arbitrary matrix of $\mathcal{L}$, $\mathfrak{J}$, $\mathcal{S}$, or $\mathcal{X}$, respectively, and t is on T.

If A is a matrix with elements a_{ij}, let the absolute value of A and the exponential and natural logarithm of A be defined[2] by the equations:

$$|A| = \left[\sum_{i,j=1}^{n} |a_{ij}|^2\right]^{1/2},$$

$$\exp A = \sum_{i=0}^{\infty} A^i/i!$$

$$\log A = \sum_{i=1}^{\infty} (-1)^{i-1}(A - E)^i/i, \quad \text{if} \quad |A - E| < 1.$$

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[1] See W. M. Whyburn, *On the fundamental existence theorems for differential systems.* Ann. of Math. (2) vol. 30 (1928–29) p. 31. We observe that equations (1.1) are equivalent to a system of $2n$ real, linear, first order differential equations satisfying all the hypotheses of this theorem.

[2] See J. v. Neumann, *Über die analytischen Eigenschaften von Gruppen linearer Transformationen und ihrer Darstellungen.* Math. Zeit. vol. 30 (1929) pp. 6, 7.

J. v. Neumann[3] has shown that if j is an integer and if $|A-E|<1$, then

$$A^j = \exp [j \log A]. \tag{1.2}$$

Hence, for any α, we define the α power function of A by the equation

$$A^\alpha = \exp (\alpha \log A), \quad \text{if} \quad |A - E| < 1.$$

If $|\exp A_i-E|<1$ $(i=1, 2, \cdots, m)$, we define μ as the set of matrix products of matrix powers,

$$\prod_{j=1}^{J} \prod_{i=1}^{m} (\exp A_i)^{\alpha_{ij}},$$

where the α_{ij} are arbitrary non-negative numbers.

Let us identify an n-by-n matrix, B, of complex numbers with the point in $2n^2$-Euclidean space, whose coordinates are the real and imaginary parts of the elements of B. The distance between two points B_1 and B_2 may be defined as $|B_1-B_2|$. A set, β, of matrices, B, is then also a point set whose closure we denote by $\bar{\beta}$.

It is the purpose of this paper to show that the sets $\bar{\lambda}$, $\bar{\iota}$, $\bar{\sigma}$, ξ and, if it exists, $\bar{\mu}$, are identical.

2. Principal theorems.

THEOREM 2.1. *The sets $\bar{\lambda}$ and $\bar{\sigma}$, defined above, are identical.*

Since any step function on T is summable on T, it follows that $\sigma \subset \lambda$. Suppose $A_{\mathrm{L}}(t)$ is a matrix of class $\mathcal{L}$ with coefficients $\rho_i(t)$. Then, for all positive δ, there exists a matrix, A_{S}, of class S with coefficients $r_i(t)$ such that

$$\sum_{k=1}^{m} \int_a^b | \rho_k(t) - r_k(t) | | a_{ijk} | \, dt < \delta \qquad (i, j = 1, 2, \cdots, n).$$

Let the corresponding solutions of (1.1) be $Y_{\mathrm{L}}(t)$ and $Y_{\mathrm{S}}(t)$. Since $Y(t)$ is a uniformly continuous functional[4] of $A(t)$, it follows that, given any positive number ϵ, δ may be so chosen that

$$Y_{\mathrm{L}}(t) - Y_{\mathrm{S}}(t) \ll \epsilon;$$

that is, the absolute value of each element of the matrix on the left is less than ϵ. Hence $\lambda \subset \bar{\sigma}$.

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[3] Loc. cit. pp. 8, 12.

[4] By this we mean that the elements of $Y(t)$ are uniformly continuous functionals of the elements of $A(t)$. See W. M. Whyburn, *Functional properties of the solutions of differential systems*. Trans. Amer. Math. Soc. vol. 32 (1930) p. 508.

THEOREM 2.2. *The sets $\bar{\sigma}$ and ξ are identical.*

Clearly $\xi \subset \sigma$.

J. v. Neumann[5] has shown that if $|A|, |B| < (1/2)\log(3/2)$, then

$$\log[\exp A \exp B] = A + B + O(|A||B|).$$

This equation, by induction, leads to the generalized equation:

$$\log \prod_{i=1}^{m} \exp A_i = \sum_{i=1}^{m} A_i + \sum_{i,j=1}^{m} O(|A_i||A_j|), \tag{2.1}$$

if $|A_i| < \delta(m)$, where it suffices to take $\delta(m) < [\log(3/2)]/2n(m-1)$. To this we may add the equation,

$$\exp(A + B) = \exp A + O(|B|) \tag{2.2}$$

which follows immediately from the definition of $\exp(A+B)$.

LEMMA 2.1. *If A is a matrix of constants, the matrix,*

$$Y(t) = Y_0 \exp[(t - a)A],$$

is the solution of the linear, matrix differential equation and initial condition

$$dY(t)/dt = Y(t)A, \qquad Y(a) = Y_0.$$

The series $\sum_{j=0}^{\infty} (t-a)^j A^j / j$ is uniformly convergent[6] on any interval $|t-a| < N$, hence the lemma may be established by term-by-term differentiation.

LEMMA 2.2. *If $A(t)$ is a matrix of summable functions, and if $A(t) \ll (M)$ on T, the solution, $Y(t)$, of equation* (1.1) *satisfies the inequality*

$$Y(t) - E \ll (1/n[\exp Mn(t - a) - 1]) \text{ on } T.$$

Slight modifications of the proof of the existence theorem given by G. D. Birkhoff and R. E. Langer[7] yield this lemma.

LEMMA 2.3. *If $B_1, B_2, \cdots, B_m$ are square matrices,*

$$\lim_{j\to\infty} \left[\prod_{i=1}^{m} \exp(B_i/j) \right]^j = \exp \sum_{i=1}^{m} B_i.$$

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[5] Loc. cit. pp. 13–15.

[6] See J. v. Neumann, loc. cit. p. 7.

[7] *The boundary problems and developments associated with a system of ordinary linear differential equations of the first order.* Proceedings of the American Academy of Arts and Sciences vol. 58 (1922–1923) pp. 59–63.

ON THE UNIQUENESS OF YOUNG'S DIFFERENTIALS

CHENG-NING YANG

Introduction. The differential of a function of several variables may be defined in a variety of ways, of which the one given by Young[1] renders the best parallelism with the case of a single variable. Stated in the way given below, his definition is applicable to a function defined in a set S of points containing limiting points at which the function is to have differentials. The question of the uniqueness of the differentials, however, arises. In this paper we shall first define, and prove two theorems concerning, the "limiting directions" which describe the directional distribution of the points of S near a limiting point. Then we proceed to show that properties of these limiting directions determine whether the differential is unique or not.

Consider the n-dimensional space. We use the notation for the scalar product of two vectors $a(a_1, a_2, \cdots, a_n)$ and $b(b_1, b_2, \cdots, b_n)$: $a \cdot b = \sum_{i=1}^{n} a_i b_i$. By $\|a\|$ we mean $+(a \cdot a)^{1/2}$; and when $\|a\| = 1$, a is called a direction.

DEFINITION. Let α be a limiting point of S (this will be assumed throughout the paper). If l is a direction such that

$$\left\{ \underset{x \text{ in } S \text{ and } \delta > \|x-\alpha\| > 0}{\text{lower bound of}} \left\| \frac{x-\alpha}{\|x-\alpha\|} - l \right\| \right\} = 0$$

for all positive δ, it is said to be a limiting direction of S at α.

THEOREM 1. *S has at least one limiting direction at α.*

THEOREM 2. *If ω is a direction perpendicular to all of the limiting directions of S at α, then*

$$\lim_{x \text{ in } S,\, x \to \alpha} \omega \cdot \frac{x-\alpha}{\|x-\alpha\|} = 0.$$

Each of the above theorems is readily derived by means of an indirect proof, with the help of the Heine-Borel theorem in the case of Theorem 1.

Let a function $f(x)$ be defined for all x in S. Young's definition of the differential may be stated in the following way:

DEFINITION OF THE DIFFERENTIAL. If there exists a vector V such that

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[1] Proc. London Math. Soc. (2) vol. 7 (1908) p. 157.

$$\lim_{x \text{ in } S,\, x\to\alpha} \frac{f(x) - f(\alpha) - V\cdot(x-\alpha)}{\|x-\alpha\|} = 0,$$

$f(x)$ is said to have a differential $df(\alpha, \eta) = V\cdot\eta$ at α.

THEOREM 3. *The differential $df(\alpha, \eta)$, if it exists, is unique when S has, at α, n linearly independent limiting directions.*

PROOF. Suppose both $U\cdot\eta$ and $V\cdot\eta$ can be written as $df(\alpha, \eta)$. Let $l_1, l_2, \cdots, l_n$ be n linearly independent limiting directions of S at α. The theorem is proved if we can show that $(U-V)\cdot l_i = 0$, $i = 1, 2, \cdots, n$.

To prove this we have

$$\lim_{x \text{ in } S,\, x\to\alpha} (U - V)\cdot\frac{x-\alpha}{\|x-\alpha\|} = \lim_{x \text{ in } S,\, x\to\alpha} \frac{f(x) - f(\alpha) - V\cdot(x-\alpha)}{\|x-\alpha\|} - \lim_{x \text{ in } S,\, x\to\alpha} \frac{f(x) - f(\alpha) - U\cdot(x-\alpha)}{\|x-\alpha\|} = 0.$$

Hence given any $\epsilon > 0$, there exists a $\delta > 0$ such that

$$\left| (U - V)\cdot\frac{x-\alpha}{\|x-\alpha\|} \right| < \epsilon$$

for all x in S satisfying $\delta \geqq \|x-\alpha\| > 0$. Since

$$\left\{ \text{lower bound of}_{\delta > \|x-\alpha\| > 0 \text{ and } x \text{ in } S} \left\| \frac{x-\alpha}{\|x-\alpha\|} - l_i \right\| \right\} = 0,$$

there must be a point x_0 in S such that

$$\delta > \|x_0 - \alpha\| > 0 \quad\text{and}\quad \left\| \frac{x_0-\alpha}{\|x_0-\alpha\|} - l_i \right\| < \epsilon.$$

Thus

$$\begin{aligned} |(U-V)\cdot l_i| &= \left| (U-V)\cdot\left[l_i - \frac{x_0-\alpha}{\|x_0-\alpha\|} + \frac{x_0-\alpha}{\|x_0-\alpha\|} \right] \right| \\ &\leqq \left| (U-V)\cdot\left(l_i - \frac{x_0-\alpha}{\|x_0-\alpha\|} \right) \right| \\ &\quad + \left| (U-V)\cdot\frac{x_0-\alpha}{\|x_0-\alpha\|} \right| \\ &\leqq \|U-V\| \left\| l_i - \frac{x_0-\alpha}{\|x_0-\alpha\|} \right\| + \epsilon \leqq [\|U-V\| + 1]\epsilon. \end{aligned}$$

But ϵ is arbitrary, hence

$$(U - V)\cdot l_i = 0.$$

THEOREM 4. *If S has $m(<n)$ linearly independent limiting directions $l_1, l_2, \cdots, l_m$ at α, and all its other limiting directions are linear combinations of them, and if $V\cdot\eta$ is a differential of $f(x)$ at α, the most general form of the differential is*

$$\left(V + \sum_{i=1}^{n-m} K_i\omega_i\right)\cdot\eta, \tag{A}$$

where K_i are arbitrary constants and $\omega_1, \omega_2, \cdots, \omega_{n-m}$ are linearly independent directions perpendicular to the l's.

PROOF. If $U\cdot\eta$ is also a differential at α, from the proof of the last theorem we have $(U-V)\cdot l_j=0, j=1, 2, \cdots, m$. Thus $(U-V)$ must be a linear combination of the vectors $\omega_1, \omega_2, \cdots, \omega_{n-m}$ which are perpendicular to the l's, that is, U is of the form (A).

To show that $(V+\sum_{i=1}^{n-m} K_i\omega_i)\cdot\eta$ is actually a differential, we have by Theorem 2

$$\begin{aligned}\lim_{x \text{ in } S,\, x\to\alpha} &\frac{f(x) - f(\alpha) - (V + \sum_{i=1}^{n-m} K_i\omega_i)\cdot(x-\alpha)}{\|x-\alpha\|} \\ &= \lim_{x \text{ in } S,\, x\to\alpha} \frac{f(x) - f(\alpha) - V\cdot(x-\alpha)}{\|x-\alpha\|} \\ &\quad - \sum_{1}^{n-m} K_i \lim_{x \text{ in } S,\, x\to\alpha} \omega_i\cdot\frac{x-\alpha}{\|x-\alpha\|} = 0.\end{aligned}$$

The theorem follows immediately.

COROLLARY. *There exists a unique differential $U\cdot\eta$ where U is in the m-dimensional space formed by $l_1, l_2, \cdots, l_m$.*

Finally I wish to express my thanks to Professor Y. Y. Tseng without whose continual encouragement this paper would never have been completed.

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RESOLUTION OF TEMPERATURE PROBLEMS BY THE USE OF FINITE FOURIER TRANSFORMATIONS

HERBERT KAPFEL BROWN[1]

1. The finite sine transformation of the convolution. The finite sine transformation and the finite cosine transformation of $U(x)$ with respect to x are defined[2] by

$$u_s(n) = \int_0^\pi U(x) \sin nx dx = S\{U(x)\}$$

and

$$u_c(n) = \int_0^\pi U(x) \cos nx dx = C\{U(x)\},$$

respectively.[3] In particular, $S\{U'\} = -nC\{U\}$, and $S\{U''\} = -n[(-1)^n U(\pi) - U(0)] - n^2 S\{U\}$. Further, if $U(x)$ and $U'(x)$ vanish at the end points of the interval $(0, \pi)$, then $S\{U''\} = -n^2 S\{U\}$, and $C\{U'\} = nS\{U\}$.

The convolution $U_1 * U_2$, or Faltung,[2] of the two functions $U_1(x)$, $-2\pi \leqq x \leqq 2\pi$, and $U_2(x)$, $-\pi \leqq x \leqq \pi$, is defined as follows:

$$U_1 * U_2 = \int_{-\pi}^{\pi} U_1(x - \xi) U_2(\xi) d\xi.$$

It is evident that the convolution of two even functions is an even function, that the convolution of two odd functions is an even function, and that the convolution of an odd function and an even function is an odd function.

The following theorem is proved by Kniess:[4]

THEOREM. *If $U_1(x)$, $-2\pi \leqq x \leqq 2\pi$, and $U_2(x)$, $-\pi \leqq x \leqq \pi$, are bounded and integrable, if U_1 is odd and periodic with period 2π, and if U_2 is even, then*

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[1] The author wishes to thank the referee and Professor R. V. Churchill for many valuable suggestions.

[2] G. Doetsch, *Integration von Differentialgleichungen vermittels der endlichen Fourier Transformation*, Math. Ann. vol. 112 (1935) pp. 52–68.

[3] The lower case letters will be used to signify the transforms of the functions designated by the corresponding capital letters. Instead of using the symbols $u_s(n)$ and $u_c(n)$ we shall use $u(n)$ for both, whenever it is evident which one is meant.

[4] Hans Kniess, *Lösung von Randwertproblemen bei Systemen gewöhnlicher Differentialgleichungen vermittels der endlichen Fourier Transformation*, Math. Zeit. vol. 44 (1938) pp. 266–291.

$$S\{U_1/2 * U_2\} = S\{U_1\}C\{U_2\}.$$

2. **Temperature distribution in a bar.** Consider a cylindrical bar of length π which has a diameter so small that the variation of temperature $U(x, t)$ over every cross section can be neglected. The thermal conductivity of the bar is a function of the time. There is a continuous internal source of heat along the bar and the initial temperature at each point of the bar is given by a prescribed function. The rate of loss of heat through the surface at each point is proportional to the temperature at that point. The temperatures at the ends $x=0$ and $x=\pi$ are determined by prescribed functions of t.

The boundary value problem for the temperature $U(x, t)$ can be written

$$\text{(A)}\quad \begin{aligned} &L(U) \equiv \partial U/\partial t - C_1(t)\partial^2 U/\partial x^2 + C_2(t)U = P(x, t), && 0 < x < \pi,\ t > 0,\\ &U(+0, t) = C_3(t), \qquad U(\pi - 0, t) = C_4(t), && t > 0,\\ &U(x, +0) = F(x), && 0 < x < \pi. \end{aligned}$$

The prescribed functions P, F, C_1, C_2, C_3, and C_4 are assumed to satisfy the following conditions: P and its first three derivatives with respect to x are continuous functions of x and t for all values of x and t in $0 \leqq x \leqq \pi$, $t \geqq 0$; F and its first three derivatives with respect to x are continuous in x for $0 \leqq x \leqq \pi$; P, F, and their first two derivatives with respect to x vanish at the end points of the interval $(0, \pi)$; C_1 and C_2 are positive and continuous for $t \geqq 0$; C_3' and C_4' are continuous for $t \geqq 0$; $C_3(0) = C_4(0) = 0$.

3. **Resolution of temperature problem.** It is obvious that the solution of our boundary value problem can be written

$$\text{(1)}\qquad U(x, t) = U_1(x, t) + U_2(x, t) + U_3(x, t) + U_4(x, t),$$

where U_1, U_2, U_3, and U_4 are solutions of the problems:

$$\text{(A}_1\text{)}\quad \begin{aligned} &L(U_1) = P(x, t), && U_1(+0, t) = 0,\\ &U_1(\pi - 0, t) = 0, && U_1(x, +0) = 0; \end{aligned}$$

$$\text{(A}_2\text{)}\quad \begin{aligned} &L(U_2) = 0, && U_2(+0, t) = C_3(t),\\ &U_2(\pi - 0, t) = 0, && U_2(x, +0) = 0; \end{aligned}$$

$$\text{(A}_3\text{)}\quad \begin{aligned} &L(U_3) = 0, && U_3(+0, t) = 0,\\ &U_3(\pi - 0, t) = C_4(t), && U_3(x, +0) = 0; \end{aligned}$$

and

$$(\mathrm{A}_4)\qquad \begin{array}{ll} L(U_4)=0, & U_4(+0,\ t)=0, \\ U_4(\pi-0,\ t)=0, & U_4(x,\ +0)=F(x). \end{array}$$

If in problem (A_2) we replace $C_3(t)$ by $C_4(t)$ and x by $\pi-x$, then $U_2(x,\ t)$ reduces to $U_3(x,\ t)$. For this reason we shall not have to consider problem (A_3) as a problem to be solved.

The finite sine transformation with respect to x applied to the first and fourth equations in problems (A_1), (A_2), and (A_4) transforms each of these boundary value problems into a problem in linear ordinary differential equations of the first order which can be written as follows:

$$(\mathrm{A}_1')\qquad \begin{array}{c} d[u_1(n,\ t)]/dt+[n^2C_1(t)+C_2(t)]u_1(n,\ t)=p(n,\ t), \\ u_1(n,\ +0)=0; \end{array}$$

$$(\mathrm{A}_2')\qquad \begin{array}{c} d[u_2(n,\ t)]/dt+[n^2C_1(t)+C_2(t)-nC_1(t)C_3(t)]u_2(n,\ t)=0, \\ u_2(n,\ +0)=0; \end{array}$$

and

$$(\mathrm{A}_4')\qquad d[u_4(n,\ t)]/dt+[n^2C_1(t)+C_2(t)]u_4(n,\ t)=0,\ u_4(n,\ +0)=f(n).$$

The solutions $u_1(n,\ t)$, $u_2(n,\ t)$, and $u_4(n,\ t)$ of the problems (A_1'), (A_2'), and (A_4'), respectively, are given as follows:

$$u_1(n,\ t)=\int_0^t p(n,\ \tau)\exp\left[-n^2\sigma_1(t,\ \tau)\right]\exp\left[-\sigma_2(t,\ \tau)\right]d\tau,$$

$$u_2(n,\ t)=\int_0^t C_3(\tau)C_1(\tau)\exp\left[-n^2\sigma_1(t,\ \tau)\right]\exp\left[-\sigma_2(t,\ \tau)\right]d\tau,$$

and

$$u_4(n,\ t)=f(n)\exp\left[-n^2s_1(t)\right]\exp\left[-s_2(t)\right],$$

where we have made the definitions: $s_i(t)=\int_0^t C_i(\xi)d\xi$, $\sigma_i(t,\ \tau)=s_i(t)-s_i(\tau)$, $0\leqq\tau\leqq t$.

It will now be shown that each of the problems (A_1), (A_2), and (A_4) can be further resolved into a single simple boundary value problem. That is to say, each of the solutions u_1, u_2, and u_4 can be expressed in terms of $u_0(n,\ t)$, the transform of $U_0(x,\ t)$, which is the solution of the following boundary value problem:

$$(\mathrm{B})\qquad \begin{array}{lr} \partial U_0/\partial t=\partial^2U_0/\partial x^2, & 0<x<\pi,\ t>0, \\ U_0(+0,\ t)=0,\quad U_0(\pi-0,\ t)=0, & t>0, \\ U_0(x,\ +0)=(\pi-x)/\pi, & 0<x<\pi. \end{array}$$

Making a formal application of the finite sine transformation to (B) gives

$$d[u_0(n, t)]/dt + n^2 u_0(n, t) = 0, \qquad u_0(n, +0) = 1/n.$$

The solution of this transformed problem is

$$u_0(n, t) = (1/n) \exp(-n^2 t).$$

From this solution we obtain the following identities:

$$\begin{aligned} \exp[-n^2 s_1(t)] &= n u_0[n, s_1(t)], \\ n \exp[-n^2 \sigma_1(t, \tau)] C_1(\tau) &= \partial u_0[n, \sigma_1(t, \tau)]/\partial\tau, \\ n \exp[-n^2 s_1(t)] C_1(t) &= -\partial u_0[n, s_1(t)]/\partial t. \end{aligned}$$

The transforms $u_1(n, t)$, $u_2(n, t)$, and $u_4(n, t)$ can now be expressed as functions of $u_0(n, t)$ as follows:

$$u_1(n, t) = \int_0^t \exp[-\sigma_2(t, \tau)] p(n, \tau) n u_0[n, \sigma_1(t, \tau)] d\tau,$$

$$u_2(n, t) = \int_0^t C_3(\tau) \exp[-\sigma_2(t, \tau)] \frac{\partial}{\partial\tau} u_0[n, \sigma_1(t, \tau)] d\tau,$$

and

$$u_4(n, t) = \exp[-s_2(t)] f(n) n u_0[n, s_1(t)].$$

If the convolution theorem is now applied to $u_1(n, t)$ and $u_4(n, t)$, a formal inverse sine transformation of $u_1(n, t)$, $u_2(n, t)$, and $u_4(n, t)$ can be performed. The result can be written

$$\begin{aligned} U_1(x, t) = \frac{1}{2} \int_0^t \exp[-\sigma_2(t, \tau)] \\ \cdot \int_{-\pi}^{\pi} P(x - \xi, \tau) \frac{\partial}{\partial\xi} U_0[\xi, \sigma_1(t, \tau)] d\xi d\tau, \end{aligned} \tag{2}$$

$$U_2(x, t) = \int_0^t C_3(\tau) \exp[-\sigma_2(t, \tau)] \frac{\partial}{\partial\tau} U_0[x, \sigma_1(t, \tau)] d\tau,$$

and

$$U_4(x, t) = \frac{1}{2} \exp[-s_2(t)] \int_{-\pi}^{\pi} F(x - \xi) \frac{\partial}{\partial\xi} U_0[\xi, s_1(t)] d\xi,$$

where P and F are the odd, periodic extensions of period 2π of the original functions P and F, respectively, and $\partial U_0/\partial\xi$ is the even extension of the original function $\partial U_0/\partial\xi$. The solution U_3 of problem (A_3) can be written

$$U_3(x, t) = \int_0^t C_4(\tau) \exp\,[-\sigma_2(t, \tau)] \frac{\partial}{\partial\tau} U_0[\pi - x, \sigma_1(t, \tau)]d\tau.$$

THEOREM. *The general temperature problem* (A) *can be resolved into the solution of the simple temperature problem* (B) *by formulas* (1) *and* (2). *That is,*

$$\begin{aligned} U(x, t) = {} & \frac{1}{2}\int_0^t \exp\,[-\sigma_2(t, \tau)] \int_{-\pi}^{\pi} P(x - \xi, \tau) \frac{\partial}{\partial\xi} U_0[\xi, \sigma_1(t, \tau)]d\xi d\tau \\ & + \int_0^t C_4(\tau) \exp\,[-\sigma_2(t, \tau)] \frac{\partial}{\partial\tau} U_0[\pi - x, \sigma_1(t, \tau)]d\tau \\ & + \int_0^t C_3(\tau) \exp\,[-\sigma_2(t, \tau)] \frac{\partial}{\partial\tau} U_0[x, \sigma_1(t, \tau)]d\tau \\ & + \frac{1}{2} \exp\,[-s_2(t)] \int_{-\pi}^{\pi} F(x - \xi) \frac{\partial}{\partial\xi} U_0[\xi, s_1(t)]d\xi. \end{aligned} \tag{3}$$

4. Verification of resolution. Instead of verifying directly that the function given by (3) is a solution of problem (A), we shall take up separately the verifications of the solutions U_1, U_2, and U_4 of the problems (A_1), (A_2), and (A_4), respectively.

Consider problem (A_1) and its solution $U_1(x, t)$ given by (2). By making use of the definitions of extensions, we can write $U_1(x, t)$ as follows:

$$\begin{aligned} U_1(x, t) = {} & \frac{1}{2}\int_0^t \exp\,(-\sigma_2) \\ & \cdot \int_0^{\pi} [P(x + \xi, \tau) + P(x - \xi, \tau)] \frac{\partial}{\partial\xi} U_0(\xi, \sigma_1)d\xi d\tau. \end{aligned}$$

Integration by parts with respect to ξ yields

$$U_1(x, t) = \frac{1}{2}\int_0^t \exp\,(-\sigma_2)H(x, t, \tau)d\tau, \tag{a}$$

where

$$H(x, t, \tau) = \int_0^{\pi} G(x, \xi, \tau)U_0[\xi, \sigma_1(t, \tau)]d\xi, \tag{b}$$

and

$$\text{(c)} \qquad G(x, \xi, \tau) = \left[\frac{\partial}{\partial x} P(x - \xi, \tau) - \frac{\partial}{\partial x} P(x + \xi, \tau)\right].$$

The function $U_0(x, t)$, the solution of problem (B), can be found by an inverse sine transformation of $u_0(n, t)$, or by the usual methods.[5] However, this problem is a special case of a more general problem solved by R. V. Churchill[6] by means of the Laplace transformation. The following properties of $U_0(x, t)$ can be derived from his paper, and are given here without the proofs: $U_0(x, t)$ is of order $O[\exp(\gamma t)]$ uniformly in x, $0 \leqq x \leqq \pi$, for all $t \geqq 0$, for some $\gamma > 0$; $U_0(x, t)$ is continuous in (x, t) when $t \geqq 0$, $0 < x \leqq \pi$; U_0, $\partial U_0/\partial x$, $\partial^2 U_0/\partial x^2$, and $\partial U_0/\partial t$ are continuous in (x, t) for each $t > 0$, $0 \leqq x \leqq \pi$; $|\partial U_0/\partial t|$ is bounded when $t \geqq 0$ and x is in the interval $0 < x_0 \leqq x \leqq \pi$.

That $H(x, t, \tau)$, $\partial H(x, t, \tau)/\partial t$, and $\partial^2 H(x, t, \tau)/\partial x^2$ are continuous functions of (x, t, τ) in R', $0 \leqq x \leqq \pi$, $0 \leqq \tau \leqq t$, can be established as follows:

$F \equiv G U_0$ is a continuous function of $(x, t, \tau, \xi) \equiv (\rho, \xi)$ in the closed region $R'': R'$, $\delta_1 \leqq \xi \leqq \pi$, $\delta_1 > 0$. Hence $|\Delta F| < \epsilon_1$ for $|\Delta \rho| < \delta'$ in R''. F is uniformly bounded and integrable in the closed region $R''': R'$, $0 \leqq \xi \leqq \delta_1$. Hence $|F| < M$ in R'''. So that

$$\begin{aligned} |H(\rho + \Delta\rho) - H(\rho)| &\leqq \int_0^{\delta_1} |\Delta F|\, d\xi + \int_{\delta_1}^{\pi} |\Delta F|\, d\xi \\ &\leqq M\delta_1 + \epsilon_1(\pi - \delta_1) < \epsilon \end{aligned}$$

for $|\Delta\rho| < \delta < \delta'$, ρ in R'. This establishes the continuity of $H(x, t, \tau)$.

Consider

$$\frac{\partial}{\partial t} H(x, t, \tau) = \frac{\partial}{\partial t} \int_0^{\pi} G(x, \xi, \tau) U_0[\xi, \sigma_1(t, \tau)]\, d\xi.$$

The derivative of the integral with respect to t yields formally

$$\begin{aligned} \frac{\partial}{\partial t} H(x, t, \tau) &= \int_0^{\pi} G(x, \xi, \tau) \frac{\partial}{\partial t} U_0[\xi, \sigma_1(t, \tau)]\, d\xi \\ &= C_1(t) \int_0^{\pi} G(x, \xi, \tau) \frac{\partial^2}{\partial \xi^2} U_0(\xi, \sigma_1)\, d\xi, \end{aligned}$$

$$\text{(d)} \qquad \frac{\partial}{\partial t} H(x, t, \tau) = C_1(t) \int_0^{\pi} \frac{\partial^2}{\partial \xi^2} G(x, \xi, \tau) U_0(\xi, \sigma_1)\, d\xi.$$

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[5] R. V. Churchill, *Fourier series and boundary value problems*, New York, McGraw-Hill Book Company, 1941, p. 102.

[6] R. V. Churchill, *On the problem of temperatures in a non-homogeneous bar with discontinuous initial temperatures*, Amer. J. Math. vol. 61 (1939) pp. 651–664.

This integral is of the type $H(x, t, \tau)$, therefore $\partial H(x, t, \tau)/\partial t$ is a continuous function of (x, t, τ) in R', provided $\partial H(x, t, t)/\partial t$ is defined by (d). The continuity of $\partial^2 H(x, t, \tau)/\partial x^2$ can be established by the same type of argument.

Consider $U_1(x, t)$ as defined by formula (a). Since the integrand is a bounded, integrable function of τ for all $t \geqq 0$, the limit of $U_1(x, t)$, as t approaches zero exists and $U_1(x, +0) = 0$.

Since $G(x, \xi, \tau)$, in (c), vanishes when x approaches both end points of $(0, \pi)$, it follows that $H(0, t, \tau) = H(\pi, t, \tau) = 0$. Hence $U_1(+0, t) = 0$ and $U_1(\pi - 0, t) = 0$.

To show that $U_1(x, t)$ satisfies the differential equation of problem (A_1), we apply Leibnitz' rule to $U_1(x, t)$, in (a), and find that

$$\begin{aligned}\frac{\partial}{\partial t} U_1(x, t) &= \frac{1}{2} \lim_{\tau \to t} \exp(-\sigma_2) H(x, t, \tau) \\ &\quad + \frac{1}{2} \int_0^t \frac{\partial}{\partial t} \{\exp[-\sigma_2(t, \tau)] H(x, t, \tau)\} d\tau \\ &= \frac{1}{2} \int_0^\pi G(x, t, t) U_0(\xi, 0) d\xi - C_2(t) U_1(x, t) \\ &\quad + \frac{1}{2} \int_0^t \exp(-\sigma_2) \frac{\partial}{\partial t} H(x, t, \tau) d\tau.\end{aligned}$$

Since $\partial^2 G/\partial x^2 = \partial^2 G/\partial \xi^2$, we have from (d) that $\partial H/\partial t = C_1 \partial^2 H/\partial x^2$. Hence

$$\begin{aligned}\frac{\partial}{\partial t} U_1(x, t) &= P(x, t) - C_2(t) U_1(x, t) \\ &\quad + C_1(t) \frac{1}{2} \int_0^t \exp(-\sigma_2) \frac{\partial^2}{\partial x^2} H(x, t, \tau) d\tau,\end{aligned}$$

so that

$$\frac{\partial}{\partial t} U_1(x, t) = P(x, t) - C_2(t) U_1(x, t) + C_1(t) \frac{\partial^2}{\partial x^2} U_1(x, t),$$

$$0 < x < \pi,\ t > 0.$$

Consider problem (A_2) and its solution $U_2(x, t)$ given by (2). Since $C_3(0) = 0$, integration by parts of $U_2(x, t)$ yields

$$\begin{aligned}U_2(x, t) &= C_3(t)((\pi - x)/\pi) \\ &\quad - \int_0^t \exp[-\sigma_2(t, \tau)] U_0[x, \sigma_1(t, \tau)] \{C_3'(\tau) + C_2(\tau) C_3(\tau)\} d\tau,\end{aligned} \tag{e}$$

so that $U_2(x, +0)=0$, $0<x<\pi$.

Now $|U_0|<\epsilon'$ when $0<x<\delta'$, $0\leqq\tau\leqq t-\epsilon''$; $|U_0|<|(\pi-x)/\pi+\epsilon'''|$ when $0<x<\delta''$, $t-\epsilon''\leqq\tau\leqq t$; so that if $E\equiv\exp(-\sigma_2)[C_3'+C_2C_3]$, $|\max E|=M$, then

$$
\begin{aligned}
|U_2(x, t)-C_3(t)| &\leqq |C_3/\pi|\,x+\left|\int_0^t EU_0d\tau\right| \\
&\leqq |C_3|\,x+\left|\int_0^{t-\epsilon''} M\epsilon'd\tau\right| \\
&\quad+\left|\int_{t-\epsilon''}^t M\,|(\pi-x)/\pi+\epsilon'''|\,d\tau\right| \\
&\leqq |C_3|\,x+M(t-\epsilon'')\epsilon'+M\,|1+\epsilon'''|\,\epsilon''<\epsilon
\end{aligned}
$$

for $0<x<\delta$, $\delta<\delta'$, δ'', $0<t\leqq T$. Hence $U_2(+0, t)=C_3(t)$, $t>0$. The same type of argument gives $U_2(\pi-0, t)=0$, since $U_0(\pi-0, t)=0$, $t>0$.

In (e), for $t>0$, $0<x<\pi$, first we take the derivative of $U_2(x, t)$ with respect to t, and then we take the second derivative of $U_2(x, t)$ with respect to x. We find that

$$
\begin{aligned}
\partial U_2(x, t)/\partial t &= C_3'(t)((\pi-x)/\pi)-((\pi-x)/\pi)[C_3'(t)+C_2(t)C_3(t)] \\
&\quad+C_2(t)\int_0^t \exp(-\sigma_2)U_0(x, \sigma_1)[C_3'(\tau)+C_2(\tau)C_3(\tau)]d\tau \\
&\quad-C_1(t)\int_0^t \exp(-\sigma_2)(\partial U_0(x, \sigma_1)/\partial\sigma_1)[C_3'(\tau)+C_2(\tau)C_3(\tau)]d\tau,
\end{aligned}
$$

and

$$
\begin{aligned}
&\partial^2U_2(x, t)/\partial x^2 \\
&\quad=-\int_0^t \exp(-\sigma_2)(\partial^2U_0(x, \sigma_1)/\partial x^2)[C_3'(\tau)+C_2(\tau)C_3(\tau)]d\tau.
\end{aligned}
$$

We form the partial differential expression $L(U_2)$:

$$
\begin{aligned}
L(U_2) &= \partial U_2/\partial t-C_1(t)\partial^2U_2/\partial x^2+C_2(t)U_2 \\
&= C_1(t)\int_0^t \exp(-\sigma_2)[\partial^2U_0(x, \sigma_1)/\partial x^2 \\
&\qquad\qquad-\partial U_0(x, \sigma_1)/\partial\sigma_1][C_3'+C_2C_3]d\tau.
\end{aligned}
$$

Hence $L(U_2)=0$, since $\partial^2U_0(x, \sigma_1)/\partial x^2-\partial U_0(x, \sigma_1)/\partial\sigma_1=0$, $0<x<\pi$, $t>0$.

The verification of $U_4(x, t)$ can be patterned after that used for $U_1(x, t)$ and, therefore, will not be discussed here.

This concludes the verification of the resolution of problem (A). The restrictions placed upon the functions $F(x)$ and $P(x, t)$ are unnecessarily severe. They were made in order to be able to condense the forms of the solutions. The same applies to the condition that $C_3(0) = C_4(0) = 0$. The verification of U_3 and U_4 can be made without them.

5. **Temperature distribution in a cube.** The method of the previous problem can be extended to a general heat flow problem in three dimensions; the same problem (B) is used in the resolution. The verification of the resolution of this problem is similar to that of the first problem and is omitted here.

The interior of a cube[7] of edge length π is filled with a homogeneous, isotropic medium. The temperatures on the faces of the cube are maintained at prescribed values, depending upon the space coordinates of the point and upon the time. There is a continuous source of heat in the medium whose thermal conductivity may be a function of the time. Further, at each point of the medium there is a sink of heat which may be a function of the time and which is proportional to the temperature at that point. The initial temperature distribution is given by a prescribed function. We take a corner of the cube as origin and the three mutually perpendicular edges as coordinate axes to locate a point $P: (x_1, x_2, x_3)$ of the cube.

Let us consider the distribution of temperatures $U(P, t)$ in the interior of such a cube. $U(P, t)$ satisfies the conditions:

$$\begin{aligned}
&\frac{\partial}{\partial t} U(P, t) - \sum_{j=1}^{3} C_j(t) \frac{\partial^2}{\partial x_j^2} U(P, t) - C_4(t) U(P, t) = F(P, t), \\
&\qquad\qquad\qquad\qquad\qquad\qquad 0 < x_j < \pi,\ t > 0, \\
&U(0, x_{2,3}, t) = G_1(x_{2,3}, t),\ U(\pi, x_{2,3}, t) = G_2(x_{2,3}, t),\ 0 \leqq x_{2,3} \leqq \pi,\ t > 0, \\
&U(0, x_{1,3}, t) = G_3(x_{1,3}, t),\ U(\pi, x_{1,3}, t) = G_4(x_{1,3}, t),\ 0 \leqq x_{1,3} \leqq \pi,\ t > 0, \\
&U(0, x_{1,2}, t) = G_5(x_{1,2}, t),\ U(\pi, x_{1,2}, t) = G_6(x_{1,2}, t),\ 0 \leqq x_{1,2} \leqq \pi,\ t > 0, \\
&\quad U(P, 0) = H(P), \qquad\qquad\qquad\qquad 0 < x_j < \pi\ (j = 1, 2, 3).
\end{aligned}$$

Problem (B), *which is a problem in heat conduction in one dimension, is sufficient to make a formal resolution of this very general problem of heat conduction in three dimensions.*

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[7] Compare with problem discussed by Carslaw: H. S. Carslaw, *Introduction to the mathematical theory of the conduction of heat in solids*, London, Macmillan and Company, 1921, p. 108.

If we use the same system of abbreviations as that used before, the solution $U(P, t)$ can be expressed as follows:

$$\begin{aligned} U(P, t) = {} & \frac{1}{8}\int_0^t \exp(-\sigma_4) \int\int\int_{-\pi}^{\pi} F(P_j, \tau) \prod_{j=1}^{3} \frac{\partial}{\partial \xi_j} U_0(\xi_j, \sigma_j) d\xi_j d\tau \\ & + \frac{1}{8} \exp(-s_4) \int\int\int_{-\pi}^{\pi} H(P_j) \prod_{j=1}^{3} \frac{\partial}{\partial \xi_j} U_0(\xi_j, s_j) d\xi_j \\ & + \frac{1}{4} \sum_{j=1}^{3} \int_0^t \exp(-\sigma_4) \frac{\partial}{\partial \tau} U_0(x_j, \sigma_j) \\ & \quad \cdot \int\int_{-\pi}^{\pi} G_{2j-1}(Q_j, \tau) \prod_{k \neq j=1}^{3} \frac{\partial}{\partial \xi_k} U_0(\xi_k, \sigma_k) d\xi_k d\tau \\ & + \frac{1}{4} \sum_{j=1}^{3} \int_0^t \exp(-\sigma_4) \frac{\partial}{\partial \tau} U_0(\pi - x_j, \sigma_j) \\ & \quad \cdot \int\int_{-\pi}^{\pi} G_{2j}(Q_j, \tau) \prod_{k \neq j=1}^{3} \frac{\partial}{\partial \xi_k} U_0(\xi_k, \sigma_k) d\xi_k d\tau, \end{aligned}$$

where $(P_j) \equiv (x_1 - \xi_1, x_2 - \xi_2, x_3 - \xi_3)$ and Q_j is of the form $(x_{j_1} - \xi_{j_1}, x_{j_2} - \xi_{j_2})$; $j_1, j_2 \neq j = 1, 2, 3$.

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A SHORT PROOF OF THE COMPLETENESS OF THE LAGUERRE FUNCTIONS

LYNN H. LOOMIS

The Laguerre functions $\phi_n(x)$, $n=0, 1, \cdots$, can be defined[1] as the coefficients of t^n in the power series expansion about $t=0$ of the generating function

$$F(x, t) = \exp\left(-\frac{x}{2}\left(\frac{1+t}{1-t}\right)\right) \Big/ (1-t).$$

THEOREM 1. *The functions* $\phi_n(x)$ *form an orthonormal set over* $(0, \infty)$.

The following proof is essentially well known.[2] From the definition of $\phi_n(x)$ we see that $\int_0^\infty \phi_m(x)\phi_n(x)dx$ is the coefficient of $t^m s^n$ in the power series expansion about $(t, s)=(0, 0)$ of

$$\int_0^\infty F(x, t)F(x, s)dx = \int_0^\infty \left(\exp\left(-x\frac{1-ts}{(1-t)(1-s)}\right) \Big/ (1-t)(1-s)\right)dx$$
$$= \frac{1}{1-ts} = \sum_{p=0}^\infty (ts)^p.$$

Hence $\int_0^\infty \phi_m(x)\phi_n(x)dx$ is 1 or 0 according as $m=n$ or $m\neq n$, as was to be proved.

The interchange of the order of integration and summation can be justified by first making the interchange for the finite integral and then letting the upper limit approach infinity.

THEOREM 2. *The functions* $\phi_n(x)$ *are complete* (*closed*) *in* $L^2(0, \infty)$.

We use a special case of the well known theorem[3] that if $f(z) = a_0+a_1z+a_2z^2+\cdots$ is analytic and bounded in $|z|<1$ with boundary function $f(e^{i\theta})$, then

$$(1) \qquad \int_0^{2\pi} |f(e^{i\theta})|^2 d\theta = 2\pi\sum_{n=0}^\infty |a_n|^2.$$

Since step functions are dense in $L^2(0, \infty)$, it is sufficient, in order to prove the completeness of the $\phi_n(x)$, to prove that the closed linear

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[1] See Kaczmarz and Steinhaus, *Theorie der Orthogonalreihen*, p. 140, line 1, (55) and [482].

[2] See, for example, G. Szegö, *Orthogonal polynomials*, Amer. Math. Soc. Colloquium Publications, vol. 23, p. 69.

[3] See Bieberbach, *Lehrbuch der Funktionentheorie*, vol. 1, p. 145.

space which they determine includes the step functions, hence that it includes the characteristic function $g_b(x)$ of the interval $0 \leqq x \leqq b$. Thus if a_n is the nth Laguerre coefficient of $g_b(x)$,

$$a_n = \int_0^\infty g_b(x)\phi_n(x)dx = \int_0^b \phi_n(x)dx, \tag{2}$$

we have to prove that

$$\int_0^\infty \left(g_b(x) - \sum_{n=0}^{N} a_n\phi_n(x)\right)^2 dx \to 0$$

as $N \to \infty$, that is, that

$$b = \sum_{n=0}^{\infty} |a_n|^2. \tag{3}$$

By (2) the Laguerre coefficients a_n of $g_b(x)$ are the coefficients of t^n in the expansion of

$$\begin{aligned} f(t) &= \int_0^b \left(\exp\left(-\frac{x}{2}\left(\frac{1+t}{1-t}\right)\right)\bigg/(1-t)\right)dx \\ &= -\frac{2}{1+t}\left[\exp\left(-\frac{b}{2}\left(\frac{1+t}{1-t}\right)\right) - 1\right]. \end{aligned}$$

This function is analytic in $|t| \leqq 1$ except at $t=1$, and is bounded in $|t| < 1$. Putting $t = e^{i\theta}$ we have

$$f(e^{i\theta}) = -(1 - i\tan(\theta/2))\left(\exp\left(-\frac{b}{2}i\cot\frac{\theta}{2}\right) - 1\right),$$

and the substitutions $y = \tan(\theta/2)$, $x = b/4y$ give

$$\int_0^{2\pi} |f(e^{i\theta})|^2 d\theta = 8\int_{-\infty}^{\infty} \sin^2(b/4y)dy = 2b\int_{-\infty}^{\infty} \frac{\sin^2 x}{x^2}dx = 2\pi b. \tag{4}$$

The desired result (3) follows from (1) and (4).

Harvard University

ON THE GROWTH OF SOLUTIONS OF LINEAR DIFFERENTIAL EQUATIONS

J. ERNEST WILKINS, JR.

1. **Introduction.** In a recent paper by Boas, Boas and Levinson [1][1] two sets of sufficient conditions were given for the existence of $\lim_{x\to\infty} y'(x)$ when $y(x)$ satisfies the differential equation

$$y'' + A(x)y = B(x). \tag{1:1}$$

We propose in this paper to use their methods and to generalize their results to the nth order linear differential equation

$$y^{(n)} + \sum_{i=1}^{n} A_i(x)y^{(n-i)} = B(x), \tag{1:2}$$

and to obtain sufficient conditions for

$$\lim_{x=\infty} y^{(n-1)}(x) \tag{1:3}$$

to exist. In case $n=2$, $A_1(x)=0$ and $A_2(x)=A(x)$, these conditions reduce to those in [1].

2. **Statements of the theorems.** In §4 we shall prove the following theorem.

THEOREM I. *If* $A_i(x)$ $(i=1, \cdots, n)$ *and* $B(x)$ *are continuous on* $0\leqq x<\infty$, *and if the integrals*

$$\int_0^\infty x^{i-1}|A_i(x)|\,dx \qquad (i = 1, \cdots, n), \tag{2:1}$$

$$\int_0^\infty B(x)dx \tag{2:2}$$

exist, then the limit (1:3) *exists for any solution* $y(x)$ *of* (1:2).

We now write each function $A_i(x)$ as the difference of two non-negative functions, $A_i(x)=A_i'(x)-A_i''(x)$, where $A_i'=(|A_i|+A_i)/2$, $A_i''=(|A_i|-A_i)/2$. Then in §5, $\cdots$, §8 we shall prove the following theorem.

THEOREM II. *If* $A_i(x)$ $(i=1, \cdots, n)$ *and* $B(x)$ *are continuous on* $0\leqq x<\infty$, *if the integrals*

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[1] Numbers in brackets refer to the Bibliography at the end of the paper.

$$\int_0^\infty x^{i-1}A_i''(x)dx \qquad (i = 1, \cdots, n), \tag{2:3}$$

$$\int_0^\infty B(x)dx \tag{2:4}$$

exist, and if we have

$$\limsup_{x=\infty} x^{i-k-1}\int_x^\infty t^k A_i'(t)dt < 2(i-k-1)!k!/n(n-1) \tag{2:5}$$

whenever $i=2$, $k=0$, *or* $i=2j-1$, $2j$, $k=i-j, \cdots, i-2$ $(j=2, \cdots, [(n+1)/2])$, *where we agree that* $A_{n+1}'(t)=0$ *if* n *is odd, then the limit* (1:3) *exists for any solution* $y(x)$ *of* (1:2). *If in addition we have*

$$\int_0^\infty \sum_{i=1}^n x^{i-1}A_i'(x)dx = \infty, \tag{2:6}$$

then $\lim_{x=\infty} y^{(n-1)}(x)=0$.

3. **Some auxiliary lemmas.** In this section we state four lemmas needed in proving the main theorems. The first of these is found in [1].

LEMMA 1. *If* $f(x)$ *is continuous on* $0 \leqq x < \infty$, *if* $M(x)$ *denotes the maximum of* $|f(t)|$ *on* $0 \leqq t \leqq x$, *and if, for some positive numbers* α *and* x_0, $|f(x)| \leqq \alpha + M(x)/2$ $(x \geqq x_0)$, *then* $f(x)$ *is bounded on* $0 \leqq x < \infty$.

LEMMA 2. *If* (2:5) *holds under the restrictions on* i *and* k *stated in Theorem* II, *then* (2:5) *also holds for* $i=2, \cdots, n$, $k=0, \cdots, i-2$.

This is manifestly true for $i=2$. If $i>2$ and if i is odd, then $i=2j-1$, $j \geqq 2$ and (2:5) holds if $k \geqq i-j=j-1$. Suppose now that $0 \leqq k<j-1$. Then $i-k-1=2j-2-k>j-1$, and

$$x^{i-k-1}\int_x^\infty t^k A_i'(t)dt \leqq x^{j-1}\int_x^\infty t^{j-1}A_i'(t)dt,$$

$$\begin{aligned}\limsup_{x=\infty} x^{i-k-1}\int_x^\infty t^k A_i'(t)dt &< 2(j-1)!^2/n(n-1)\\ &< 2(2j-2-k)!k!/n(n-1).\end{aligned}$$

The reasoning when i is even is quite similar.

LEMMA 3. *If* $f(x)$ *is continuous and bounded on* $0 \leqq x < \infty$, *and if the functions* $A_i(x)$ *satisfy the hypotheses of Theorem* I, *then all integrals of the form*

$$\int_0^\infty f(x)x^pA_i(x)dx, \qquad \int_0^\infty x^p\,|\,A_i(x)\,|\,dx \qquad (p = 0, \cdots, i-1)$$

exist. Under the hypotheses of Theorem II, *the same conclusion is valid provided that* $i=2, \cdots, n$, $p=0, \cdots, i-2$, *or that* $A_i(x)$ *be replaced by* $A_i'(x)$.

The proof of the first sentence of the lemma is immediate, and the second sentence follows similarly as soon as we refer to the preceding lemma.

LEMMA 4. *If* $y(x)$ *is of class* $C^{(m+q)}$ *on* $0 \leqq x < \infty$, *where* m *and* q *are non-negative integers, then*

$$q!\limsup_{x\to\infty} x^{-q}\,|\,y^{(m)}(x)\,| \leqq \limsup_{x\to\infty}\,|\,y^{(m+q)}(x)\,|. \tag{3:1}$$

To prove Lemma 4 we use Taylor's Theorem in the form

$$\begin{aligned} y^{(m)}(x) &= \sum_{k=0}^{q-1}(x-x_0)^k y^{(m+k)}(x_0)/k! \\ &\quad + \int_{x_0}^{x}\int_{x_0}^{t_{q-1}}\cdots\int_{x_0}^{t_1} y^{(m+q)}(t_0)dt_0dt_1\cdots dt_{q-1}. \end{aligned} \tag{3:2}$$

Let $M=\limsup_{x\to\infty}|y^{(m+q)}(x)|$ and pick $e>0$. Then take x_0 so large that

$$|\,y^{(m+q)}(x)\,| < M + e \qquad (x \geqq x_0).$$

It follows from (3:2) that if $x \geqq x_0$

$$|\,y^{(m)}(x)\,| \leqq \sum_{k=0}^{q-1} x^k\,|\,y^{(m+k)}(x_0)\,|\,/k! + (M+e)x^q/q!,$$

$$q!\limsup_{x\to\infty} x^{-q}\,|\,y^{(m)}(x)\,| \leqq M + e.$$

Since e is arbitrary, the statement of the lemma follows at once.

4. **Proof of Theorem I.** By virtue of (2:1) we can pick x_0 such that

$$\int_{x_0}^\infty \sum_{i=1}^n x^{i-1}\,|\,A_i(x)\,|\,/(i-1)!dx < 1/2. \tag{4:1}$$

If in (1:2) we substitute for $y^{(n-i)}$ $(i=2, \cdots, n)$ the values obtained from (3:2) by replacing m by $n-i$ and q by $i-1$, and solve the resulting equation for $y^{(n)}$, we get an equation which upon integration between the limits x_0 and x gives

$$
\begin{aligned}
y^{(n-1)}(x) = y^{(n-1)}(x_0) &- \sum_{i=2}^{n}\sum_{k=0}^{i-2}\int_{x_0}^{x} A_i(t)\,\frac{(t-x_0)^k}{k!}\,y^{(n-i+k)}(x_0)dt \\
&- \sum_{i=1}^{n}\int_{x_0}^{x}\int_{x_0}^{t_{i-1}}\cdots\int_{x_0}^{t_1} A_i(t_{i-1})y^{(n-1)}(t_0)dt_0\cdots dt_{i-1} \qquad (4:2)\\
&+ \int_{x_0}^{x} B(t)dt.
\end{aligned}
$$

Define the quantities B and α and the function $M(x)$ by the equations

$$
\begin{aligned}
B &= \max_{x_0 \leqq x \leqq \infty}\left|\int_{x_0}^{x} B(t)dt\right|, \\
\alpha &= \left| y^{(n-1)}(x_0)\right| \\
&\quad + \sum_{i=2}^{n}\int_{x_0}^{\infty}\left| A_i(t)\right| \sum_{k=0}^{i-2}\frac{t^k}{k!}\left| y^{(n-i+k)}(x_0)\right| dt + B, \qquad (4:3)\\
M(x) &= \max_{0\leqq t\leqq x}\left| y^{(n-1)}(t)\right|.
\end{aligned}
$$

B exists by virtue of (2:2) and α exists by virtue of Lemma 3. From (4:2) and (4:1) we now get

$$
\left| y^{(n-1)}(x)\right| \leqq \alpha + M(x)/2 \qquad (x \geqq x_0). \qquad (4:4)
$$

It follows from Lemma 1 that $y^{(n-1)}(x)$ is bounded on $0 \leqq x < \infty$. In this event we use Lemma 3 to see that the integrals involving $A_i(t)$ on the right side of (4:2) approach limits as $x \to \infty$. By (2:2) the integral of $B(x)$ approaches a limit. Therefore, $y^{(n-1)}(x)$ has a limit, proving Theorem I.

5. **Proof of Theorem II when $y^{(n-1)}(x)$ does not change sign for large values of x.** Then we may assume without loss of generality that x_0 is so large that $y^{(n-1)}(x) \geqq 0$ for $x \geqq x_0$ and that

$$
\int_{x_0}^{\infty}\sum_{i=1}^{n} x^{i-1}A_i''(x)/(i-1)!dx < 1/2. \qquad (5:1)
$$

Since $y^{(n-1)}(t) \geqq 0$ on $t \geqq x_0$ and $-A_i(t) = A_i''(t) - A_i'(t) \leqq A_i''(t)$, we then have from (4:2) and (5:1) that (4:4) holds, the integrals in α existing by virtue of the second part of Lemma 3. It follows from Lemma 1 that $y^{(n-1)}(x)$ is bounded. Set $A_i = A_i' - A_i''$ in (4:2). Since $y^{(n-1)}(x)$ is bounded we see from Lemma 2 that all of the terms in (4:2) on the right side approach limits as $x \to \infty$ with the possible exception of

$$(5:2)\qquad -\sum_{i=1}^{n}\int_{x_0}^{x}\int_{x_0}^{t_{i-1}}\cdots\int_{x_0}^{t_1} A_i'(t_{i-1})y^{(n-1)}(t_0)dt_0\cdots dt_{i-1}.$$

Since $A_i' \geqq 0$, $y^{(n-1)}(t_0) \geqq 0$ for $t_0 \geqq x_0$, this term is a nonincreasing function of x which is bounded below since all the other terms in (4:2) are bounded. Hence it also approaches a limit. Therefore, the limit (1:3) exists.

6. Proof of Theorem II when $y^{(n-1)}(x)$ changes sign infinitely many times. Suppose first that $y^{(n-1)}(x)$ is bounded but that the limit (1:3) does not exist. Then we may assume without loss of generality that

$$\limsup |\, y^{(n-1)}(x)\,| = \limsup y^{(n-1)}(x) = M > 0.$$

Let x_m be a monotone sequence of points such that $x_m \to \infty$, $y^{(n-1)}(x_m) > 0$, $y^{(n-1)}(x_m) \to M$. Let a_m be the first point to the left of x_m such that $y^{(n-1)}(a_m) = 0$. We can suppose that a_1 is so large that for some $c < 1$ and for $i = 2, \cdots, n$, $k = 0, \cdots, i-2$ we have

$$(6:1)\qquad x^{i-1-k}\int_x^{\infty} t^k A_i'(t)dt < 2c(i-1-k)!k!/n(n-1) \qquad (x \geqq a_1).$$

By (4:2) with x_0 replaced by a_m and x replaced by x_m we have, if we observe that $y^{(n-1)}(t) \geqq 0$ on $a_m \leqq t \leqq x_m$,

$$(6:2)\qquad \begin{aligned} y^{(n-1)}(x_m) \leqq{} & \sum_{i=1}^{n}\int_{a_m}^{x_m}\int_{a_m}^{t_{i-1}}\cdots\int_{a_m}^{t_1} A_i''(t_{i-1})y^{(n-1)}(t_0)dt_0\cdots dt_{i-1} \\ & + \sum_{i=2}^{n}\sum_{k=0}^{i-2}\int_{a_m}^{x_m}\frac{t^k}{k!}A_i''(t)\,|\, y^{(n-i+k)}(a_m)\,|\,dt \\ & + \sum_{i=2}^{n}\sum_{k=0}^{i-2}\int_{a_m}^{x_m}\frac{t^k}{k!}A_i''(t)\,|\, y^{(n-i+k)}(a_m)\,|\,dt \\ & + \left|\int_{a_m}^{x_m} B(t)dt\right|. \end{aligned}$$

Since $y^{(n-1)}(t)$ is bounded, we see from Lemma 3 that the first sum on the right of (6:2) approaches zero as $m \to \infty$. By virtue of (2:4) so does the last term in (6:2). If we use Lemma 4 we discover that the upper limit of the third sum in (6:2) can not exceed

$$\begin{aligned} & \limsup_{m=\infty}\sum_{i=2}^{n}\sum_{k=0}^{i-2}\frac{|\, y^{(n-i+k)}(a_m)\,|}{k!a_m^{i-1-k}}\int_{a_m}^{x_m} t^{i-1}A_i''(t)dt \\ & \qquad \leqq \limsup_{x=\infty}\sum_{i=2}^{n}\sum_{k=0}^{i-2}\frac{|\, y^{(n-1)}(x)\,|}{k!(i-1-k)!}\int_{x}^{\infty} t^{i-1}A_i''(t)dt = 0. \end{aligned}$$

Finally we use (6:1) and Lemma 4 to see that the upper limit of the second sum in (6:2) cannot exceed

$$\limsup_{m=\infty} \sum_{i=2}^{n} \sum_{k=0}^{i-2} \int_{a_m}^{\infty} \frac{t^k}{k!} A_m'(t) \left| y^{(n-i+k)}(a_m) \right| dt$$

$$\leqq \limsup_{x=\infty} \sum_{i=2}^{n} \sum_{k=0}^{i-2} \frac{2c(i-1-k)! \left| y^{(n-i+k)}(x) \right|}{n(n-1)x^{i-1-k}}$$

$$\leqq \limsup_{x=\infty} \sum_{i=2}^{n} \sum_{k=0}^{i-2} 2c \left| y^{(n-1)}(x) \right| / n(n-1) = cM < M.$$

Referring to (6:2) we see that we have reached a contradiction of our choice of the points x_m.

7. **Proof that $y^{(n-1)}(x)$ must be bounded.** To complete the proof of the first part of Theorem II, it is sufficient to prove that $y^{(n-1)}(x)$ must be bounded under the hypotheses (2:3), (2:4), (2:5) and the assumption that $y^{(n-1)}(x)$ changes sign infinitely many times. Suppose on the contrary that $y^{(n-1)}$ is unbounded. Then we can pick a sequence $x_m \to \infty$ such that

$$\left| y^{(n-1)}(x_m) \right| \geqq \left| y^{(n-1)}(x) \right| \qquad (x \leqq x_m), \tag{7:1}$$

$y^{(n-1)}(x_m)$ has the same sign, which we may suppose to be positive, and $y^{(n-1)}(x_m) \to \infty$. Let a_m be defined for x_m as in §6, and suppose that a_1 is so large that (6:1) holds. Using (7:1) and Taylor's Theorem (3:2) with m replaced by $n-i+k$, q replaced by $i-1-k$, x_0 replaced by 0, and x replaced by a_m, we find that

$$\left| y^{(n-i+k)}(a_m) \right| \leqq y^{(n-1)}(x_m) \frac{a_m^{i-k-1}}{(i-k-1)!} + \sum_{h=0}^{i-k-2} \left| y^{(n-i+k+h)}(0) \right| \frac{a_m^h}{h!}.$$

It now follows from (6:2) and (6:1) that

$$y^{(n-1)}(x_m) \leqq y^{(n-1)}(x_m) \left\{ \sum_{i=1}^{n} \int_{a_m}^{\infty} \frac{t^{i-1} A_i''(t)}{(i-1)!} dt + c \right.$$

$$\left. + \sum_{i=2}^{n} \sum_{k=0}^{i-2} \int_{a_m}^{\infty} \frac{t^{i-1} A_i''(t)}{k!(i-k-1)!} dt \right\} + \left| \int_{a_m}^{x_m} B(t) dt \right|$$

$$+ \sum_{i=2}^{n} \sum_{k=0}^{i-2} \sum_{h=0}^{i-k-2} \frac{\left| y^{(n-i+k+h)}(0) \right|}{k!h!} \int_{a_m}^{\infty} t^{k+h} A_i(t) dt.$$

Since all of the integrals on the right of this last inequality approach zero as $m \to \infty$ and $0 < c < 1$, we reach a contradiction.

8. Proof of the second part of Theorem II. Suppose on the contrary that $y^{(n-1)}(x)$ does not approach zero. Without loss of generality we may assume that

$$\lim_{x=\infty} y^{(n-1)}(x) = 2a > 0.$$

Then there exists an x_0 such that

$$3a > y^{(n-1)}(x) > a \qquad (x \geqq x_0). \tag{8:1}$$

Now set $A_i = A_i' - A_i''$ in (4:2) and let $x \to \infty$. Then all of the terms on the right approach limits with the possible exception of the term (5:2). Since $y^{(n-1)}(x)$ approaches a limit, so must (5:2). But by (8:1) we have

$$\int_{x_0}^{t_{i-1}} \cdots \int_{x_0}^{t_1} y^{(n-1)}(t_0)dt_0 \cdots dt_{i-2} > a(t_{i-1} - x_0)^{i-1}/(i-1)!.$$

Consequently, the term (5:2) is greater than

$$a\int_{x_0}^{x} \sum_{i=1}^{n} (t - x_0)^{i-1}A_i'(t)/(i-1)!dt.$$

By (2:6) and Lemma 3 this last integral becomes infinite as $x \to \infty$, so that (5:2) cannot approach a limit. This contradiction completes the proof of Theorem II.

Added in proof. Since the submission of this paper to the editors, it has come to the author's attention that Theorem I was proved by Otto Haupt, *Über das asymptotische Nerhalten der Lösungen gewisser linearer gewöhnlicher Differentialgleichungen*, Math. Zeit. vol. 48 (1942) pp. 282–292. Our proof, based on Lemma 1, seems distinctly simpler and certainly more elementary than that of Haupt. To the best of our present knowledge, Theorem II is new.

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TUSKEGEE INSTITUTE

LIMIT POINTS OF SUBSEQUENCES

R. CREIGHTON BUCK[1]

1. **Introduction.** In a previous paper [2],[2] it was shown that for simple sequences of real numbers, divergence of a sequence implies divergence of almost every subsequence. The proof given there required in an essential way that the space be metric. The purpose of this note is to show that the above result holds for multiple sequences in an L^* space. If the space is compact separable metric, even more is true: *the set of limit points of almost every subsequence coincides with the set of limit points of the original sequence.*

2. **Notation.** We denote by $x=x[n]=x[n_1, n_2, \cdots, n_r]$, where $n_k=1, 2, \cdots$, an arbitrary r-tuple sequence with terms in a space M. Likewise, $x'=x_\lambda[n]=x[\lambda_1(n_1), \lambda_2(n_2), \cdots, \lambda_r(n_r)]$ will denote an arbitary subsequence of x; here, for any fixed k, $\lambda_k(1), \lambda_k(2), \cdots$ form an increasing sequence of integers. This is the natural generalization of subsequences x_{λ_n} of the simple sequence x_n.

Let $\mathcal{S}$ be the set of all sequences $s=(s_\alpha)$ composed of 0's and 1's, containing infinitely many 1's. This represents in the usual manner the class of subsequences of a simple sequence [1, p. 788] that is $s_n=1$ if x_n is chosen, and 0 if x_n is omitted. A product measure can be defined in $\mathcal{S}$ [4, p. 420; 5, p. 144].[3]

Then

$$\mathfrak{S} = \mathcal{S} \times \mathcal{S} \times \cdots \times \mathcal{S} \qquad (r \text{ factors})$$

is the class of all subsequences of an r-tuple sequence. Measure is defined in $\mathfrak{S}$ as the product measure over $\mathcal{S}$.

We assume that some definition of convergence is given for sequences in M. Limit points are then defined as

$$Px = p[\lim x' = p \text{ for some subsequence } x' \text{ of } x].$$

We recall the defining conditions of the Fréchet limit spaces [3, p. 77]:

(L) $\lim x=p$ *implies* $\lim x'=p$ *for every subsequence* x' *of* x.

(L^*) $\lim x=p$ *if and only if* $p\in Px'$ *for every subsequence* x' *of* x.

L^* is the "star-convergence" of Urysohn.

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[1] Society of Fellows, Harvard University.

[2] Numbers in brackets refer to the references listed at the end of the paper.

[3] $\mathcal{S}\subseteq\mathcal{S}^*=S\times S\times S\times S\times\cdots$, where $S=\{0, 1\}$. Both $\mathcal{S}$ and $\mathcal{S}^*$ have measure 1. $\mathcal{S}$ can be mapped, with measure preserved, on the real numbers $0<t\leqq 1$.

3. Limit points. The following lemma is essential.

LEMMA. *Almost every $s \in \mathcal{S}$ coincides with any fixed $t \in \mathcal{S}$ in an infinite number of* 1 *coordinates.*

Set $s=(s_\alpha)$ and $t=(t_\alpha)$; t_α is 1 for an infinite set of α's. Let

$$F_n = s[s_\alpha t_\alpha = 0 \text{ for all } \alpha > n],$$
$$C_n = s\left[\sum s_\alpha t_\alpha = n\right].$$

Then F_n is an elementary subset of $\mathcal{S}$, and furthermore $m(F_n)=0$; since $C_n \subseteq \bigcup_{k\geq n} F_k$, $m(C_n)=0$. If $C=\bigcup C_n$, then $m(C)=0$ so that $m(C')=1$ where

$$C' = s\left[\sum s_\alpha t_\alpha = \infty\right].$$

THEOREM 1. *If M is an L space and $p \in Px$, then $p \in Px'$ for almost every subsequence x' of x.*

If $p \in Px$, then for some subsequence x'' of x, $\lim x''=p$. Let τ be the point of $\mathfrak{S}$ corresponding to x'', and $\tau=(t^1, t^2, t^3, \cdots, t^r)$. By the lemma, for each k there is a set Σ_k having measure 1, composed of points $s^k=(s^k_\alpha)$ of $\mathcal{S}$ whose coordinates coincide with those of $t^k=(t^k_\alpha)$ at an infinite number of 1's. Then

$$\Sigma = \Sigma_1 \times \Sigma_2 \times \cdots \times \Sigma_r$$

is a subset of $\mathfrak{S}$ composed of points $\sigma=(s^1, s^2, s^3, \cdots, s^r)$ each of which coincides with τ in an infinite number of components. Σ then corresponds to a set of subsequences x' of x each of which has a subsequence in common with x''. By (L), this common subsequence converges to p, and $p \in Px'$. This is true for almost every subsequence x', for $m(\Sigma)=1$.

THEOREM 2. *If M is an L^* space with unique limits, then if x is divergent, so is almost every subsequence x' of x.*[4]

Either Px is void, or $Px=\{p\}$, or Px contains at least two distinct points. If Px is void, then *every* subsequence of x is divergent. If Px contains distinct points p and q, then by the previous theorem, these are limit points of almost every subsequence, and since limits are unique, these are in turn divergent. Finally, if x is divergent and $Px=\{p\}$, there must be a subsequence x'' of x for which Px'' is void. As in the proof of Theorem 1, almost every subsequence x' of x has a subsequence in common with x'', and must therefore be divergent.

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[4] The statements of this and the following theorem were altered at the suggestion of Dr. M. M. Day.

We now suppose that closure of subsets of M is defined.

THEOREM 3. *If M is an L space and Px is separable, then $\bar{P}x' = \bar{P}x$ for almost every subsequence x' of x.*

Choose a countable dense set $E \subset Px$. Let A_p be the class of subsequences of x having p as a limit point. Theorem 1 states that if $p \in Px$, $m(A_p) = 1$. Set $A = \bigcap_{p \in E} A_p$; since E is countable, $m(A) = 1$. Thus, for almost every subsequence x' of x, $E \subset Px' \subset Px$. Closing these sets, we have the theorem.

If M is a neighborhood space, and we define convergence in the usual way, M becomes an L^* space. Thus, Theorem 1 holds for multiple sequences in any topological space. If, further, M obeys the Hausdorff separation axiom, limits are unique and Theorem 2 holds. If M is a topological space with second countability, then Px is separable, and Theorem 3 holds.

COROLLARY 1. *If M is a topological space with second countability, such that the interior of any neighborhood of a point contains a neighborhood of that point, then for simple sequences x, $Px = Px'$ for almost all subsequences x' of x.*

The additional assumption implies that for simple sequences, the set of limit points is closed. (It might be noted that, as subsequences and limit points are here defined, the set of limit points of a multiple sequence need not be closed!)

COROLLARY 2. *If, in addition to the assumptions above, M is also a group, then almost every bracketing of the series $\sum a_n$ leaves the set of limit points unchanged.*

As in [2], bracketing a series is equivalent to selecting a subsequence.

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HARVARD UNIVERSITY

CLUSTER POINTS OF SUBSEQUENCES[1]

M. M. DAY

In the preceding paper [1][2] Buck defines a class of "subsequences" of a multiple sequence and shows that "almost all" of such subsequences have certain properties. This note is essentially based on a different choice of the definition of "subsequences"; that is, this paper and [1] are generalizations in different directions of a preceding paper by Buck and Pollard (reference 2 of [1]). In this discussion countability is the important property of the index systems such as the integers underlying the simple sequences or the n-tuples of integers underlying the multiple sequences. Countability is a slightly stronger condition than is necessary since the results will be shown to hold as well for functions of n variables as for multiple sequences; some other special cases are mentioned at the end of this paper. Also I modify Buck's approach by considering cluster points in neighborhood spaces rather than limit points in convergence spaces [3]. It may be mentioned that even for multiple sequences Theorems 1 and 2 of these papers are independent since Buck's set of "subsequences" is a set of measure zero in the set of "subsequences" considered here; my Theorem 3 contains the corresponding theorem of [1] as a special case. Lemma 1 and its corollary, Lemma 3, are the fundamental results on which the theorems rest; Lemma 3 is the generalization appropriate to this paper of the lemma in §3 of [1].

1. **Preliminaries.** If R is any set, a product measure can be defined in the set of characteristic functions of subsets of R [1, footnote 2] and this in turn induces a measure $|\cdots|$ for subsets of the set $\mathcal{E}$ of all subsets E of R; this measure is non-negative, completely additive, and (if R is infinite) takes all values between 0 and 1 inclusive; its other principal characteristic is that if $r_1, \cdots, r_k \in R$, then[3] $\{E \mid \text{no } r_i \in E\}$ is of measure 2^{-k}; hence if E_0 is an infinite subset of R and $A = \{E \mid E \cap E_0 \text{ is empty}\}$, $|A| = 0$.

An *index system* $\mathfrak{R} = (R, \geqq)$ is a set R and a binary relation $\geqq$ such that $\geqq$ is transitive and every element r_0 has a successor $r_1 > r_0$ such that $r_0 \not> r_1$. (In the language of [4] $\mathfrak{R}$ is oriented and has no terminal

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[1] Considerations suggested by the preceding paper of R. C. Buck.

[2] Numbers in brackets refer to the Bibliography at the end of the paper.

[3] The usual notation of $\cup$ and $\cap$ will be used for union and intersection of sets; $\{p \mid P\}$ will mean the set of all p having the property P.

elements.) A set E in $\mathfrak{R}$ is called *cofinal* in $\mathfrak{R}$ if for every r in R there exists $r' \geqq r$ with r' in E. Let $E^* = \{r \mid r \geqq \text{some } r' \text{ in } E\}$.

Note that if $\mathfrak{R}$ is the system of integers ordered by magnitude, then the cofinal subsets of $\mathfrak{R}$ are the infinite subsets; for g defined on a general index system $\mathfrak{R}$ it is clear that reducing the domain of definition of g to a cofinal subset E of $\mathfrak{R}$ is a generalization of the process of selecting a subsequence in case $\mathfrak{R}$ is the system of integers.

A subsystem $\mathfrak{R}' = (R', \geqq)$ of $\mathfrak{R} = (R, \geqq)$ is a subset R' of R with the order relation between points of R' defined by that in R; if R' is cofinal in the index system $\mathfrak{R}$, then $(R', \geqq)$ is also an index system. Cofinality is transitive in a transitive system; that is, if R' is cofinal in $(R, \geqq)$ and R'' is cofinal in $(R', \geqq)$, then R'' is cofinal in $(R, \geqq)$.

We may note that if $\mathfrak{R}$ is the set of n-tuples of integers (the case studied in [1]), where $(i_1, \cdots, i_n) \geqq (j_1, \cdots, j_n)$ if and only if $i_k \geqq j_k$ for every $k \leqq n$, then the product subsets defined by Buck are cofinal in $\mathfrak{R}$ but are very sparsely distributed in the set of all cofinal subsets of $\mathfrak{R}$; to be precise, such sets form a set of measure 0 if $n \geqq 2$. A product set in the set of n-tuples of integers, $R = I \times I \times \cdots \times I$, is a set of the form $E_1 \times E_2 \times \cdots \times E_n$, $E_i \subset I$; these product sets define the class of "subsequences" used by Buck. If E_k is the set of elements of R with all coordinates not greater than k and if A_k is the class of all subsets E of R such that $E \cap E_k$ is a product set in E_k, then $A = \bigcap_k A_k$. It is easily seen that if $E' \subset E_k$, $\{E \mid E \cap E_k = E'\}$ is of measure 2^{-k^n}; since there are $(2^k - 1)^n + 1 < 2^{nk}$ product sets in E_k, it follows that $|A_k| < 2^{nk - k^n}$; if $n \geqq 2$, this tends to zero as k increases so $|A| = 0$.

LEMMA 1. *If $\mathfrak{R}$ has a countable cofinal subset and $\mathcal{C}$ is the set of all cofinal subsets of $\mathfrak{R}$, then $|\mathcal{C}| = 1$.*

Let R' be a countable cofinal subset of $\mathfrak{R}$ and suppose $E \notin \mathcal{C}$; then there exists r in R such that $(r)^* \cap E$ is empty. Since there exists r' in R' such that $r' \geqq r$, it follows that $(r')^* \cap E$ is empty. Since r' has an infinite number of distinct successors in R', the set $A_{r'} = \{E \mid (r')^* \cap E$ is empty$\}$ is of measure zero. Since $\mathcal{E} - \mathcal{C} = \bigcup_{r' \in R'} A_{r'}$, $|\mathcal{E} - \mathcal{C}| = 0$ so $|\mathcal{C}| = 1$.

By means of this lemma we can define a measure in $\mathcal{C}$ by taking the measure in $\mathcal{E}$ of elements of $\mathcal{C}$; since $|\mathcal{C}| = 1$, we can talk meaningfully about almost all cofinal subsets of $\mathfrak{R}$ [5, Theorem 1.1]. Note that cofinality of E is not affected by adding or removing a finite set, so $\mathcal{C}$ is a "homogeneous" subset of $\mathcal{E}$ and therefore if it is measurable must have measure 0 or 1; which case occurs when $\mathfrak{R}$ does not have a countable cofinal subset, I do not know.

X is a *neighborhood space* [3] if for each x in X is defined a non-empty family of subsets of X, the neighborhoods of x. If g is a function defined on an index system $\mathfrak{R}$ with values in a neighborhood space X, x is a *limit point* of g (symbol: $x=\lim_{(R,\geqq)} g$) if for each neighborhood N of x and every r_0 in R there exists $r_1\geqq r_0$ such that $g(r)\in N$ whenever $r\geqq r_1$. (This definition is due to Alaoglu and Birkhoff [2]; in case $\mathfrak{R}$ is directed it reduces to the standard simpler form: $x=\lim_{(R,\geqq)} g$ if for each N there exists r_N in R such that $g(r)\in N$ whenever $r\geqq r_N$. $\mathfrak{R}$ is directed if every pair of elements has a common successor.) A point x is called a *cluster point* of g if for every neighborhood N of x and every r_0 in R there exists $r_1\geqq r_0$ such that $g(r_1)\in N$. Clearly every limit point of g is a cluster point of g, but not conversely. (See Lemma 2 below.) If g is a function from $\mathfrak{R}$ into X and E is a cofinal subset of $\mathfrak{R}$, let g_E be the function g reduced onto E and let Qg_E be the set of cluster points of g_E; the function g_E and the set Qg_E will play a role here analogous to that played by the subsequence x' and the set Px' in [1]. Clearly $x=\lim_{(R,\geqq)} g$ implies $x=\lim_{(E,\geqq)} g_E$ for every E cofinal in $\mathfrak{R}$. Let Pg_E be the set of limit points of functions $g_{E'}$ for E' cofinal in E; that is, $x\in Pg_E$ if and only if there exists E' cofinal in $(E,\geqq)$ such that $x=\lim_{(E',\geqq)} g_{E'}$.

Recall that X is said to satisfy Hausdorff's *first countability condition* if for each x in X there is a countable set $\{N_i\}$ of neighborhoods of x such that each neighborhood of x contains an N_i. The next lemma shows the connection between Qg and Pg.

LEMMA 2. *If $\mathfrak{R}$ has a countable cofinal subsystem, if X satisfies the first countability condition, and if the intersection of each pair of neighborhoods of each point x of X contains a third neighborhood of x, then x is a cluster point of g if and only if there exists E cofinal in $\mathfrak{R}$ such that $x=\lim_{(E,\geqq)} g_E$; that is, $Qg=Pg$.*

$Qg\supset Pg$ with no restriction on $\mathfrak{R}$ or X, for $x=\lim_{(E,\geqq)} g_E$ and N a neighborhood of x imply that if $r\in R$, there exists r_1 in E with $r_1\geqq r$ and then an r_2 in E such that $r_2\geqq r_1$ and $g(r_2)\in N$. If $\mathfrak{R}$ and X are restricted as above and if x is a cluster point of g, there exists a sequence $\{N_i'\}$ of neighborhoods of x such that each neighborhood of x contains an N_i'. By the other condition there exists a decreasing sequence of such neighborhoods $N_1\supset N_2\supset\cdots\supset N_i\supset\cdots$. Enumerate R in a sequence $\{r_i'\}$; then let r_1 be a point of $g^{-1}(N_1)$ which follows r_1'; let r_2 and r_3 be points of $g^{-1}(N_2)$ which, respectively, follow r_1 and r_2'; let r_4, r_5, r_6 be points of $g^{-1}(N_3)$ which follow r_2, r_3, and r_3', and so on. Then $E=\{r_i\}$ contains a successor of every element of R', so is cofinal in $(R',\geqq)$ and hence cofinal in $\mathfrak{R}$. If N is a neighborhood of x, there is an $N_j\subset N$ and there exists n such that $g(r_i)\in N_j$ if $i\geqq n$.

Since the set of all r which do not precede any r_i, $i<n$, is cofinal in R and contains all successors of each of its elements, its intersection with E is a set of the same sort in E; this shows that E has the desired property; that is, that $x=\lim_{(E,\geq)} g_E$.

Note that no such relation holds for multiple sequences if the cofinal sets of $\mathfrak{R}$ which are used are restricted as in [1] to be product sets.

We used Lemma 1 to show that "almost everywhere" has meaning in $\mathcal{C}$; a simple application of the same proof gives the next result which can be regarded as an extension of the lemma of [1, §3].

LEMMA 3. *If $\mathfrak{R}$ has a countable cofinal subsystem, if E_0 is cofinal in $\mathfrak{R}$, and if $A=\{E\mid E\cap E_0$ is not cofinal in $E_0\}$, then $|A|=0$; that is, almost every E of $\mathcal{C}$ meets E_0 in a set cofinal in $\mathfrak{R}$.*

Let E_1 be a countable subset of E_0 cofinal in $(E_0, \geq)$; then $E\cap E_0$ not cofinal in E_0 means that there exists r_E in E_1 such that $E\cap E_0\cap(r_E)^*$ is empty. For fixed r in E_1 let $A_r=\{E\mid E\cap E_0\cap(r)^*$ is empty$\}$; since $E_0\cap(r)^*$ is infinite, $|A_r|=0$; since $A=\bigcup_{r\in E_1}A_r$, $|A|=0$.

2. **Cluster points.** We now proceed to the analogues of the theorems of [1].

THEOREM 1. *If $\mathfrak{R}$ is an index system with a countable cofinal subset, if X satisfies the first countability condition, if g is a function from $\mathfrak{R}$ into X, and if $x\in Qg$, then $x\in Qg_E$ for almost every E of $\mathcal{C}$; that is, each cluster point of g is a cluster point of almost every g_E.*

x is a cluster point of g if and only if $g^{-1}(N)$ is cofinal in $\mathfrak{R}$ for every neighborhood N of x. If $\{N_i\}$ is an equivalent sequence of neighborhoods of x, let $A_i=\{E\mid E\cap g^{-1}(N_i)$ is not cofinal in $g^{-1}(N_i)\}$; then, by Lemma 3, $|A_i|=0$. Setting $A=\mathcal{C}-\bigcup_i A_i$, $|A|=|\mathcal{C}|=1$. If $E\in A$ and N is a neighborhood of x, there is an $N_i\subset N$; since $E\notin A_i$, $E\cap g^{-1}(N_i)$ is cofinal in $g^{-1}(N_i)$ and hence cofinal in $\mathfrak{R}$. Since $E\cap g^{-1}(N)\supset E\cap g^{-1}(N_i)$, $E\cap g^{-1}(N)$ is also cofinal in $\mathfrak{R}$ and therefore is cofinal in $(E, \geq)$; that is, if N is a neighborhood of x and $E\in A$, $g^{-1}(N)\cap E$ is cofinal in $(E, \geq)$, that is, x is a cluster point of g_E if $E\in A$.

Limit points have an analogous property.

THEOREM 1'. *If $\mathfrak{R}$ and X satisfy the hypotheses of Theorem 1, then $x=\lim_{(R,\geq)} g$ if and only if $x=\lim_{(E,\geq)} g_E$ for almost every E in $\mathcal{C}$.*

If $x=\lim_{(R,\geq)} g$, then $x=\lim_{(E,\geq)} g_E$ for every E in $\mathcal{C}$. If $x\neq\lim_{(R,\geq)} g$, there exists a neighborhood N of x and an r_0 in $\mathfrak{R}$ such that every $r_1 > r_0$ has a successor $r_2 > r_1$ for which $g(r_2) \notin N$; let $E_0 = \{r \mid g(r) \in X - N \text{ and } r > r_0\}$; then if E_1 is so chosen that E_0 and E_1 have no common successors and $E_0 \cup E_1$ is cofinal in $\mathfrak{R}$, by Lemma 3 the set $A = \{E \mid E \cap (E_0 \cup E_1) \text{ is cofinal in } \mathfrak{R}\}$ is of measure 1. For any such E, $E \cap E_0$ is cofinal in $(E_0, \geqq)$ so $x \neq \lim_{(E,\geqq)} g_E$ if $E \in A$; that is, $x \neq \lim_{(R,\geqq)} g$ implies $x \neq \lim_{(E,\geqq)} g_E$ for almost every E in $\mathcal{C}$.

Say that g is *divergent* if $x = \lim_{(R,\geqq)} g$ is false for every x in X.

COROLLARY. *Let X and $\mathfrak{R}$ satisfy the conditions of the theorem and suppose that g is divergent; then for each x in X the set $A_x = \{E \mid x = \lim_{(E,\geqq)} g_E\}$ is of measure zero. Hence if almost every g_E has a limit point, then Pg is uncountable.*

The first statement follows immediately from the theorem. For the second, $\{E \mid g_E \text{ has a limit point}\} = \bigcup_{x \in P_g} A_x$; since $|A_x| = 0$ and $|\bigcup_{x \in P_g} A_x| = 1$, Pg is uncountable.

The next two results are related to Theorem 1′ but stronger hypotheses enable us to draw stronger conclusions.

THEOREM 2. *If X and $\mathfrak{R}$ satisfy the conditions of Theorem 1 and if each pair of distinct points of X has a pair of disjoint neighborhoods,*[4] *then g is divergent if and only if almost every g_E is divergent.*

If g has the limit x, so does every g_E. If g is divergent, by the corollary $|A_x| = 0$ for every x. By the first statement in the proof of Lemma 2, if $x_1 = \lim_{(E_1,\geqq)} g_{E_1}$, then x_1 is a cluster point of g; by Theorem 1, x_1 is a cluster point of almost every g_E. Let $A = \{E \mid x_1 \neq \lim_{(E,\geqq)} g_E$ but g_E has a limit point$\}$. If E is in A, let $x = \lim_{(E,\geqq)} g_E$; since there exist disjoint neighborhoods N_1 of x_1 and N of x and since g_E plunges eventually into N, there is an r_1 in E such that $g_E(r) \notin N_1$ if $r > r_1$ and $r \in E$. Hence $g_E^{-1}(N_1)$ is not cofinal in $(E, \geqq)$, so x_1 is not a cluster point of g_E when $E \in A$. Hence $|A| = 0$ by Theorem 1; since $|A_{x_1}| = 0$ also, we see that $|\{E \mid g_E \text{ has a limit}\}| = |A| + |A_{x_1}| = 0$.

Buck notes that the proof of Theorem 1′ can easily be modified to prove another theorem with the same conclusion as that of Theorem 2.

THEOREM 2′. *If $\mathfrak{R}$ has a countable cofinal subset and if X satisfies Hausdorff's second countability condition,*[5] *then g is divergent if and only if almost every g_E is divergent.*

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[4] This is the separation condition in a Hausdorff space; however X need not satisfy the other axioms of such a space.

[5] In this system the second countability condition becomes: There exists a countable subset $\{N_i\}$ of subsets of X such that for each x and each neighborhood N of x there is an i such that N_i contains a neighborhood of x and $N_i \subset N$.

LEMMA 4. *If every neighborhood N of x contains a neighborhood N' of x such that for every y in N' there is a neighborhood N_y of y with $N_y \subset N$, then Qg is closed in X.*

If x is in the closure of Qg, then for every neighborhood N of x there is a point y in $N' \cap Qg$; then for every r_0 in R there exists $r_1 \geqq r_0$ such that $g(r_1) \in N_y \subset N$ so $x \in Qg$.

THEOREM 3. *If $\mathfrak{R}$ and X satisfy the hypotheses of Theorem 1 and Lemma 4, and if Qg is separable, then $Qg = Qg_E$ for almost every E in $\mathfrak{C}$.*

Take a countable dense subset X' of Qg and follow the proof of Theorem 3 of [1], using Theorem 1 and Lemma 4 at the appropriate points. This is much stronger than the corresponding theorem of [1]; the principal extension is that this formulation is valid for all essentially countable index systems rather than for the integers alone. In case $\mathfrak{R}$ is the system of integers, this result includes that of [1] since the hypotheses of [1, Theorem 3] imply the hypotheses of Theorem 1 and Lemmas 2 and 4; Theorems 1 and 2 are not generalizations of the corresponding results of [1] but rather are generalizations in a slightly different direction from the case $n=1$ of those theorems.

Note that a metric space satisfies all the hypotheses on X except that on Qg in Theorem 3; there the requirement that X is separable would be a sufficient additional condition. Hence with X metric and $\mathfrak{R}$ having a countable cofinal subset, the set Qg used in this paper is equal to the set Pg analogous to Px of [1]. Any countable index system will do for $\mathfrak{R}$ as will the system of real numbers ordered by magnitude or the system of n-tuples of real numbers ordered by $(a_1, \cdots, a_n) \geqq (b_1, \cdots, b_n)$ if $a_i \geqq b_i$ for all $i \leqq n$. Another such system is the system of n-tuples $(r_1, \cdots, r_n)$ where $r_i \in \mathfrak{R}_i$, an index system with a countable cofinal subset, and where $(r_1, \cdots, r_n) > (r_1', \cdots, r_n')$ if $r_1 > r_1'$ or $r_1 = r_1'$ and $r_2 > r_2'$ or, for some $j \leqq n$, $r_i = r_i'$ for $i < j$ while $r_j > r_j'$. (This is the so-called ordinal or lexicographic product of the systems $\mathfrak{R}_i$.) Still another example is the system of pairs of integers where $(i_1, i_2) \geqq (j_1, j_2)$ means that $i_1 = i_2$ and $j_1 \geqq j_2$.

It may be noted by means of Lemmas 1 and 3 that the proofs of Theorems 1 and 2 of [1] also hold when the set $I \times I \times \cdots \times I$ used in [1] as the domain of the function $x = x[i_1, i_2, \cdots, i_n]$ is replaced by $\mathfrak{R}_1 \times \mathfrak{R}_2 \times \cdots \times \mathfrak{R}_n$, where the $\mathfrak{R}_i$ are any index systems with countable cofinal subsystems, providing that $\mathfrak{S}$ is then defined as $\mathfrak{C}_1 \times \mathfrak{C}_2 \times \cdots \times \mathfrak{C}_n$ where $\mathfrak{C}_i$ is the family of cofinal subsets of $\mathfrak{R}_i$. The theorems thus obtained almost include the corresponding results of both papers, the case $n=1$ giving analogues of the present theorems with stronger hypotheses, the case where all $\mathfrak{R}_i = I$ giving those of [1].

An open question is whether the existence of a countable cofinal subset is needed to derive the conclusions of Lemmas 1 and 3; if not, some weakening of the hypotheses of the theorems would be possible.

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A THEOREM ON THE UNIT GROUPS OF SIMPLE ALGEBRAS

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1. **Introduction.** Let k be an algebraic number field of finite degree m and let A be a normal simple algebra of degree n, order n^2, over k. Our object is to prove the following theorem.

THEOREM. *If A is an R-algebra, that is, if $n>2$ or at least one infinite prime place of k is unramified in A when $n=2$, then any two distinct maximal orders of A have distinct groups of units.*

There are essential arithmetical differences between algebras which satisfy the R-condition (R-algebras) and those which do not, especially with regard to class-number properties (Eichler [1, 2, 3]).[1] The meaning of the R-condition in the case $n=2$ is as follows. Both k and A are simple algebras over the field k_0 of rational numbers, of orders m and $4m$, respectively, over k_0. Suppose k_0 is extended to the field k_1 of real numbers. Then the extended algebra $k\times k_1$ is the direct sum of fields, each of which is isomorphic either to k_1 or to the field k_2 of complex numbers. This decomposition of $k\times k_1$ involves the decomposition of $A\times k_1$ into a direct sum of simple algebras over k_1, the centers of which are the corresponding summands of $k\times k_1$. Each summand of $A\times k_1$ is either (1) a matrix algebra of degree 2 over k_1, (2) a matrix algebra of degree 2 over k_2, or (3) the division algebra of quaternions over k_1. With each summand of $k\times k_1$ is associated an infinite prime place of k which is said to be ramified or unramified in A according as the corresponding summand of $A\times k_1$ is (3) or is either (1) or (2). The R-condition for $n=2$ is thus equivalent to requiring that not all summands of $A\times k_1$ be (3), in other words, that A over k is not a totally definite quaternion algebra. The condition is in general indispensable in our theorem. For example, the unit groups of all maximal orders in certain definite quaternion algebras over k_0 consist of the units ± 1 only.

The proof of the theorem will be based on the following Hilfssatz due to Eichler [3, p. 239, Hilfssatz 9].

Let A be an R-algebra, let $\mathfrak{O}$ be a maximal order of A, and let $\mathfrak{F}$ be a two-sided $\mathfrak{O}$-ideal. If $\mathcal{A}$ is an element of $\mathfrak{O}$ whose reduced norm $N(\mathcal{A})$ is congruent modulo $\mathfrak{F}$ to a unit of k, then $\mathfrak{O}$ contains a unit $\mathcal{E}\equiv\mathcal{A}$ modulo $\mathfrak{F}$.

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[1] Numbers in brackets refer to the references cited at the end of the paper.

By local (p-adic) methods we shall first reduce the proof of the theorem to that for a special case. The lemmas proved for this purpose are independent of the R-condition and may be of interest also in another connection not discussed here, namely, in the problem of imbedding non-maximal orders in maximal orders. Next, also by local methods, we prove the theorem in the special case by applying the Hilfssatz. Finally, we indicate briefly an application of the theorem to the determination of the structure of the Brandt groupoid of normal ideals of A.

2. **Reduction to a special case.** Let A' be a normal simple algebra over a p-adic number field k'. Let ν be the index of A', $n=\nu\kappa$. Then A' is the algebra of matrices of degree κ with elements in a division algebra B' of degree ν over k'. If $\kappa=1$, the unique maximal order $\mathfrak{o}'$ in B' is the only maximal order in $A'=B'$. If $\kappa>1$, the set

$$\mathfrak{O}' = \sum e_{ij}\mathfrak{o}' \qquad (i, j = 1, \cdots, \kappa), \tag{1}$$

where the e_{ij} are ordinary matric units in A', is a maximal order in A'. Conversely, every maximal order in A' is of the form (1) with the appropriate e_{ij}.

Let $\mathfrak{p}'$ be the prime ideal of B'. Then $\mathfrak{p}'$ is a principal ideal: $\mathfrak{p}'=\pi\mathfrak{o}'=\mathfrak{o}'\pi$, where π is a prime element in B'. The ideal $\mathfrak{P}'=\mathfrak{O}'\mathfrak{p}'=\mathfrak{p}'\mathfrak{O}'=\sum e_{ij}\mathfrak{p}'$ is the $\mathfrak{O}'$-prime-ideal in A'. If $\mathfrak{L}'$ is any integral $\mathfrak{O}'$-left-ideal in A', the matric units can be further selected (Hasse [1, p. 524]) so that (1) holds and also

$$\mathfrak{L}' = \mathfrak{O}'\Lambda = \sum e_{ii}\pi^{(a_1+\cdots+a_i)}, \tag{2}$$

where the a's are non-negative integers. The right order of $\mathfrak{L}'$ is

$$\Lambda^{-1}\mathfrak{O}'\Lambda = \sum e_{ij}\mathfrak{p}'^{(a_1+\cdots+a_j)-(a_1+\cdots+a_i)}. \tag{3}$$

The intersection of $\mathfrak{O}'$ and $\Lambda^{-1}\mathfrak{O}'\Lambda$ is

$$\mathfrak{O}' \cap \Lambda^{-1}\mathfrak{O}'\Lambda = \sum_{i\geqq j} e_{ij}\mathfrak{o}' + \sum_{i<j} e_{ij}\mathfrak{p}'^{(a_{i+1}+\cdots+a_j)}. \tag{4}$$

Equations (3) and (4) are readily verified by displaying matrices in the usual way.

With these preliminaries we are ready to prove the following lemma.

LEMMA 1. *Let A' be a normal simple algebra over a p-adic number field k', and let $\mathfrak{O}'_1$ and $\mathfrak{O}'_2$ be any two distinct maximal orders in A', whose distance is $\mathfrak{D}'_{12}=(\mathfrak{O}'_2\mathfrak{O}'_1)^{-1}$. Then, either $\mathfrak{D}'_{12}$ is irreducible, or there exists a maximal order $\mathfrak{O}'_3$ in A' such that $\mathfrak{O}'_1\cap\mathfrak{O}'_3\supset\mathfrak{O}'_1\cap\mathfrak{O}'_2$ and such that $\mathfrak{D}'_{13}=(\mathfrak{O}'_3\mathfrak{O}'_1)^{-1}$ is irreducible and divides $\mathfrak{D}'_{12}=\mathfrak{D}'_{13}\mathfrak{D}'_{32}$.*

By a remark above $\kappa>1$, since otherwise A' would have only one maximal order. Let $\mathfrak{O}_1'=\mathfrak{O}'$ in (1) and $\mathfrak{D}_{12}'=\mathfrak{L}'$ in (3). Then $\mathfrak{L}'\neq\mathfrak{O}'$, since $\mathfrak{O}_2'\neq\mathfrak{O}'$, and hence at least one a_i in (2) is not zero. If a_1 were not zero, $\mathfrak{L}'$ would be divisible by the two-sided ideal $\mathfrak{P}'$, contrary to the properties of a distance ideal.[2] It follows that there is a fixed $r\geqq 2$ for which $a_1=a_2=\cdots=a_{r-1}=0$, $a_r\neq 0$. If $r=\kappa$, $a_r=1$, $\mathfrak{D}_{12}'=\mathfrak{L}'$ is irreducible and the first alternative in Lemma 1 holds.

Assume that $r<\kappa$ or $a_r>1$ if $r=\kappa$. Then $\mathfrak{L}'$ has the irreducible left divisor

$$\mathfrak{J}_r' = \mathfrak{O}'\Lambda_r, \qquad \Lambda_r = 1+(\pi-1)e_{rr}. \tag{5}$$

For, $\Lambda=\Lambda_r\Lambda_0$, where Λ_0 is in $\mathfrak{O}'$. Writing $\mathfrak{O}_3'=\Lambda_r^{-1}\mathfrak{O}\Lambda_r$, $\mathfrak{O}_3'\Lambda_0=\mathfrak{D}'$, we have $\mathfrak{L}'=\mathfrak{J}_r'\mathfrak{D}'$, $\mathfrak{D}'=\mathfrak{O}_3'\Lambda_0$. Since $\mathfrak{J}_r'$ is irreducible and has $\mathfrak{O}'$ and $\mathfrak{O}_3'$ as its left and right orders, $\mathfrak{J}_r'$ is the distance $\mathfrak{D}_{13}'$. It can be shown easily, but is not required in what follows, that $\mathfrak{D}'=(\mathfrak{O}_2'\mathfrak{O}_3')^{-1}=\mathfrak{D}_{32}'$. By a simple computation, we obtain

$$\mathfrak{O}_3' = \Lambda_r^{-1}\mathfrak{O}'\Lambda_r = \sum_{i,j\neq r} e_{ij}\mathfrak{o}' + \sum_{j<r} e_{rj}\mathfrak{p}'^{-1} + \sum_{i<r} e_{ir}\mathfrak{p}' + e_{rr}\mathfrak{o}', \tag{6}$$

and

$$\mathfrak{O}'\cap\mathfrak{O}_3' = \sum_{j\neq r} e_{ij}\mathfrak{o}' + e_{rr}\mathfrak{o}' + \sum_{i\neq r} e_{ir}\mathfrak{p}'. \tag{7}$$

Comparison of (4) and (7) shows that $\mathfrak{O}'\cap\mathfrak{O}_3'\supset\mathfrak{O}'\cap\mathfrak{O}_2'$. This completes the proof of Lemma 1.

The reduction in the large, corresponding to that in Lemma 1, is based on the following lemma.

LEMMA 2. *Let A be a normal simple algebra over an algebraic number field k and let $\mathfrak{O}_1$ and $\mathfrak{O}_2$ be any two distinct maximal orders in A, whose distance is $\mathfrak{D}_{12}=(\mathfrak{O}_2\mathfrak{O}_1)^{-1}$. Let p be a prime ideal of k which divides the reduced norm $N(\mathfrak{D}_{12})$. Then, either $\mathfrak{D}_{12}$ is irreducible, in which case $N(\mathfrak{D}_{12})=p$, or there exists a maximal order $\mathfrak{O}_3$ in A such that $\mathfrak{O}_1\cap\mathfrak{O}_3\supset\mathfrak{O}_1\cap\mathfrak{O}_2$ and such that $\mathfrak{D}_{13}=(\mathfrak{O}_3\mathfrak{O}_1)^{-1}$ is irreducible, has norm p, and divides $\mathfrak{D}_{12}=\mathfrak{D}_{13}\mathfrak{D}_{32}$.*

In proving Lemma 2, we write $\mathfrak{O}$ for $\mathfrak{O}_1$ and use primed symbols, without subscripts, to refer to p-components for a fixed p which divides $N(\mathfrak{D}_{12})$.

The ideal $\mathfrak{O}p$ is a power of a two-sided $\mathfrak{O}$-prime-ideal P:

$$\mathfrak{O}p = p\mathfrak{O} = P^{\nu}, \qquad n = \nu\kappa, \qquad N(P) = p^{\kappa}, \tag{8}$$

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[2] See Deuring [1, chap. VI, VII] for definitions and properties when explicit reference elsewhere is not made.

where $\nu=\nu_p$ and $\kappa=\kappa_p$ are the order of ramification and the capacity, respectively, of p or of P in A. In A, P decomposes into a product of κ irreducible ideals, each of norm p. In accord with (8), the index of A', over k', is ν. The p-component $\mathfrak{D}'_{12}$ is the distance $(\mathfrak{O}'_2\mathfrak{O}'_1)$ (Hasse [1, p. 529]; cf. Deuring [1, p. 104]). Moreover, $\mathfrak{D}'_{12}\neq\mathfrak{O}'$, since $p \mid N(\mathfrak{D}_{12})$. Hence $\mathfrak{O}'_2\neq\mathfrak{O}'$, from which it follows that $\kappa\geqq 2$, $\nu<n$. If $\mathfrak{D}_{12}$ is irreducible its norm is necessarily p, and we have nothing to prove. We now assume that $\mathfrak{D}_{12}$ is not irreducible and obtain $\mathfrak{D}_{13}$ and $\mathfrak{O}_3$ of the lemma by the usual method of specifying their components at all finite prime places of k.

Let q_1 denote any prime ideal of k, other than p, which divides $N(\mathfrak{D}_{12})$. If there is no q_1 we omit (10) and the q_1-components in (13), $\cdots$, (16), below, but the argument is not essentially altered. Let q_2 be any prime ideal of k which does not divide $N(\mathfrak{D}_{12})$. By Lemma 1, we have

$$\mathfrak{D}'_{12}=\mathfrak{D}'_{13}\mathfrak{D}'_{32}, \qquad \mathfrak{O}'_1\cap\mathfrak{O}'_3\supset\mathfrak{O}'\cap\mathfrak{O}'_2, \tag{9}$$

where $\mathfrak{D}'_{13}$ is the distance $(\mathfrak{O}'_3\mathfrak{O}'_1)^{-1}$ and is irreducible. Also, for any q_1 and any q_2,

$$\mathfrak{D}'_{12q_1}=\mathfrak{O}'_{q_1}\mathfrak{D}'_{12q_1}, \qquad \mathfrak{O}'_{q_1}\cap\mathfrak{O}'_{q_1}\supset\mathfrak{O}'_{q_1}\cap\mathfrak{O}'_{2q_1}, \tag{10}$$

and

$$\mathfrak{D}'_{12q_2}=\mathfrak{O}'_{q_2}\mathfrak{O}'_{q_2}, \qquad \mathfrak{O}'_{q_2}\cap\mathfrak{O}'_{q_2}=\mathfrak{O}'_{q_2}\cap\mathfrak{O}'_{q_2}, \tag{11}$$

where the first statement in (11) follows from

$$\mathfrak{O}'_{q_2}=\mathfrak{O}'_{2q_2}=\mathfrak{D}'_{12q_2}, \qquad q_2 \nmid N(\mathfrak{D}_{12}). \tag{12}$$

Consider the sets

$$\mathfrak{D}'_{13},\ \mathfrak{O}'_{q_1},\ \mathfrak{O}'_{q_2},\ \text{for all } q_1 \text{ and all } q_2, \tag{13}$$

$$\mathfrak{O}',\ \mathfrak{O}'_{q_1},\ \mathfrak{O}'_{q_2},\ \text{for all } q_1 \text{ and all } q_2, \tag{14}$$

$$\mathfrak{O}'_2,\ \mathfrak{O}'_{2q_1},\mathfrak{O}'_{2q_2},\ \text{for all } q_1 \text{ and all } q_2, \tag{15}$$

and

$$\mathfrak{O}'_3,\ \mathfrak{O}'_{q_1},\ \mathfrak{O}'_{q_2},\ \text{for all } q_1 \text{ and all } q_2. \tag{16}$$

The intersection of (13) and A is an irreducible left divisor $\mathfrak{D}_{13}$ of $\mathfrak{D}_{12}$, that of (14) and A is $\mathfrak{O}$, that of (15) and A is $\mathfrak{O}_2$, and that of (16) and A is $\mathfrak{O}_3$, the right order of $\mathfrak{D}_{13}$. The irreducibility of $\mathfrak{D}_{13}$ follows at once from its definition by (13) and the irreducibility of $\mathfrak{D}'_{13}$. That $\mathfrak{D}_{13}$ divides $\mathfrak{D}_{12}$ follows from the first statements in (9), (10) and (11).

The statements concerning (14), (15) and (16) are evident. Moreover, $\mathfrak{D}_{13}=(\mathfrak{O}_3\mathfrak{O}_1)^{-1}$ since $\mathfrak{D}_{13}$ is irreducible and $\mathfrak{O}_1$ and $\mathfrak{O}_3$ are its left and right orders, respectively. We write $\mathfrak{D}_{12}=\mathfrak{D}_{13}\mathfrak{D}_{32}$ but omit the proof that $\mathfrak{D}_{32}=(\mathfrak{O}_2\mathfrak{O}_3)^{-1}$.

Let $\mathcal{A}$ be any element of $\mathfrak{O}\cap\mathfrak{O}_2$. Then $\mathcal{A}$ is in the p-, q_1-, and q_2-components of both $\mathfrak{O}$ and $\mathfrak{O}_2$, for all q_1 and all q_2. By the second statements in (9), (10) and (11), $\mathcal{A}$ is in the corresponding components of $\mathfrak{O}_3$. Hence $\mathcal{A}$ is in $\mathfrak{O}_3$ and $\mathfrak{O}\cap\mathfrak{O}_3$, since $\mathcal{A}$ is in A. This proves that $\mathfrak{O}\cap\mathfrak{O}_3\supset\mathfrak{O}\cap\mathfrak{O}_2$, and completes the proof of Lemma 2.

3. Proof of the theorem. By virtue of Lemma 2, in order to prove the theorem it evidently suffices to prove that *if* $\mathfrak{D}_{13}=(\mathfrak{O}_3\mathfrak{O}_1)^{-1}$ *is irreducible, there is a unit in* $\mathfrak{O}_1$ *which is not in* $\mathfrak{O}_3$.

Let $N(\mathfrak{D}_{13})=p$, and take p to be the fixed prime ideal of k in §2. As before, we use primed symbols, without subscripts, to refer to p-components, and we write $\mathfrak{O}$ for $\mathfrak{O}_1$. We employ the representation (1) of $\mathfrak{O}'$, and have $\kappa\geqq 2$ by an argument used in the proof of Lemma 2. The "canonical" form of $\mathfrak{D}_{13}'$, corresponding to (2), is

$$\mathfrak{D}_{13}'=\mathfrak{O}'\Delta,\qquad \Delta=1+(\pi-1)e_{\kappa\kappa}, \tag{17}$$

and we assume that the e_{ij} are chosen so that (17) holds. The ideal $\mathfrak{D}_{13}'$ divides the two-sided ideal $\mathfrak{O}'P=\mathfrak{P}'$, where P is defined in (8).

Consider the element $\mathcal{A}_p=1+e_{\kappa-1,\kappa}$ of $\mathfrak{O}'$. The reduced norm of an element of $\mathfrak{O}'$ is the determinant of the matrix representing it in (1), with the understanding[3] that the coefficients, in B', are replaced by their corresponding matrices in a reduced representation of B'. It follows that $N(\mathcal{A}_p)=1\equiv 1 \pmod{\mathfrak{O}'P}$. Since $\mathfrak{O}/P$ and $\mathfrak{O}'/\mathfrak{O}'P$ are isomorphic, there exists an element $\mathcal{A}$ in $\mathfrak{O}$ such that $\mathcal{A}\equiv\mathcal{A}_p \pmod{\mathfrak{O}'P}$ and $N(\mathcal{A})\equiv 1 \pmod{P}$. Using, at last, the assumption that *A is an R-algebra*, by Eichler's Hilfssatz, $\mathfrak{O}$ contains a unit $\mathcal{E}\equiv\mathcal{A} \pmod{P}$. We shall prove that $\mathcal{E}$ is not in $\mathfrak{O}_3$.

Using (17), we get

$$\mathfrak{O}_3'=\Delta^{-1}\mathfrak{O}'\Delta=\sum_{i,j<\kappa}e_{ij}\mathfrak{o}'+e_{\kappa\kappa}\mathfrak{o}'+\sum_{j<\kappa}e_{\kappa j}\mathfrak{p}'^{-1}+\sum_{i<\kappa}e_{i\kappa}\mathfrak{p}'. \tag{18}$$

Comparison of (1) and (18) shows that $\mathcal{A}_p$ is not in $\mathfrak{O}_3'$. Since $\mathcal{E}\equiv\mathcal{A}_p \pmod{\mathfrak{O}'P}$, $\mathcal{E}$ is not in $\mathfrak{O}_3'$ and hence $\mathcal{E}$ is not in $\mathfrak{O}_3$.

4. An application of the theorem. The author was led to the theorem in attempting to answer the following question proposed to him some time ago by Professor R. Baer. The normal ideals of an algebra

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[3] This is essential if $\nu>1$, since then B is noncommutative.

A form an infinite groupoid (Brandt [1]) with respect to proper multiplication. The normal ideals having the same left order form, similarly, a *Mischgruppe* (Loewy [1]). Baer [1] has shown that the theory of the latter, and hence the closely related theory of groupoids, is essentially subordinate to the theory of a group G and a non-invariant subgroup H. Indeed, with one trivial exception, every Mischgruppe is isomorphic to the factor-Mischgruppe $(G\|H)_L$ of cosets $H\mathcal{G}$, $\mathcal{G}\in G$, for suitable G and H. Similarly, every Brandt groupoid, with the corresponding exception, is a factor-groupoid $(G\|H)_B$ of cosets $\mathcal{G}_1^{-1}H\mathcal{G}_2$, $\mathcal{G}_i\in G$. The question is: *Are there groups G and H in an algebra A such that $(G\|H)_B$ and $(G\|H)_L$ are isomorphic to the groupoid and Mischgruppe, respectively, of normal ideals, described above?*

By means of our theorem a partial answer to this question can be given. Let $\mathfrak{O}$ be any order of A, U the unit group of $\mathfrak{O}$, A^* the multiplicative group of nonsingular elements of A. Then *if A is an R-algebra, the Mischgruppe of principal $\mathfrak{O}$-left-ideals $\mathfrak{O}\mathcal{A}$, $\mathcal{A}\in A^*$, and the groupoid of ideals $\mathcal{A}_1^{-1}\mathfrak{O}\mathcal{A}_2$, $\mathcal{A}_i\in A^*$ are isomorphic to $(A^*\|U)_L$ and $(A^*\|U)_B$, respectively.*

The Mischgruppe and groupoid described are natural generalizations of the group of principal ideals of k, and hence the above isomorphism, for R-algebras, is a natural generalization of the isomorphism of the group of principal ideals of k with the factor group k^*/u, where u is the group of units of k. However, Baer's formal subordination of the theory of the Brandt groupoid to the theory of groups does not seem to make the concept of a groupoid, as such, any less essential to a satisfactory arithmetical theory of simple algebras. The above isomorphism for R-algebras evidently fails for the rational definite quaternion algebras cited in the introduction since $(A^*\|U)_L$ and $(A^*\|U)_B$ are both groups if U consists of ± 1 only.

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ON STRONG SUMMABILITY OF A FOURIER SERIES

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Let $s_n(x)=a/2\sum_{\nu=1}(a_\nu \cos \nu x+b_\nu \sin \nu x)$ be the partial sum of the Fourier series of an integrable periodic function $f(t)$ of period 2π, and let $\phi(t)=\{f(x+t)+f(x-t)-2s\}/2$. We shall establish the following result (Hardy-Littlewood [1][1]).

THEOREM. *If*

$$\int_0^t |\phi(u)|\{1+\log^+|\phi(u)|\}du=o(t), \qquad \text{as } t\to 0, \tag{1}$$

then $\sum_{\nu=0}^{n}|s_\nu(x)-s|^2=o(n\log\log n)$, *as* $n\to\infty$.

To prove this theorem we require the following lemmas.

LEMMA 1. *If*

$$\int_0^t |\phi(u)|du=o(t), \qquad \text{as } t\to 0, \tag{2}$$

then

$$\sum_{\nu=0}^{n}|s_\nu(x)-s|^2=\frac{1}{\pi^2}\int_{1/n}^{\delta}\frac{\phi(t)}{t^2}dt\int_{1/n}^{t}\phi(u)\,\frac{\sin n(u-t)}{u-t}du+o(n).$$

PROOF. By (2), for $\nu\leqq n$,

$$s_\nu(x)-s=\frac{2}{\pi}\int_{1/n}^{\delta}\phi(t)\,\frac{\sin \nu t}{t}dt+o(1).$$

Hence

$$\begin{aligned}\sum_{\nu=0}^{n}|s_\nu(x)-s|^2&=\frac{4}{\pi^2}\int_{1/n}^{\delta}\int_{1/n}^{\delta}\frac{\phi(u)\phi(t)}{ut}\left\{\sum_{\nu=1}^{n}\sin\nu t\sin\nu u\right\}dudt+o(n)\\&=\frac{2}{\pi^2}\int_{1/n}^{\delta}\int_{1/n}^{\delta}\frac{\phi(u)\phi(t)}{ut}\,\frac{\sin n(u-t)}{u-t}dudt\\&\quad+\frac{2}{\pi^2}\int_{1/n}^{\delta}\int_{1/n}^{\delta}\frac{\phi(u)\phi(t)}{ut}\,\frac{\sin n(u+t)}{u+t}dudt+o(n)\\&=J_1+J_2+o(n).\end{aligned}\tag{3}$$

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[1] Numbers in brackets refer to the references listed at the end of the paper.

Set $\Phi(t)=\int_0^t|\phi(u)|\,du$. Then

$$J_2 = O\left\{\int_{1/n}^{\delta}\frac{|\phi(t)|}{t^2}dt\int_{1/n}^{t}\frac{|\phi(u)|}{u}du\right\}.$$

But

$$\int_{1/n}^{\delta}\frac{|\phi(t)|}{t^2}dt\int_{1/n}^{\delta}\frac{|\phi(u)|}{u}du = O\left\{\int_{1/n}^{\delta}\frac{|\phi(t)|}{t^2}\log ntdt\right\},$$

and

$$\begin{aligned}\int_{1/n}^{\delta}\frac{|\phi(t)|}{t^2}\log ntdt &= \left[\frac{\Phi(t)}{t^2}\log nt\right]_{1/n}^{\delta} + 2\int_{1/n}^{\delta}\frac{\Phi(t)}{t^3}\log ntdt \\ &\quad - \int_{1/n}^{\delta}\frac{\Phi(t)}{t^3}dt = o(n) + o\left(n\int_1^{\delta n}\frac{\log t}{t^2}dt\right) \\ &\quad + o\left(\int_{1/n}^{\delta}\frac{dt}{t^2}dt\right) = o(n).\end{aligned}$$

Hence

$$J_2 = o(n). \tag{4}$$

By inversion of the order of integration, and the resolution of $1/t(t-u)$ into partial fractions, we get

$$\begin{aligned}J_1 &= \frac{2}{\pi^2}\int_{1/n}^{\delta}\frac{\phi(t)}{t^2}dt\int_{1/n}^{t}\phi(u)\frac{\sin n(n-t)}{u-t}du \\ &\quad + O\left(\int_{1/n}^{\delta}\frac{|\phi(t)|}{t^2}dt\int_{1/n}^{t}\frac{|\phi(u)|}{u}du\right) \\ &= \frac{1}{\pi^2}\int_{1/n}^{\delta}\frac{\phi(t)}{t^2}dt\int_{1/n}^{t}\phi(u)\frac{\sin n(u-t)}{u-t}du + o(n).\end{aligned} \tag{5}$$

Lemma 1 follows from (4) and (5).

Lemma 2. *If $P(u)$ and $F(u)$ are non-negative integrable functions in (a, b), and $\psi(x)$ is a convex function in* $(0, \infty)$,

$$\psi\left\{\int_a^b P(u)F(u)du\Big/\int_a^b P(u)du\right\} \leq \int_a^b P(u)\psi\{F(u)\}du\Big/\int_a^b P(u)du.$$

This is the well known Jensen inequality (Hardy-Littlewood-Pólya [2]).

PROOF OF THE THEOREM. Since $1+\log^+|\phi(u)|\geqq 1$,

$$\int_0^t |\phi(u)|\,du \leqq \int_0^t |\phi(u)|\,\{1+\log^+|\phi(u)|\,\}du = o(t).$$

Hence by Lemma 1 it suffices to prove that

$$G_n = \int_{1/n}^{\delta} \frac{\phi(t)}{t^2}dt \int_{1/n}^{t} \phi(u)\, \frac{\sin n(u-t)}{u-t}\,du = o(n \log\log n).$$

Now

$$\begin{aligned} G_n &= \int_{1/n}^{n^{-1}\log n} \frac{\phi(t)}{t^2}dt \int_{1/n}^{t} \phi(u)\, \frac{\sin n(u-t)}{u-t}\,du \\ &\quad + \int_{n^{-1}\log n}^{\delta} \frac{\phi(t)}{t^2}dt \int_{1/n}^{t-n^{-1}\log n} \phi(u)\, \frac{\sin n(u-t)}{u-t}\,du \qquad (6) \\ &\quad + \int_{n^{-1}\log n}^{\delta} \frac{\phi(t)}{t^2}dt \int_{t-n^{-1}\log n}^{t} \phi(u)\, \frac{\sin n(u-t)}{u-t}\,du \\ &= I_1 + I_2 + I_3. \end{aligned}$$

By (2) we have

$$I_1 = \int_{1/n}^{n^{-1}\log n} \frac{|\phi(t)|}{t^2} O(nt)dt = o(n \log\log n), \qquad (7)$$

and

$$I_2 = \int_{n^{-1}\log n}^{\delta} \frac{|\phi(t)|}{t^2} O\left(\frac{n}{\log n}t\right)dt = o(n). \qquad (8)$$

Now set

$$P(u) = \left|\frac{\sin n(u-t)}{u-t}\right|,$$

$$k = \int_{t-n^{-1}\log n}^{t} P(u)du \sim \log\log n,$$

and

$$J_n(t) = \int_{t-n^{-1}\log n}^{t} |\phi(u)|\,P(u)du.$$

Applying Lemma 2 with $\psi(x)=x(1+\log^+ x)$, we find

$$\psi\left\{\frac{J_n(t)}{k}\right\} \leqq \frac{\int_{t-n^{-1}\log n}^{t} |\phi(u)|\,\{1+\log^+|\phi(u)|\,\}P(u)du}{k} \leqq \frac{nt}{k}.$$

Hence

$$J_n(t) \leqq nt\left\{1 + \log^+ \frac{J_n(t)}{k}\right\}^{-1}.$$

If we suppose now that

$$J_n(t) \geqq A_1 nt/\log ntk^{-1}, \tag{9}$$

we find $1+\log^+ (J_n(t)/k) \geqq A \log (nt/k)$, for $nt/k \geqq \log n/\log \log n$. Hence

$$J_n(t) \leqq A_2 nt/\log ntk^{-1}, \tag{10}$$

for all values of n and t such that (9) and the relation $nt \geqq \log n$ hold true.

If (9) does not hold,

$$J_n(t) \leqq A_1 nt/\log ntk^{-1}. \tag{11}$$

By (10) and (11) we have then

$$J_n(t) \leqq A_3 nt/\log ntk^{-1} \tag{12}$$

for all values of n and t for which $nt \geqq \log n$.

By (12)

$$\begin{aligned}
|I_3| &\leqq \int_{n^{-1}\log n}^{\delta} \frac{|\phi(t)|}{t^2} J_n(t)dt \leqq A \int_{n^{-1}\log n}^{\delta} \frac{|\phi(t)|}{t^2} \frac{n}{\log (n/\log\log n)} dt \\
&\leqq An\left\{\left[\frac{\Phi(t)}{t} \frac{1}{\log (nt/\log \log n)}\right]_{n^{-1}\log n}^{\delta}\right. \\
&\quad + \int_{n^{-1}\log n}^{\delta} \frac{\Phi(t)}{t^2} \frac{dt}{\log (nt/\log\log n)} \\
&\quad \left. + \int_{n^{-1}\log n}^{\delta} \frac{\Phi(t)}{t^2} \frac{dt}{(\log (nt/\log \log n))^2}\right\} \\
&= o(n \log \log n).
\end{aligned}$$

From this inequality and (6), (7), and (8) we draw

$$G_n = o(n \log \log n).$$

In conclusion I remark that by the above method I established in a recent paper the following result, which is a generalization of Hardy and Littlewood's theorem [1].

If

$$\int_0^t |\phi(u)|\, du = O(t) \quad \text{and} \quad \int_0^t \phi(u)du = o(t) \quad \text{as } t \to 0,$$

then

$$\sum_{\nu=0}^{n} | s_\nu(x) - s |^2 = o(n \log n), \qquad \text{as } n \to \infty.$$

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A NOTE ON RIESZ SUMMABILITY OF THE TYPE e^{n^α}

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Recently I proved the following result in the case $r=2$ (Wang [4][1]).

Let $\sigma_n^{(r)}$ *be the rth Cesàro mean of the series* $\sum_{n=0}^{\infty} a_n$. *If* $\sigma_n^{(r)} - s = o(n^{-r\alpha})$ $0<\alpha<1$, *as* $n\rightarrow\infty$, *where r is a positive integer, and* $a_n > -Kn^{\alpha-1}$, *the series converges to sum s.*

For the case $r=1$ this result has been established by Boas [1]. His argument, however, does not seem to be applicable in any simple way to the general case.

The object of this note is to prove a theorem on Riesz summability of type e^{n^α}, and then to deduce the result above from a Tauberian theorem of Hardy [2].

Let us put $C_\tau(\omega) = a_0 e^{\tau\omega^\alpha} + \sum_{n<\omega} (e^{\omega^\alpha} - e^{n^\alpha})^\tau a_n$. A series $\sum_{n=0}^{\infty} a_n$ is said to be summable (e^{n^α}, τ) to the sum s if

$$C_\tau(\omega) = s e^{\tau\omega^\alpha} + o(e^{\tau\omega^\alpha}). \tag{1}$$

The result by Hardy which is to be called upon is the following: If the series $\sum_{n=0}^{\infty} a_n$, with terms $a_n \geqq -Kn^{\alpha-1}$, $0<\alpha<1$, is summable (e^{n^α}, τ), it is convergent. We shall now prove the following theorem.

THEOREM. *If* $\sigma_n^{(r)} - s = o(n^{-r\alpha})$, $0<\alpha<1$, *as* $n\rightarrow\infty$, *the series* $\sum_{n=0}^{\infty} a_n$ *is summable* (e^{n^α}, τ) *to the sum s, where* $\tau > r/(1-\alpha)$.

To prove this let $\beta_n = (e^{\omega^\alpha} - e^{n^\alpha})^\tau$, $\Delta\beta_n = \beta_n - \beta_{n+1}$, $\Delta^{r+1}\beta_n = \Delta^r\beta_n - \Delta^r\beta_{n+1}$ and

$$s_n^{(r)} = \sum_{\nu=0}^{n} \binom{n-\nu+r}{n-\nu} a_\nu,$$

$m=[\omega]$. Then, by successive Abel's transformations we have

$$\begin{aligned} C_\tau(\omega) &= a_0 e^{\tau\omega\alpha} + \sum_{n=1}^{m} \beta_n a_n \\ &= a_0 e^{\tau\omega\alpha} + \sum_{n=1}^{m-r+1} s_n^{(r)} \Delta^{r+1}\beta_n + \sum_{i=0}^{r} s_{m-i}^{(i)} \Delta^i \beta_{m-i} - \sum_{i=0}^{r} s_0^{(i)} \Delta^i \beta_1 \qquad (2) \\ &= a_0 e^{\tau\omega\alpha} + J_1 + J_2 - J_3. \end{aligned}$$

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[1] Numbers in brackets refer to the references listed at the end of the paper.

Since $\beta_{m-i}=(e^{\omega^\alpha}-e^{(m-i)^\alpha})^\tau=O(e^{\tau\omega^\alpha}\omega^{\tau(\alpha-1)})$, it follows that

$$\Delta^{(i)}\beta_{m-i}=O(e^{\tau\omega^\alpha}\omega^{\tau(\alpha-1)}) \qquad \text{for } 0\leqq i\leqq r.$$

By a familiar theorem on Cesàro sums we get

$$s_{m-i}^{(i)}=O(\omega^r), \qquad \text{for } 0\leqq i\leqq r,$$

and from this

$$J_2=O(e^{\tau\omega^\alpha}\omega^{\tau(\alpha-1)+r})=o(e^{\tau\omega^\alpha}). \tag{3}$$

Since $\Delta^i\beta_1=O(e^{\tau\omega^\alpha}\omega^{i(\alpha-1)})$, for $1\leqq i\leqq r$, and $\beta_1=(e^{\omega^\alpha}-e)^\tau$, $s_0^{(i)}=s_0=a_0$ we get

$$J_3=e^{\tau\omega^\alpha}a_0+o(e^{\tau\omega^\alpha}). \tag{4}$$

By the hypothesis of the theorem we have

$$\begin{aligned} J_1=\sum_{n=1}^{m-r-1}s_n^{(r)}\Delta^{r+1}\beta_n &= s\sum_{n=1}^{m-r-1}\binom{n+r}{n}\Delta^{r+1}\beta_n \\ &\quad+o\left(\sum_{n=1}^{m-r-1}n^{r(1-\alpha)}\left|\Delta^{r+1}\beta_n\right|\right). \end{aligned} \tag{5}$$

It follows by mathematical induction that

$$\Delta^{r+1}\beta_n=(-1)^{r+1}\int_n^{n+1}dx_1\int_{x_1}^{x_1+1}dx_2\cdots\int_{x_r}^{x_r+1}B^{(r+1)}(x_{r+1})dx_{r+1},$$

where

$$\beta^{(r+1)}(x)=\frac{d^{r+1}}{dx^{r+1}}\left\{(e^{\omega^\alpha}-e^{x^\alpha})^\tau\right\}.$$

By direct differentiation we have

$$B^{(r+1)}(x)=\sum_{j=1}^{r+1}\sum_{i=1}^{\tau}c_{ij}e^{(\tau-i)\omega^\alpha}e^{ix^\alpha}x^{j\alpha-r-1},$$

where τ is taken as a positive integer and the constants c_{ij} depend upon i, j, τ, r, α. Hence we get

$$\Delta^{r+1}\beta_n=\sum_{i=1}^{\tau}O(e^{(\tau-i)\omega^\alpha}e^{in^\alpha}n^{(r+1)(\alpha-1)}). \tag{6}$$

It is easily verified by Abel's transformation that

$$\sum_{n=1}^{m-r-1}\binom{n+r}{n}\Delta^{r+1}\beta_n=e^{\tau\omega^\alpha}+o(e^{\tau\omega^\alpha}). \tag{7}$$

Hence by (5), (6), and (7)

$$
\begin{aligned}
J_1 &= se^{r\omega^\alpha} + o(e^{r\omega^\alpha}) + o\left(\sum_{i=1}^{r}\sum_{n=1}^{m-r-1} e^{(r-i)\omega^\alpha} e^{in^\alpha} n^{\alpha-1}\right) \\
&= se^{r\omega^\alpha} + o(e^{r\omega^\alpha}).
\end{aligned}
\tag{8}
$$

The proof of the theorem follows from (4), (2), (3), and (8).

I conclude by observing that the theorem is the best possible of its kind (Wang [4]).

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ON RIESZ SUMMABILITY OF FOURIER SERIES BY EXPONENTIAL MEANS

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Let $f(t)$ be an integrable periodic function with the period 2π. Let its Fourier series be

$$(1) \qquad f(t) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nt + b_n \sin nt$$

and let

$$\phi(t) = \{f(x+t) + f(x-t) - 2s\}/2,$$

$$\phi_\beta(t) = (1/\Gamma(\beta)) \int_0^t (t-u)^{\beta-1}\phi(u)du,$$

$$A_n = a_n \cos nx + b_n \sin nx.$$

We shall prove the following result.[1]

If $A_n > -Kn^{-\beta/\gamma}$ $(\gamma > \beta > 0)$ *and*

$$(2) \qquad \phi_\beta(t) = o(t^\gamma) \qquad \text{as } t \to 0,$$

the Fourier series (1) *converges to* s *at* $t = x$.

Set $\alpha = 1 - \beta/\gamma$, and

$$(3) \qquad C_\tau(\omega) = a_0 e^{\tau\omega^\alpha}/2 + \sum_{n<\omega} (e^{\omega^\alpha} - e^{n^\alpha})^\tau A_n.$$

The Fourier series (1) is said to be summable (e^{n^α}, τ) to the sum s if[2]

$$C_\tau(\omega) = se^{\tau\omega^\alpha} + o(e^{\tau\omega^\alpha}) \qquad \text{as } \omega \to \infty.$$

Concerning this kind of summability we have the following theorem.

THEOREM.[3] *If* (2) *holds and* τ *is a positive integer greater than* $\gamma + 1$ *the Fourier series* (1) *is summable* (e^{n^α}, τ) *to the sum* s *at* $t = x$.

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[1] G. H. Hardy and J. E. Littlewood [2], F. T. Wang [6]. Numbers in brackets refer to the references listed at the end of the paper.

[2] G. H. Hardy and M. Riesz [3].

[3] Under the hypotheses of the Theorem I have established that the Fourier series (1) is summable $(e^{n^\alpha}, \gamma+\delta)(\delta>0)$ to the sum s at $t = x$, but the proof is very complicated. See F. T. Wang [6].

The convergence criterion above is deducible from this theorem by the use of a result of Hardy,[4] namely *if the series* $\sum_{n=0}^{\infty} a_n$, *with terms* $a_n \geqq -Kn^{\alpha-1}$, $0<\alpha<1$, *is summable* (e^{n^α}, τ), *it is convergent.*

To prove the theorem we write

$$E_\tau(\omega, t) = \tau\alpha \int_0^\omega (e^{\omega^\alpha} - e^{x^\alpha})^{\tau-1} e^{x^\alpha} x^{\alpha-1} \frac{\sin xt}{t} dx.$$

Then we have[5] $C_\tau(\omega) = (2/\pi)\int_0^1 \phi(t) E_\tau(\omega, t)dt + se^{\tau\omega^\alpha} + o(e^{\tau\omega^\alpha})$, and if we take $\omega_1 = 2^{-1/\alpha}\omega$, then

$$\begin{aligned} E_\tau(\omega, t) &= \alpha\tau \int_{\omega_1}^\omega (e^{\omega^\alpha} - e^{x^\alpha})^{\tau-1} e^{x^\alpha} x^{\alpha-1} \frac{\sin xt}{t} dx + o(e^{\tau\omega^\alpha}) \\ &= F_\omega(t) + o(e^{\tau\omega^\alpha}). \end{aligned}$$

Hence

$$C_\tau(\omega) = (2/\pi) \int_0^1 \phi(t) F_\omega(t) dt + se^{\tau\omega^\alpha} + o(e^{\tau\omega^\alpha}). \tag{4}$$

By setting $n = [\beta]+1$ and differentiating under the integral sign we get

$$\begin{aligned} F_\omega^{(n)}(t) &= \frac{d^n}{dt^n}\left\{\alpha\tau \int_{\omega_1}^\omega (e^{\omega^\alpha} - e^{x^\alpha})^{\tau-1} e^{x^\alpha} x^{\alpha-1} \frac{\sin xt}{t} dx\right\} \\ &= \sum_{i=0}^{n} K_i \int_{\omega_1}^\omega (e^{\omega^\alpha} - e^{x^\alpha})^{\tau-1} e^{x^\alpha} x^{\alpha-1+n-i} \frac{\sin (xt-a)}{t^{i+1}} dx, \end{aligned} \tag{5[6]}$$

where $a = (n-i)\pi/2$.

By mathematical induction we can easily establish the formula

$$\begin{aligned} &(d^\tau/dx^\tau)\left\{(e^{\omega^\alpha} - e^{x^\alpha})^{\tau-1} e^{x^\alpha} x^{\alpha-1+n-i}\right\} \\ &\qquad = \sum_{j=1}^{\tau}\sum_{p=0}^{\tau} K_{ip} e^{(\tau-j)\omega^\alpha} e^{jx^\alpha} x^{(p+1)(\alpha-1)+n-i-(\tau-p)}. \end{aligned} \tag{6}$$

Then by the use of (5) and (6) and an integration by parts we find

$$\begin{aligned} F_\omega^{(n)}(t) = \sum_{i=0}^{n}\sum_{j=1}^{\tau}\sum_{p=c}^{\tau-1} K_{ijp} e^{(\tau-j)\omega^\alpha} \int_{\omega_1}^\omega e^{jx^\alpha} x^c \frac{\sin (xt-b)}{t^{i+\tau}} dx \\ + O(e^{(\tau-1/2)\omega^\alpha} \omega^{k_1} t^{-k_2}), \end{aligned} \tag{7}$$

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[4] G. H. Hardy [4].

[5] F. T. Wang [6].

[6] Throughout this paper we use K or $K_i \cdots$ as a constant different in different occurrences.

where $c=(p+1)(\alpha-1)+n-i-(\tau-1-p)$, and $b=(n-i-\tau+1)\pi/2$.

Put $t=1$ in (7). Then $F_\omega^{(m)}(0)$ is finite for $1\leq m<n$, and $F_\omega^{(m)}(1) =o(e^{\tau\omega^\alpha})$. Successive integration of (4) by parts yields

$$C_\tau(\omega) = (2/\pi)\int_0^1 \phi_n(t\, F_\omega^{(n)}(t)dt + se^{\tau\omega^\alpha} + o(e^{\tau\omega^\alpha}). \tag{8}$$

By a theorem on the fractional integral[7]

$$\phi_n(t) = (1/\Gamma(n-\beta))\int_0^t (t-u)^{n-\beta-1}\phi_\beta(u)du,$$

we have

$$C_\tau(\omega) = (2/\pi)\int_0^1 \phi_\beta(u)H_\omega(u)du + se^{\tau\omega^\alpha} + o(e^{\tau\omega^\alpha}), \tag{9}$$

where

$$\begin{aligned} H_\omega(u) &= (1/\Gamma(n-\beta))\int_u^1 (t-u)^{n-\beta-1}F_\omega^{(n)}(t)dt \qquad (n>\beta>n-1) \\ &= F_\omega^{(n)}(u) \qquad (n=\beta). \end{aligned} \tag{10}$$

Concerning $H_\omega(u)$ we require the following two lemmas.

LEMMA 1. *For* $\omega>K$ *and* $0<u<1$,

$$\begin{aligned} H_\omega(u) = \sum_{i=0}^{n-1} O(e^{\tau\omega^\alpha}\omega^{\beta-i}u^{-i-1}) + \sum_{i=0}^{n-1} O(e^{\tau\omega^\alpha}\omega^{n-i-1}u^{n-\beta-i-2}) \\ + O(e^{\tau\omega^\alpha}\omega^{n-1}(1-u)^{n-\beta-1}) + O(e^{\tau\omega^\alpha}u^{-\beta-1}). \end{aligned}$$

PROOF. From (10) and (5) we have

$$\begin{aligned} H_\omega(u) = \sum_{i=0}^{n} K_i\int_{\omega_1}^{\omega} (e^{\omega^\alpha} - e^{x^\alpha})^{\tau-1}e^{x^\alpha}x^{\alpha-1+n-i}dx \\ \cdot\int_u^1 (t-u)^{n-\beta-1}\frac{\sin(xt-a)}{t^{i+1}}dt. \end{aligned} \tag{11}$$

Now

$$\int_1^\infty (t-u)^{n-\beta-1}\frac{\sin(xt-a)}{t^{i+1}}dt = O\{(1-u)^{n-\beta-1}x^{-1}\}. \tag{12}$$

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[7] L. S. Bosanquet [1].

It follows from a change of variable, the second mean value theorem, and a theorem on the Γ function,[8] that

$$
\begin{aligned}
&\int_u^\infty (t-u)^{n-\beta-1}\frac{\sin(xt-a)}{t^{i+1}}\,dt \\
&\qquad = u^{n-\beta-i-1}\int_0^\infty v^{n-\beta-1}\sin\left(x^u(1+v)-a\right)dv + O(u^{n-\beta-i-2}x^{-1}) \qquad (13)\\
&\qquad = O(u^{-i-1}x^{\beta-n}) + O(u^{n-\beta-i-2}x^{-1}).
\end{aligned}
$$

The lemma is proved by (11), (12), (13) and an easy estimate of the term $i=n$ in (11).

Lemma 2. *For* $\omega>K$ *and* $0<u<1$,

$$
\begin{aligned}
H_\omega(u) = {} & \sum_{i=0}^{n} O(e^{\tau\omega^\alpha}u^{-\tau-i-1}\omega^{\tau(\alpha-1)+\beta-i}) + O(e^{(\tau-1/2)\omega^\alpha}\omega^{k_1}u^{-k_2}) \\
& + \sum_{i=0}^{n} O(e^{\tau\omega^\alpha}u^{n-\beta-\tau-i-1}\omega^{(\tau-1)(\alpha-1)+n-i-1}) \\
& + O(e^{\tau\omega^\alpha}\omega^{(\tau-1)(\alpha-1)+n-1}(1-u)^{n-\beta-1}).
\end{aligned}
$$

Proof. From (13) we get

$$
\begin{aligned}
H_\omega(u) = {} & \sum_{i=0}^{n}\sum_{j=1}^{\tau}\sum_{p=0}^{\tau-1} K_{ijp}e^{(\tau-j)\omega^\alpha}\int_{\omega_1}^{\omega} e^{jx^\alpha}x^c\,dx \qquad (14)\\
& \cdot\int_u^1 (t-u)^{n-\beta-1}\frac{\sin(xt-b)}{t^{i+\tau}}\,dt + O(e^{(\tau-1/2)\omega^\alpha}\omega^{k_1}u^{-k_2}),
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_u^1 (t-u)^{n-\beta-1}\frac{\sin(xt-b)}{t^{i+\tau}}\,dt \\
&\qquad = u^{n-\beta-i-\tau}\int_0^\infty v^{n-\beta-1}\sin\left\{xu(1+v)-b\right\}dv \qquad (15)\\
&\qquad\quad + O((1-u)^{n-\beta-1}x^{-1}) + O(u^{n-\beta-i-\tau-1}x^{-1}) \\
&\qquad = u^{-\tau-i}x^{\beta-n}\Gamma(\beta-n)\sin(xu-b') \\
&\qquad\quad + O((1-u)^{n-p-1}x^{-1}) + O(u^{n-\beta-i-\tau-1}x^{-1}).
\end{aligned}
$$

From (14) and (15), Lemma 2 follows.

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[8] E. C. Titchmarsh [5, p. 107].

PROOF OF THE THEOREM. By Lemma 1 and (2)

$$\int_0^{\omega^{\alpha-1}} \phi_\beta(u) H_\omega(u) du = o(e^{\tau\omega^\alpha}) \qquad \text{as } \omega \to \infty, \tag{16}$$

and by (2) and Lemma 2

$$\int_{\omega^{\alpha-1}}^{1} \phi_\beta(u) H_\omega(u) du = o(e^{\tau\omega^\alpha}) \qquad \text{as } \omega \to \infty. \tag{17}$$

By (9), (16), and (17), then,

$$C_\tau(\omega) = se^{\tau\omega^\alpha} + o(e^{\tau\omega^\alpha}) \qquad \text{as } \omega \to \infty.$$

Thus the theorem is proved.

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NATIONAL UNIVERSITY OF CHEKIANG

ON THE ASYMPTOTIC EXPANSIONS OF ENTIRE FUNCTIONS DEFINED BY MACLAURIN SERIES

H. K. HUGHES

1. Introduction. Let $f(z)$ be an entire function defined by a Maclaurin series of the form

$$\sum_{n=0}^{\infty} g(n) z^n \tag{1}$$

with infinite radius of convergence. We wish, in this paper, to make some observations of a general nature relative to the asymptotic behavior of this function in the neighborhood of the point at infinity. Some restrictions will, of course, be placed on the coefficient $g(n)$ of z^n.

To begin with, we shall suppose $g(n)$ to be of such character that we can replace n by the complex variable $w=x+iy$. Then we shall assume that the resulting function $g(w)$ satisfies the following two conditions when it is considered throughout any arbitrary right half-plane $x>x_0$:

(a) it is single-valued and analytic;

(b) corresponding to any positive number ϵ, there exists a positive constant K, dependent only on ϵ and x_0, such that

$$|g(x+iy)| < Ke^{(\gamma\pi+\epsilon)|y|} \tag{2}$$

where γ is a non-negative constant.

Under these conditions, if k is a positive integer such that $k \geqq \gamma$, then it is known that we may express $f(z)$ in the form

$$\begin{aligned} f(z) = {} & \int_{-l-1/2}^{\infty} g(x)[(-1)^{k+1}z]^x \frac{\sin k\pi x}{\sin \pi x}\, dx \\ & - \sum_{n=1}^{l} g(-n) z^{-n} + \xi(z, l), \end{aligned} \tag{3}$$

where l is any positive integer, and where if $|\arg [(-1)^{k+1}z]| < \pi$, we have $\lim_{|z|\to\infty} z^l \xi(z, l) = 0$.

This theorem was first proved for the special case in which $k=1$ by W. B. Ford [1, p. 30].[1] The general form (3) is due to C. V.Newsom [4]. Both Ford and Newsom have used their results in finding the asymptotic expansions of several special types of functions [1, chap-

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[1] Numbers in square brackets refer to references at the end of the paper.

ters 6, 7; 5; 6]. The theorem was also used in [2]. A study of the examples appearing in these investigations has suggested to the writer the generalities which are noted in the following sections.

2. A fundamental lemma. It will be observed that equation (3) does not furnish complete information regarding the asymptotic expansions of the function $f(z)$, due to the presence of the integral in the right-hand member. By finding the asymptotic expansions of this integral under suitable restrictions on $g(w)$, we shall arrive at a general theorem concerning such expansions of $f(z)$.

The proof of the theorem rests upon a lemma regarding the asymptotic character of a certain definite integral. This lemma is as follows:

LEMMA. *Consider the function*

$$F(z) = \int_{-\lambda}^{\infty} H(x) z^x dx, \tag{4}$$

where λ *is a sufficiently large positive number.*[2] *Suppose that the function* $H(w)$, *where* $w = x + iy$, *is single-valued and analytic in the finite w-plane and, when considered throughout the arbitrary right half-plane* $x > x_0$, *can be expressed in the form*

$$H(w) = \frac{c_0}{\Gamma(w+1)} + \frac{c_1}{\Gamma(w+2)} + \cdots + \frac{c_s + \delta(w, s)}{\Gamma(w+s+1)},$$

where the c's are constants, and $\lim_{|w|\to\infty} \delta(w, s) = 0$; $s = 0, 1, 2, \cdots$. *Then for values of z of large modulus such that* $|\arg z| < \pi$, *we have*

$$F(z) \sim \begin{cases} e^z \sum_{n=0}^{\infty} c_n z^{-n}; & |\arg z| \leqq \pi/2, \\ 0; & |\arg z| > \pi/2. \end{cases} \tag{5}$$

The proof of this lemma is omitted here since it requires only slight changes in one of the discussions appearing in the work of Ford [1, art. 20].

3. A general theorem. We proceed now to study the function $f(z)$ defined by the series (1). We shall establish the following general theorem regarding its asymptotic expansions.

THEOREM. *Given the entire function*

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[2] The result also holds if λ is a complex number whose real part is positive and sufficiently large. The integration then takes place from $-\lambda$ to $+\infty$ along the line parallel to the real axis.

$$f(z) = \sum_{n=0}^{\infty} g(n) z^n, \tag{6}$$

suppose that the function $g(w)$, *where* $w = x + iy$, *is single-valued and analytic in the finite w-plane and, when considered throughout the arbitrary right half-plane* $x > x_0$, *can be expressed in the form*

$$g(w) = \sigma^w \left\{ \frac{c_0}{\Gamma(\alpha w + t)} + \frac{c_1}{\Gamma(\alpha w + t + 1)} + \cdots + \frac{c_s + \delta(\alpha w, s)}{\Gamma(\alpha w + t + s)} \right\}, \tag{7}$$

where σ *and* α *are positive, the c's and t are constants, and* $\lim_{|w| \to \infty} \delta(\alpha w, s) = 0$; $s = 0, 1, 2, \cdots$. *Then, when considered for values of z of large modulus such that* $-\pi < \arg z \leqq \pi$, *the function* $f(z)$ *is developable asymptotically in the form*

$$f(z) \sim \frac{1}{\alpha} \sum_{\mu} \left\{ Y_\mu^{(1-t)} e^{Y_\mu} \sum_{n=0}^{\infty} c_n Y_\mu^{-n} \right\} - \sum_{n=1}^{\infty} g(-n) z^{-n}, \tag{8}$$

where Y_μ *denotes the expression* $\sigma^{1/\alpha} z^{1/\alpha} e^{2\pi i \mu/\alpha}$, *and the symbol* $\sum_\mu$ *denotes summation over those integral values of* μ *which satisfy* $|\arg z + 2\pi\mu| \leqq \pi\alpha/2$.

PROOF. Under the conditions postulated, when $x > x_0$, we may write

$$|g(x + iy)| < K e^{(\pi\alpha/2 + \epsilon)|y|}$$

where ϵ and K have the meanings described in §1.[3] Consequently, the theorem of Ford and Newsom is applicable to the function $f(z)$ if k is properly chosen. If a proper choice be made, then, for the corresponding range of arg z, we have

$$f(z) = \phi(z) - \sum_{n=1}^{l} g(-n) z^{-n} + \xi(z, l), \tag{9}$$

where $\phi(z)$ denotes the integral

$$\int_{-l-1/2}^{\infty} g(x) [(-1)^{k+1} z]^x \frac{\sin k\pi x}{\sin \pi x} dx.$$

In particular, if k is selected as the smallest odd integer $2p+1$ such that $k \geqq \alpha/2$, then the factor $[(-1)^{k+1} z]^x$ in the integrand of $\phi(z)$ becomes z^x, and (9) is then valid when $|\arg z| < \pi$. On the other hand, if k is taken as $2p+2$, then this factor becomes $(-z)^x$, and (9) holds for $0 < \arg z = \pi + \arg(-z) < 2\pi$, and hence in particular when $\arg z = \pi$.

---

[3] For the special property of the gamma function from which this inequality results, see [1, p. 61].

Now we have

$$\frac{\sin k\pi x}{\sin \pi x} = \begin{cases} \displaystyle\sum_{\mu=-p}^{p} e^{2\pi i\mu x}; & k = 2p+1, \\ \displaystyle e^{\pi i x}\sum_{\mu=-p-1}^{p} e^{2\pi i\mu x}; & k = 2p+2. \end{cases}$$

And since $e^{\pi i x}\cdot(-z)^x = z^x$, it follows that for either of the two values of k mentioned, the integral $\phi(z)$ is equal to the sum of k integrals of the form

$$h_\mu(z) = \int_{-l-1/2}^{\infty} g(x)(z\, e^{2\pi i\mu})^x dx, \tag{10}$$

where we now have $-\pi < \arg z \leqq \pi$, and the integer μ takes on the values indicated in the above identities.

We shall next obtain the asymptotic expansions of the integral in (10). First, we recall that $g(x)$ is expressible in form (7). Hence, if we change the variable of integration in (10) from x to x' through the transformation $\alpha x + t = x' + 1$, and then drop the primes, (10) assumes the form

$$h_\mu(z) = \frac{1}{\alpha} Y_\mu^{(1-t)} \int_{-\lambda}^{\infty} H(x) Y_\mu^x dx, \tag{11}$$

where Y_μ is as already defined, $\lambda = \alpha(l+1/2)+1-t$, and $H(x)$ is of the form

$$\frac{c_0}{\Gamma(x+1)} + \frac{c_1}{\Gamma(x+2)} + \cdots + \frac{c_s + \delta(x, s)}{\Gamma(x+s+1)}.$$

We may therefore apply the fundamental lemma of §2 to the integral appearing in (11), assuming that l has been chosen sufficiently large. In (5), z is to be replaced by Y_μ, and $\arg z$ by $(\arg z + 2\pi\mu)/\alpha$. Thus, for values of z such that $-\pi < \arg z \leqq \pi$ and $|\arg z + 2\pi\mu| < \pi\alpha$, we have

$$h_\mu(z) \sim \begin{cases} \displaystyle\frac{1}{\alpha} Y_\mu^{(1-t)} e^{Y_\mu} \sum_{n=0}^{\infty} c_n (Y_\mu)^{-n}; \\ 0, \end{cases} \tag{12}$$

the first result holding when $|\arg z + 2\pi\mu| \leqq \pi\alpha/2$, and the second otherwise.

Having established (12), we have only to sum the expansions of

$h_\mu(z)$ over the proper values of μ in order to arrive at the expansion of the integral $\phi(z)$. Evidently this will yield the expansion

$$\phi(z) \sim \frac{1}{\alpha} \sum_{\mu} \left(Y_\mu^{(1-t)} e^{Y_\mu} \sum_{n=0}^{\infty} c_n Y_\mu^{-n} \right), \tag{13}$$

where $\sum_\mu$ has the meaning already stated.

Finally, if we replace $\phi(z)$ in (9) by its asymptotic expansion given in (13), and write the result in asymptotic notation, we arrive at (8). This concludes the proof of the theorem.

4. **Special cases. Dominant terms.** We shall now note the simplified forms assumed by (8) for certain values of α and of $\arg z$. At the same time, we shall pick out the dominant terms in each form. First, if α is such that $0<\alpha<2$, then the condition $|\arg z+2\pi\mu| \leqq \pi\alpha/2$ is satisfied only by $\mu=0$ when $|\arg z| \leqq \pi\alpha/2$. Hence for such α and $\arg z$, we have

$$f(z) \sim \frac{1}{\alpha} (\sigma z)^{(1-t)/\alpha} \exp (\sigma z)^{1/\alpha} \sum_{n=0}^{\infty} c_n (\sigma z)^{-n/\alpha} - \sum_{n=1}^{\infty} g(-n) z^{-n}. \tag{14}$$

Here we see that the first series is dominant when $\arg z$ is confined to the narrower range $|\arg z| <\pi\alpha/2$, but not when $|\arg z| =\pi\alpha/2$. Furthermore, if $\alpha=2$, and $|\arg z| <\pi$, expansion (14) still holds, and the first series is dominant.

On the other hand, if we have $0<\alpha<2$ and $|\arg z| >\pi\alpha/2$, but still such that $-\pi<\arg z\leqq\pi$, then no integral value of μ satisfies $|\arg z+2\pi\mu| \leqq\pi\alpha/2$. Hence, for such values of z, (8) reduces to the form

$$f(z) \sim - \sum_{n=1}^{\infty} g(-n) z^{-n}. \tag{15}$$

It is to be noted that this result is consistent with another theorem due to Ford [1, p. 4] according to which (15) holds throughout the open sector $\pi\alpha/2 <\arg z< (2-\alpha/2)\pi$.

Finally, we note that when $\alpha \geqq 2$, the terms affected by $\sum_\mu$ are dominant over the last series in (8), this being true for all values of $\arg z$ under consideration.

5. **Supplementary remarks.** Several supplementary observations relative to the theorem established in §3 are now to be noted. In the first place, we have required that the function $g(w)$ have no finite singularities. Suppose, however, that it does have a singularity at $w=w_0$, while (7) continues to hold for all values of $|w|$ sufficiently

large. Then if w_0 is not a negative integer, the theorem still holds provided that we subtract from the right member of (8) the asymptotic expansion of the loop integral

$$\frac{1}{2i}\int_C \frac{g(w)[(-1)^{k+1}z]^w}{e^{ki\pi w}\sin \pi w}\,dw. \tag{16}$$

The loop C surrounds the point w_0 and extends to infinity in any convenient direction lying in either the third or the fourth quadrant. If w_0 is a negative integer, say $-m$, then the term $g(-m)z^{-m}$ in (8) is suppressed, and the expansion of (16) is put in its place. For justification of these statements, the reader is referred to [3], where a method for finding the expansion of (16) is also discussed.

Secondly, the theorem which we have here established narrows down the difficulties involved in determining the asymptotic expansion of a given entire function of form (1). For the problem of finding the expansion of the integral appearing in (3) is solved if it can be shown that the function $g(w)$ can be written in the form (7).

Thirdly, the theorem can evidently be applied to a large variety of entire functions. In many instances in which the coefficient $g(n)$ involves the gamma function, relation (7) is satisfied. In this connection, we note that if the value of σ in (7) is unity, then $f(z)$ is necessarily the function

$$F_\alpha(z, t) = \sum_{n=0}^{\infty} \frac{h(n)}{\Gamma(\alpha n + t)} z^n$$

which was discussed in [2]. The writer plans to apply the theorem to certain further important types of functions, and to report the results in later papers.

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PURDUE UNIVERSITY

SCHUR'S THEOREMS ON COMMUTATIVE MATRICES

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In 1905 I. Schur[1] proved that the maximum number $N(n)$ of linearly independent commutative matrices of n rows and columns is given by the formula $N(n)=[n^2/4]+1=\nu^2+1$ if $n=2\nu$ and $=\nu(\nu-1)+1$ if $n=2\nu-1$. Schur also determined the sets of linearly independent commutative matrices containing $N(n)$ elements. In this note we give a simpler derivation of Schur's results and an extension of these results from algebraically closed fields to arbitrary fields.

If $A_1, \cdots, A_{N(n)}$ is a set of linearly independent commutative matrices, the set $\mathfrak{A}$ of matrices $\sum A_i\phi_i$ where ϕ_i is arbitrary in the underlying field Φ is a commutative subalgebra containing the identity of the matrix algebra Φ_n. Hence $N(n)$ is the maximal dimensionality of commutative subalgebras of Φ_n. It is easy to see that $N(n)\geqq[n^2/4]+1$. For consider the set $\mathfrak{Z}_n$ of matrices

$$\begin{pmatrix}0 & A\\ 0 & 0\end{pmatrix} \tag{1}$$

where if $n=2\nu$, A is arbitrary in Φ_ν and if $n=2\nu-1$, A is an arbitrary matrix of ν rows and $\nu-1$ columns. Thus $\dim \mathfrak{Z}_n=[n^2/4]$. It may be verified that $\mathfrak{Z}_n$ is a zero algebra. Hence the algebra $\mathfrak{B}_n$ obtained by adjoining 1 to $\mathfrak{Z}_n$ is a commutative algebra of dimensionality $[n^2/4]+1$. We remark also that if $n=2\nu-1$ we may replace $\mathfrak{Z}_n$ by the algebra $\overline{\mathfrak{Z}}_n$ of matrices of the form (1) in which A is an arbitrary matrix of $\nu-1$ rows and ν columns. We denote by $\overline{\mathfrak{B}}_n$ the extension of $\overline{\mathfrak{Z}}_n$ obtained by adjoining 1.

To prove that $N(n)\leqq[n^2/4]+1$ it suffices to assume that Φ is algebraically closed. For if $A_1, \cdots, A_{N(n)}$ are linearly independent and commutative in Φ_n, then they have these properties in Σ_n for any extension field Σ of the field Φ. Thus $N(n, \Phi)\leqq N(n, \Sigma)$. We shall therefore assume that Φ is algebraically closed. Let $\mathfrak{A}$ be a commutative subalgebra of Φ_n containing the identity and let N be the dimensionality of $\mathfrak{A}$ over Φ. We suppose first that $\mathfrak{A}$ is an indecomposable algebra of matrices. Then it is known that by replacing $\mathfrak{A}$ by a similar set we may suppose that the matrices of $\mathfrak{A}$ have the form

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[1] *Zur Theorie vertauschbaren Matrizen*, J. Reine Angew. Math. vol. 130 (1905) pp. 66–76.

$$\begin{pmatrix} \alpha & & & & * \\ & \alpha & & & \\ & & \cdot & & \\ & & & \cdot & \\ 0 & & & & \alpha \end{pmatrix}. \tag{2}$$

Thus $\mathfrak{A}=(1)+\mathfrak{N}$ where $\mathfrak{N}$ is a nilpotent algebra of matrices in proper triangular form, that is, of the form (2) in which $\alpha=0$. Evidently $\dim \mathfrak{N}=N-1$.

Let the k_1th column ($k_1>1$) be the first column for which there exists a matrix U_{1k_1} in $\mathfrak{N}$ with element in the $(1, k_1)$ position not equal to 0. We may suppose that the element in the $(1, k_1)$ position of U_{1k_1} is 1. We normalize U_{1k_1} further by using the following lemma.

Lemma 1. *Let $U\in\Phi_n$ and let V be the matrix obtained from U by adding the kth column multiplied by θ to the lth column ($k\neq l$) and then subtracting the lth row multiplied by θ from the kth row. Then U and V are similar.*

We have $V=S^{-1}US$ where $S=1+e_{kl}\theta$, e_{kl} the matrix with 1 in the (k, l) position and 0's elsewhere.

We may apply this lemma to U_{1k_1} and replace it by a matrix whose first row is $e_{k_1}=(0, \cdots, 1, 0, \cdots, 0)$ where the 1 is in the k_1th column. The operations required for this purpose are additions of multiples of the k_1th column to later columns and additions to the k_1th row of later rows. These operations replace $\mathfrak{N}$ by a properly triangular set of matrices $\mathfrak{N}'$ similar to $\mathfrak{N}$ such that all the elements in the $(1, j)$ position with $j<k_1$ in $\mathfrak{N}'$ are 0 and such that $\mathfrak{N}'$ contains a matrix V_{1k_1} (similar to U_{1k_1}) whose first row is e_{k_1}. Now let $\mathfrak{P}'$ be the subspace of $\mathfrak{N}'$ of matrices in which the elements in the $(1, k_1)$ position are 0 and suppose that the k_2th column ($k_2>k_1$) is the first column for which there is a matrix U_{1k_2} in $\mathfrak{P}'$ with element in the $(1, k_2)$ place not equal to 0. Evidently any matrix in $\mathfrak{N}'$ has the form $V_{1k_1}\beta_1+P'$, P' in $\mathfrak{P}'$. We now apply to U_{1k_2} the process used before for U_{1k_1} and replace it by a matrix V_{1k_2} similar to it and having e_{k_2} for first row. The set $\mathfrak{N}'$ will be transformed into a set $\mathfrak{N}''$ of properly triangular matrices and V_{1k_1} changed into a new matrix which we shall again denote as V_{1k_1} with first row e_{k_1}. Any matrix in $\mathfrak{N}''$ has the form $A=V_{1k_1}\beta_1+P''$, P'' in $\mathfrak{P}''$, the transform of the set $\mathfrak{P}'$. It is clear that the elements in the $(1, j)$ position, $j<k_2$, for any matrix in $\mathfrak{P}''$ are 0. Hence $A=V_{1k_1}\beta_1+V_{1k_2}\beta_2+S''$ where S'' is in the subspace $\mathfrak{S}''$ of $\mathfrak{N}''$ of matrices having 0 in the $(1, j)$ position with $j\leq k_2$. This process may be continued and proves the following lemma.

Lemma 2. *The set $\mathfrak{N}$ is similar to a set $\mathfrak{N}^{(r)}$ of properly triangular* *matrices that contain matrices* $V_{1k_1}, \cdots, V_{1k_r}$ *such that the first row of* V_{1k_i} is e_{k_i}, $1<k_1<k_2< \cdots <k_r$, *and such that any matrix in* $\mathfrak{N}^{(r)}$ *has the form* $\sum V_{1k_i}\beta_i+Z$, *where* Z *has first row* 0.

Now let $\mathfrak{N}_2$ be the subset of $\mathfrak{N}^{(r)}$ of matrices Z having first row 0. Evidently $\mathfrak{N}^{(r)}=\{V_{1k_1}, \cdots, V_{1k_r}\}+\mathfrak{N}_2$ and the V_{1k_i} are linearly independent. Hence $\dim \mathfrak{N}^{(r)}=N-1=r+\dim \mathfrak{N}_2$. Now we note that if $Z\in\mathfrak{N}_2$, the first row of $V_{1k_i}Z$ is the k_ith row of Z and the first row of ZV_{1k_i} is 0. Hence the k_ith row of every matrix Z in $\mathfrak{N}_2$ is 0.

We now repeat the argument for $\mathfrak{N}_2$. Then $\mathfrak{N}_2$ may be replaced by a set $\mathfrak{N}_2^{(s)}$ similar to $\mathfrak{N}_2$ such that (1) $\mathfrak{N}_2^{(s)}$ is properly triangular, (2) $\mathfrak{N}_2^{(s)}$ contains matrices $V_{2l_1}, \cdots, V_{2l_s}$ having first row 0 and second row $e_{l_1}, \cdots, e_{l_s}$, respectively, such that any matrix in $\mathfrak{N}_2^{(s)}$ has the form $\sum V_{2l_i}\beta_i+Z$ where Z is a matrix with first two rows 0. Let $\mathfrak{N}_3$ denote the set of matrices Z. We assert that if $s=l_i$ or $s=k_j$ then the sth row of $\mathfrak{N}_3$ is 0. This is clear if $s=l_i$. Hence suppose that $s=k_j\neq$any l_i. Then the matrices of $\mathfrak{N}_2$ all have k_jth row 0 and the operations performed in passing from $\mathfrak{N}_2$ to $\mathfrak{N}_2^{(s)}$ do not affect this row. Hence the k_jth row of every matrix in $\mathfrak{N}_2^{(s)}$ is 0. Evidently $N-1=r+s+\dim \mathfrak{N}_3$.

We now write $k_i=k_{1i}$, $l_i=k_{2i}$, $r=r_1$, $s=r_2$. Then if we continue this process we see that $N-1$ is equal to the number of matrices in the following set

$$
\begin{array}{c}
e_{1k_{11}}, \cdots, e_{1k_{1r_1}} \\
e_{2k_{21}}, \cdots, e_{2k_{2r_2}} \\
\cdots\cdots\cdots
\end{array}
\tag{3}
$$

where $1<k_{11}< \cdots <k_{1r_1}$, $2<k_{21}<k_{22}< \cdots <k_{2r_2}, \cdots$, and $r_i=0$ if $i=k_{jl}$ with $j<i$. Let $s_1, s_2, \cdots, s_m$ be the complete set of integers k_{ij} arranged in increasing order. Then it is clear that $N-1 \leqq N(s_1, s_2, \cdots, s_m)$, the number of matrices in the set

$$
\begin{array}{l}
e_{1s_1}, e_{2s_1}, \cdots, e_{s_1-1,s_1} \\
e_{1s_2}, e_{2s_2}, \cdots, e_{s_1-1,s_2}, e_{s_1+1,s_2}, \cdots, e_{s_2-1,s_2} \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots .
\end{array}
\tag{4}
$$

Evidently

$$
\begin{aligned}
N(s_1, s_2, \cdots, s_m) &= (s_1-1)+(s_2-2)+\cdots+(s_m-m) \\
&= \sum s_i - m(m+1)/2.
\end{aligned}
\tag{5}
$$

Hence we have

$$N-1 \leqq N(s_1, \cdots, s_m) \leqq N(n-m+1, \cdots, n) \tag{6}$$
$$= m(n-m).$$

Now $m(n-m)$ attains its maximum value for $m=[n/2]$. If $n=2\nu$ this maximum is ν^2 and if $n=2\nu-1$, it is $\nu(\nu-1)$. Thus the maximum value is $[n^2/4]$. This proves for indecomposable algebras $\mathfrak{A}$ the following theorem.

THEOREM 1. *If* $\mathfrak{A}$ *is a commutative subalgebra of* Φ_n, dim $\mathfrak{A} \leqq [n^2/4]+1$.

If $\mathfrak{A}$ is decomposable we suppose that the matrices of $\mathfrak{A}$ have the form

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$$

where $A \in \Phi_{n_1}$ and $B \in \Phi_{n_2}$, $n_i \geqq 1$, $n_1+n_2=n$. We may assume that the theorem holds for the Φ_{n_i}.

Case 1. $n=2\nu-1$, $n_1=2\nu_1-1$, $n_2=2\nu_2$. Here $\nu=\nu_1+\nu_2$ and $N \leqq \nu_1(\nu_1-1)+1+\nu_2^2+1 \leqq \nu(\nu-1)+1$. Equality holds between the last two terms only when $n=3$.

Case 2. $n=2\nu$, $n_1=2\nu_1-1$, $n_2=2\nu_2-1$. Here $\nu=\nu_1+\nu_2-1$ and $N \leqq \nu_1(\nu_1-1)+1+\nu_2(\nu_2-1)+1 \leqq \nu^2+1$. Equality holds only if $n=2$.

Case 3. $n=2\nu$, $n_1=2\nu_1$, $n_2=2\nu_2$. Here $\nu=\nu_1+\nu_2$ and $N=\nu_1^2+1+\nu_2^2+1<\nu^2+1$. Thus the theorem is proved.

We have also proved the following theorem.

THEOREM 2. *The maximum number* $N(n)$ *of linearly independent commutative matrices of* n *rows and columns is given by the formula* $N(n)=[n^2/4]+1$.

We shall investigate next the form of commutative subalgebras $\mathfrak{A}$ of Φ_n of the maximum dimensionality $N(n)$. Suppose first that $\mathfrak{A}$ has the structure $\mathfrak{A}=(1)+\mathfrak{N}$ where $\mathfrak{N}$ is a nilpotent algebra. Then it is known that by replacing $\mathfrak{A}$ by a similar set we may suppose that the matrices of $\mathfrak{N}$ are properly triangular. We may apply the above considerations to $\mathfrak{N}$. By (3), (4), (5) and (6) we see that if $n=2\nu$ we must have $k_{11}=k_{21}= \cdots =k_{\nu 1}=\nu+1, \cdots, k_{1\nu}=k_{2\nu}= \cdots =k_{\nu\nu}=n$ as the set of k's in (3). If $n=2\nu-1$ the set of k's is either $k_{11}= \cdots =k_{\nu 1}=\nu+1, \cdots, k_{1\,\nu-1}= \cdots =k_{\nu\,\nu-1}=n$ or $k_{11}= \cdots =k_{\nu-1\,1}=\nu, \cdots, k_{1\nu}= \cdots =k_{\nu-1\,\nu}=n$. Suppose first that n is even. Let $\mathfrak{N}^{(r)}$ $(r=\nu)$ and $\mathfrak{N}_2$ be determined as before. It is clear that $\mathfrak{N}^{(r)}$ is similar to $\mathfrak{N}$ by a matrix in Φ_n and we need not assume here that Φ is algebraically closed. The matrices of $\mathfrak{N}_2$ have the form

$$B=\left(\begin{array}{c|c} 0\cdots 0 & \overbrace{0\cdots 0}^{\nu} \\ R & A \\ \hline 0 & 0 \end{array}\right)\left.\vphantom{\begin{array}{c}0\\ R\end{array}}\right\}\nu. \tag{7}$$

Since $k_{21}=\nu+1$ it is clear that the second row of R is 0. Moreover the operations used to pass from $\mathfrak{N}_2$ to $\mathfrak{N}_3$ affect only the last ν rows and last ν columns of $\mathfrak{N}_2$. Hence the third row of R is the same as the third row of the corresponding matrix in $\mathfrak{N}_3$. Since $k_{31}'=\nu+1$ the third row of R is 0. Similarly the other rows of R are 0, and $R=0$ in (7). Now $\dim \mathfrak{N}_2=\nu^2-\nu$. Hence $\mathfrak{N}_2$ consists of all matrices of the form (7) in which $R=0$ and A is arbitrary. Let

$$V_{1j}=\left(\begin{array}{c|c} 0 & 0\cdots 0 \quad 1 \quad 0\cdots 0 \\ & V_j \\ \hline & T_j \end{array}\right), \qquad j=\nu+1,\cdots,n,$$

where the 1 is in the jth column and T_j is a properly triangular matrix. Since $V_{1j}B=BV_{1j}$ the following holds in Φ_ν:

$$\begin{pmatrix} 0\cdots 0 \\ A \end{pmatrix} T_j=0.$$

Since A is arbitrary, $T_j=0$. Thus $\mathfrak{N}^{(r)}$ is the set $\mathfrak{Z}_n$ and $\mathfrak{A}$ is similar to the algebra $\mathfrak{B}_n$ defined before. If n is odd a similar argument shows that $\mathfrak{A}$ is similar either to $\mathfrak{B}_n$ or to $\overline{\mathfrak{B}}_n$.

We suppose now that $\mathfrak{A}$ is arbitrary. Evidently $\mathfrak{A}$ contains the identity matrix. Since $n>3$ by the proof of Theorem 1, $\mathfrak{A}$ is indecomposable. Moreover if Ω is the algebraic closure of Φ then $\mathfrak{A}_\Omega$ is an indecomposable algebra containing the identity. It follows that $\mathfrak{A}_\Omega$ is similar to a set of matrices of the form (1). Hence $\mathfrak{A}_\Omega=(1)+\mathfrak{Z}$ where $\mathfrak{Z}$ is nilpotent and so $\mathfrak{A}_\Omega$ is similar to either $\mathfrak{B}_n(\Omega)$ or $\overline{\mathfrak{B}}_n(\Omega)$. Thus $\mathfrak{Z}$ is a zero algebra. Now let $\mathfrak{N}$ be the radical of the algebra $\mathfrak{A}$ and consider the semi-simple algebra $\overline{\mathfrak{A}}=\mathfrak{A}-\mathfrak{N}$. The extension $\overline{\mathfrak{A}}_\Omega$ is a homomorphic image of $\mathfrak{A}_\Omega$. Hence $\overline{\mathfrak{A}}_\Omega=(1)+\overline{\mathfrak{Z}}$ where $\overline{\mathfrak{Z}}$ is a zero algebra. The structure of $\overline{\mathfrak{A}}$ is given by the following lemma.

LEMMA 3. *If $\overline{\mathfrak{A}}$ is a semi-simple commutative algebra such that $\overline{\mathfrak{A}}_\Omega=(1)+\overline{\mathfrak{Z}}$ where $\overline{\mathfrak{Z}}$ is a zero algebra, then either $\overline{\mathfrak{A}}=(1)$ or Φ is an imperfect field of characteristic 2 and $\overline{\mathfrak{A}}=\Phi(x)$ where $x^2=\xi$, a non-square in Φ.*

Since $\mathfrak{A}$ is semi-simple, $\overline{\mathfrak{A}}$ is a direct sum of fields, but since $\overline{\mathfrak{A}}_\Omega$ has only one idempotent element, $\overline{\mathfrak{A}}$ is a field. Let $\overline{\mathfrak{A}}>(1)$. Then $\overline{\mathfrak{A}}$ has no separable subfields, for if Σ were such a subfield Σ_Ω is a direct sum of fields and $\overline{\mathfrak{A}}_\Omega$ would contain more than one idempotent element. Thus Φ has characteristic $p\neq 0$ and $\overline{\mathfrak{A}}$ contains an element x such that $x^p=\xi$ is in Φ where ξ is not a pth power in Φ. Now there exists an element η in Ω such that $\eta^p=\xi$ and hence the element $z=x-\eta$ in $\overline{\mathfrak{A}}_\Omega$ is nilpotent of index p. Since $\overline{\mathfrak{Z}}$ is a zero algebra, $p=2$. It follows readily that in this case $\mathfrak{A}=\Phi(x)$, $x^2=\xi$.

This lemma shows that unless Φ is an imperfect field of characteristic 2 any commutative subalgebra $\mathfrak{A}$ of $\Phi_n(n>3)$ of maximum dimensionality has a difference algebra with respect to its radical $\mathfrak{N}$ of dimensionality 1. Since $\mathfrak{A}$ contains the identity, $\mathfrak{A}=(1)+\mathfrak{N}$. As we have seen, this implies that $\mathfrak{A}$ is similar to either $\mathfrak{B}_n$ or to $\overline{\mathfrak{B}}_n$.

THEOREM 3. *Suppose that Φ is not an imperfect field of characteristic 2 and let $n>3$. Then if $\mathfrak{A}$ is a subalgebra of Φ_n of maximum dimensionality $N(n)$, $\mathfrak{A}$ is similar to $\mathfrak{B}_n$ if $n=2\nu$ and $\mathfrak{A}$ is similar to either $\mathfrak{B}_n$ or $\overline{\mathfrak{B}}_n$ if $n=2\nu-1$.*[2]

As a consequence we have the following theorem.

THEOREM 4. *Let Φ, n and $\mathfrak{A}$ be as in Theorem 3. Then $\mathfrak{A}=(1)+\mathfrak{N}$ where $\mathfrak{N}$ is a zero algebra.*

We remark finally that if n is odd the sets $\mathfrak{B}_n$ and $\overline{\mathfrak{B}}_n$ are not similar. This may be seen by considering the sets $\mathfrak{Z}_n$ and $\overline{\mathfrak{Z}}_n$. Let $\mathfrak{S}(\overline{\mathfrak{S}})$ be the space determined by the columns of the matrices of $\mathfrak{Z}_n(\overline{\mathfrak{Z}}_n)$. Then $\dim \mathfrak{S}=\nu$ and $\dim \overline{\mathfrak{S}}=\nu-1$. On the other hand if $\mathfrak{Z}_n$ were similar to $\overline{\mathfrak{Z}}_n$ we would have $\dim \mathfrak{S}=\dim \overline{\mathfrak{S}}$. It follows that $\mathfrak{Z}_n$ and $\overline{\mathfrak{Z}}_n$ are not similar and hence $\mathfrak{B}_n$ and $\overline{\mathfrak{B}}_n$ are not similar. Thus in this case there are for $n=2\nu-1>3$ two distinct classes in the sense of similarity of commutative subalgebras of dimensionality $N(n)$.

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[2] If $n=2, 3$, $\mathfrak{A}$ may be decomposable. The determination of these algebras is readily obtained.

PROJECTIVE INVARIANTS OF INTERSECTION OF CERTAIN PAIRS OF SURFACES

CHUAN-CHIH HSIUNG

1. Introduction. In a recent paper [2][1] the author has shown the existence, together with metric and projective characterizations,[2] of a unique projective invariant determined by the neighborhood of the second order of two surfaces S_1, S_2 at an ordinary point O in ordinary space when the tangent planes τ_1, τ_2 of the surfaces S_1, S_2 at the point O are distinct and the line t of intersection of the two tangent planes τ_1, τ_2 does not coincide with any one of the asymptotic tangents of the surfaces S_1, S_2 at the point O. On the other hand, with regard to the coincidences of the line t and the asymptotic tangents of the surfaces S_1, S_2 at the point O two essentially different cases can arise. The object of this note is to derive some projective invariants in these cases. Noticing that no projective invariant can be determined by the neighborhood of the second order of the surfaces S_1, S_2 at the point O, we obtain all projective invariants determined by the neighborhood of the second order of one surface and that of the third order of the other at the point O.

I. TWO SURFACES WITH DISTINCT TANGENT PLANES AND DISTINCT ASYMPTOTIC TANGENTS AT AN ORDINARY POINT

2. Derivation of invariants. Let us first consider two surfaces S_1, S_2 in ordinary space intersecting at an ordinary point O with distinct tangent planes τ_1, τ_2, whose line of intersection t is supposed to be an asymptotic tangent of the surface S_1 at the point O. Let t_1 be the other asymptotic tangent of the surface S_1 at the point O, and t_2 the harmonic conjugate line of t with respect to the asymptotic tangents of the surface S_2 at the point O. If we choose the point O to be the origin, the lines t, t_2, t_1 to be respectively the axes x, y, z of a general nonhomogeneous projective coordinate system, then the power series expansions of the surfaces S_1, S_2 in the neighborhood of the point O may be written in the form

$$S_1:\ y = lxz + px^3 + rx^2z + sxz^2 + qz^3 + \cdots, \tag{1}$$

$$S_2:\ z = mx^2 + ny^2 + \cdots. \tag{2}$$

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[1] Numbers in square brackets refer to the references at the end of the paper.

[2] An extension of these results to two hypersurfaces has been made by Professor Su. See his paper [6].

In order to find projective invariants of the surfaces S_1, S_2 at the point O, we have to consider the most general projective transformation of coordinates which shall leave the lines t, t_1, t_2 unchanged:

$$
\begin{aligned}
x &= a_{22}x^*/(1 + a_{12}x^* + a_{13}y^* + a_{14}z^*),\\
y &= a_{33}y^*/(1 + a_{12}x^* + a_{13}y^* + a_{14}z^*),\\
z &= a_{44}z^*/(1 + a_{12}x^* + a_{13}y^* + a_{14}z^*),
\end{aligned}
\tag{3}
$$

where a_{ik} are arbitrary constants. The effect of this transformation on equations (1), (2) is to produce two other equations of the same form whose coefficients, indicated by stars, are given by the formulas

$$
\begin{gathered}
a_{33}l^* = a_{22}a_{44}l, \qquad a_{33}p^* = a_{22}^3 p, \qquad a_{33}q^* = a_{44}^3 q,\\
2a_{12}a_{33}l^* + a_{33}r^* = a_{12}a_{22}a_{44}l + a_{22}^2 a_{44}r,\\
2a_{14}a_{33}l^* + a_{33}s^* = a_{14}a_{22}a_{44}l + a_{22}a_{44}^2 s;\\
a_{44}m^* = a_{22}^2 m, \qquad a_{44}n^* = a_{33}^2 n.
\end{gathered}
\tag{4}
$$

Further elimination of a_{ik} from equations (4) gives immediately that *the quantities*

$$
I = lm/p, \qquad J = p^2q^4m^3/l^{12}n^3 \tag{5}
$$

are projective invariants associated with the surfaces S_1, S_2, at the point O.

3. Projective characterizations of the invariants I, J. It is well known that the tangent plane τ_1 intersects the surface S_1 in a curve with a node at the point O, the nodal tangents being the asymptotic tangents t, t_1. The expansion of the branch C_1 of this curve which is tangent to the tangent t, $z=y=0$, is easily found to be

$$
z = -(p/l)x^2 + \cdots . \tag{6}
$$

On the other hand, the curve C_2 in which the tangent plane τ_1 intersects the surface S_2 is given by the equations

$$
z = mx^2 + \cdots, \qquad y = 0. \tag{7}
$$

It is thus easy to reach the conclusion that *the projective invariant I associated with the point O of the surfaces S_1, S_2 is, except for sign, equal to the projective invariant of contact*[3] *of the two plane curves C_1, C_2 at the point O.*

To characterize projectively the other invariant J it is useful to consider the asymptotic curves Γ, Γ_1 of the surface S_1 at the point O. The expansions of these asymptotic curves at the point O are found, after a simple calculation, to be

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[3] The projective invariant of contact of two plane curves having ordinary contact at a point was first found by H. J. S. Smith [5] and R. Mehmke [3]; and its simple projective characterization was later given by C. Segre [4].

$$(8) \qquad \Gamma: \quad y = -2px^3 + \cdots, \qquad z = -(3p/l)x^2 + \cdots;$$

$$(9) \qquad \Gamma_1: \quad y = -2qz^3 + \cdots, \qquad x = -(3q/l)z^2 + \cdots.$$

Let K be any five-point quadric cone of the asymptotic curve Γ at the point O with O for vertex and with an asymptotic tangent of the surface S_2 at the point O for a generator, then making use of equations (8) we can easily obtain its equation, namely,

$$(10) \qquad z^2 + (9p/2l^2)xy \pm (9p/2l^2)(-n/m)^{1/2}y^2 + kxz = 0,$$

where k is arbitrary. Similarly, we have a five-point quadric cone K_1 of the asymptotic curve Γ_1 at the point O such that it has O for vertex and the line t_2 for a generator and is tangent to the plane τ_2, $z=0$, along t_2. The equation of this cone K_1 is

$$(11) \qquad x^2 + (9q/2l^2)yz = 0.$$

From equations (10), (11) it follows that the cone projecting the curve of intersection of the cones K, K_1 from the line t_2 consists of the four planes:

$$(12) \qquad 9q^2z^4 - 18kl^2qx^2z^2 - 9pqx^3z \pm 2l^2p(-n/m)^{1/2}x^4 = 0,$$

which are intersected by the plane τ_1 in four lines λ_i $(i=1, \cdots, 4)$ with equations

$$(13) \qquad z + \alpha_i x = 0, \qquad y = 0 \qquad (i = 1, \cdots, 4),$$

where the coefficients α_i satisfy the relation

$$(14) \qquad \alpha_1\alpha_2\alpha_3\alpha_4 = \pm (2l^2p/9q^2)(-n/m)^{1/2}.$$

On the other hand, associated with the point O of the surface S_1 there are three tangents of Darboux, whose equations may be found by making use of equations (8), (9) and Bompiani's result [1] that they are the three principal lines of the asymptotic curves Γ, Γ_1 at the point O. The result is

$$(15) \qquad px^3 + qz^3 = 0, \qquad y = 0.$$

Let d be any one of the tangents (15) of Darboux, and let D_i denote the cross ratio of the four lines t, t_1, d, λ_i $(i=1, \cdots, 4)$; then

$$(16) \qquad D_i = (tt_1, d\lambda_i) = (1/\alpha_i)(p/q)^{1/3} \qquad (i = 1, \cdots, 4).$$

A reference to equations (5), (14), (16) suffices to substantiate that *the invariant J can be expressed in terms of the four cross ratios D_1, D_2, D_3, D_4 as follows*:

$$J = -(2/9)^6(D_1D_2D_3D_4)^6. \tag{17}$$

II. TWO SURFACES WITH DISTINCT TANGENT PLANES AND A COMMON ASYMPTOTIC TANGENT AT AN ORDINARY POINT

4. Derivation of an invariant. Finally, suppose that S_1, S_2 be two surfaces in ordinary space intersecting at an ordinary point O with distinct tangent planes τ_1, τ_2, whose line of intersection t stands for one asymptotic tangent of both surfaces S_1, S_2 at the point O; and that t_1, t_2 be respectively the other asymptotic tangents of the surfaces S_1, S_2 at the point O. If we choose the point O to be the origin, the lines t, t_2, t_1 to be respectively the axes x, y, z of a general nonhomogeneous projective coordinate system, then the power series expansions of the surfaces S_1, S_2 in the neighborhood of the point O may be written in the form (1) and

$$S_2:\quad z = mxy + \cdots . \tag{18}$$

In a way similar to the foregoing we can easily show that *the quantity*

$$I = p^4q^2/l^9m^3 \tag{19}$$

is a projective invariant associated with the surfaces S_1, S_2 at the point O.

5. A projective characterization of the invariant I. Let Γ, Γ_1 be the asymptotic curves of the surface S_1 at the point O whose tangents are t, t_1 respectively. Among the four-point quadrics of the asymptotic curve Γ at the point O we can determine a two-parameter family such that every one of the family has t_2 for a generator and has at the point O contact of the second order with the surface S_2. By means of equations (8), (18) it is easy to obtain the equation of a general quadric of this family, namely,

$$z + (3p/l)x^2 - mxy + Eyz + Fz^2 = 0, \tag{20}$$

where E, F are arbitrary. The quadric (20) is cut by the plane τ_2 in the line t_2 and a line with equations

$$3px - lmy = 0, \qquad z = 0. \tag{21}$$

If any five-point quadric cone, with the point O for vertex, of the asymptotic curve Γ at O passes through the line (21), then it must have the equation:

$$z^2 + (9p/2l^2)xy - (3m/2l)y^2 + kyz = 0, \tag{22}$$

where k is arbitrary. The cone projecting the curve of intersection of the cones (11), (22) from the line t_2 consists of the four planes:

$$27q^2z^4 - 6kl^2qx^2z^2 - 27pqx^3z - 2l^3mx^4 = 0, \tag{23}$$

which are intersected by the plane τ_1 in four lines μ_i $(i=1, \cdots, 4)$:

$$z + \beta_i x = 0, \qquad y = 0 \qquad (i = 1, \cdots, 4), \tag{24}$$

where the coefficients β_i satisfy the relation

$$\beta_1\beta_2\beta_3\beta_4 = -2l^3m/27q^2. \tag{25}$$

If d be any one of the three tangents of Darboux associated with the point O of the surface S_1 and if D_i denotes the cross ratio of the four lines t, t_1, d, μ_i $(i=1, \cdots, 4)$, then from equations (15), (24) it follows that

$$D_i = (tt_1, d\mu_i) = (1/\beta_i)(p/q)^{1/3} \qquad (i = 1, \cdots, 4). \tag{26}$$

A reference to equations (19), (25), (26) suffices to show that *the invariant I can be expressed in terms of the four cross ratios D_1, D_2, D_3, D_4 as follows*:

$$I = -(2/27)^3(D_1D_2D_3D_4)^3. \tag{27}$$

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SOME REMARKS ON CONNECTED SETS

P. ERDÖS

This note will consist of a few disconnected remarks on connected sets.

i. Swingle[1] raised the question whether the plane is the sum of c disjoint biconnected sets. The answer as we shall show is affirmative.

First we construct a biconnected set A with a dispersion point x such that any two points of $A-x$ can be separated. (The first such set was constructed by Wilder.[2] Our construction will be very similar to that of Burton Jones.[3])

Our biconnected set will contain the origin $\mathfrak{O}$, at most one other point on lines through the origin with irrational slopes, and no point other than the origin on lines with rational slopes. Further it will contain at least one point on every cut of the plane. It is easy to see that such a set exists. Every cut of the plane contains a closed subset which also cuts the plane, and the power of closed sets is known to be c. Let us well order the closed cuts $C_1, C_2, \cdots, C_\gamma, \cdots, \gamma<\Omega_\alpha$, where Ω_α is the least ordinal number of power c. We construct A as follows: $\mathfrak{O}$ belongs to A. We shall choose a point x_γ on C_γ and we shall have $A=\bigcup x_\gamma$. We shall determine x_γ by transfinite induction. Suppose we have already determined x_δ, $\delta<\gamma$, we determine x_γ as follows: if C_γ contains $\mathfrak{O}$ then $x_\gamma=\mathfrak{O}$. If C_γ does not contain $\mathfrak{O}$ then clearly C_γ has to intersect c lines through $\mathfrak{O}$. Therefore we can find a point $x_\gamma \in C_\gamma$ such that $(\mathfrak{O}, x_\gamma)$ has irrational slope and does not go through any other point x_δ, $\delta<\gamma$. (We denote by (a, b) the line through the points a and b.) This way we construct A. Clearly A is biconnected. First of all A is connected since it intersects every cut of the plane. Also any two points of $A-\mathfrak{O}$ can be separated since the two points x_1 and x_2, say, are on different irrational lines through the origin, and a rational line, which of course does not intersect $A-\mathfrak{O}$, will separate them. This completes the proof. The origin we call the center of A.

Now we shall split the plane into the sum of c such disjoint sets A_γ, $\gamma<\Omega_\alpha$, Ω_α the smallest ordinal of power c. Let us well order the points x_γ of the plane and the closed cuts C_γ of the plane. Our first step is to select x_1 as the center of A_1 and a suitable point of A_1 on C_1.

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[1] P. M. Swingle, Amer. J. Math. vol. 54 (1932) p. 532.

[2] R. L. Wilder, Bull. Amer. Math. Soc. vol. 33 (1927) p. 423.

[3] Burton Jones, Bull. Amer. Math. Soc. vol. 48 (1942) p. 115.

(By suitable we mean that it does not lie on any line with rational slope through x_1.) Our second step is to select the first point which has not yet been selected as center of A_2, then select suitable points on C_1 and C_2 for A_2 and on C_2 for A_1. Suppose that this construction has been already carried out for all $\delta<\gamma$. Then the γth step consists of the following construction: First we take the first x_ϵ which has not yet been chosen as center of A_γ. Then we choose suitable points on all the C_δ, $\delta\leqq\gamma$, for A_γ and suitable points on C_γ for all the A_δ, $\delta<\gamma$. It is easy to see that this construction can be carried out since before the γth step we have chosen less than c points x_j, and a cut C_γ if it does not go through the origin intersects c lines through the origin. (If the cut C goes through the origin of one or more of the A_γ's then of course we can choose the origin.) Thus the construction can be carried out for all ordinals $\gamma<\Omega_\alpha$. Hence we get c sets A_γ where $\bigcup A_\gamma$ is the whole plane, $A_{\gamma_1}\cap A_{\gamma_2}=0$ and the A_γ are all biconnected, which completes the proof of the theorem.

A simple modification of this proof would give the following result: Let $m\leqq c$ be any cardinal number greater than 2, then the plane can be expressed as the sum of m disjoint biconnected sets.[4]

ii. Let C be any connected set. Knaster and Kuratowski[5] proved that there exists a proper connected subset of C. (Single points, of course, do not count as connected sets.) Their proof is very simple. Let $p\in C$ be arbitrary. If $C-p$ is connected our result is proved. If $C-p$ is not connected it can be written in the form $U+V$ where U and V are separated. But then it is easy to see that both $U+p$ and $V+p$ are connected.

It seems likely that the following result is true: every connected set C contains a connected subset C' such that $C-C'$ has power c. I am unable to prove this. It is easy to see that if a set C does not satisfy this conjecture it has to be both biconnected and widely connected, thus must be rather pathological.

A. Stone[6] proved that every connected set C has a connected subset C' such that $C-C'$ is infinite. Proof: First we can assume that $C-p$ is connected for every $p\in C$, for if not then at least one of the sets $U+p$ or $V+p$ has a complement of power c. (U and V were defined in the previous paragraph.) Similarly for every sequence $p_1, p_2, \cdots, p_n$, $C-p_1-p_2-\cdots-p_n$ is connected. Choose $p_1, p_2, \cdots$

---

[4] Swingle proved (ibid.) that the Euclidean n space is the sum of r biconnected sets if r is any finite number not less than $n+1$, also that it is the sum of $\aleph_0$ biconnected sets. He also showed that n-space is not the sum of n or less biconnected sets.

[5] Knaster-Kuratowski, Fund. Math. vol. 2 (1921) p. 206.

[6] Arthur Stone, oral communication.

such that it will converge to a point $p\in C$. Then $C-\sum_{i=1}^{\infty}p_i$ is connected. For if $\sum_{i=1}^{\infty}p_i$ would separate C, a closed subset of it would also separate C. But a closed subset of $\sum_{i=1}^{\infty}p_i$ is finite, and so by assumption can not separate C, and this completes the proof.

In this connection it might be of some interest to construct a connected set A with the following properties: (1) no finite subset disconnects A, (2) every countable dense subset totally disconnects A. For our construction we shall have to use the continuum hypothesis.

First we need the following lemma: Let S_1 and S_2 be two countable disjoint sets. S_1 is everywhere dense in the plane. Then there exist countably many closed Jordan curves J_r such that $J_r\cap S_1$ is dense on J_r, $J_r\cap S_2=0$, and to any two points p and q there exists an r such that J_r separates p and q. The proof does not present any difficulties and can be omitted.

And now to our construction.[7] First we well order the countable dense subsets of the plane, $D_1, D_2, \cdots, D_\alpha, \cdots$ (their number is clearly $c=\aleph_1$; the continuum hypothesis has been assumed). Also we well order all the closed connected cuts of the plane C_δ, again clearly $\delta<\Omega_1$ (it is well known that every cut of the plane contains a closed connected cut). We shall now construct our set A by transfinite induction. Let Δ be any given countable dense set, put $\Delta\subset A$. Our first step is to define a countable subset P_1 of C_1. If $\Delta\cap C_1$ is infinite we put $P_1=0$, if $C_1\cap\Delta$ is finite, we choose an arbitrary countably infinite $P_1\subset C_1$ and put $P_1\subset A$. Consider next the smallest α such that $D_\alpha\subset\Delta+P_1$. Put $S_1=D_\alpha$ and $S_2=\Delta+P_1-D_\alpha$. Then by our lemma there exists a countable collection of closed Jordan curves $J_r^{(1)}$ such that $J_r^{(1)}\cap D_\alpha$ is dense on $J_r^{(1)}$ and $J_r^{(1)}\cap\Delta+P_1-D_\alpha=0$ and any two points p and q are separated by some $J_r^{(1)}$. Now we make the condition that no point of any P_β $(\beta>1)$ should lie on any J_r^β. Suppose we carried out our construction for any $\gamma<\delta$, we shall show that we can carry it out for $\gamma=\delta$. Consider C_δ; if $C_\delta\cap\Delta+\sum_{\beta<\delta}P_\beta$ is infinite, then $P_\delta=0$. If $C_\delta\cap\Delta+\sum_{\beta<\delta}P_\beta$ is finite, then we choose a countably infinite subset P_δ of C_δ, such that P_δ does not meet $\sum_{\lambda<\delta}\sum_{r=1}^{\infty}J_r^\lambda$. In other words P_δ does not meet any of countably many Jordan curves (we have assumed the continuum hypothesis). We have to prove that such a choice of P_δ is possible. By assumption $(\Delta+\sum_{\beta<\delta}P_\beta)\cap C_\delta$ is finite. For each of the Jordan curves J_r^λ (with $\lambda<\delta$) $J_r^\lambda\cap\Delta+\sum_{\beta<\delta}P_\beta$ is dense on J_r^λ, therefore $C_\delta\cap J_r^\lambda$ is nowhere dense on C_δ (for if not, since C_δ is closed, it would contain a portion of J_r^λ and therefore $C_\delta\cap(\Delta+\sum_{\beta<\delta}P_\beta)$ would be infinite, as $\Delta+\sum_{\beta<\delta}P_\beta$ is dense on J_r^λ).

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[7] Our construction will be similar to Miller's construction of a biconnected set without a dispersion point. (Fund. Math. vol. 29 (1937) p. 123.)

Therefore $C_\delta \cap \sum_{\lambda<\delta} \sum_{r=1}^{\infty} J_r^\lambda$ is of first category on C_δ, and thus there are c points of C not belonging to $\sum_{\lambda<\delta} \sum_{r=1}^{\infty} J_r^\lambda$; therefore we can choose an infinite $P_\delta \subset C_\delta$, $P_\delta \cap \sum_{\lambda<\delta} \sum_{r=1}^{\infty} J_r^\lambda = 0$ without difficulty. Next consider the smallest α such that $D_\alpha \subset \Delta + \sum_{\beta \leq \delta} P_\beta$ and D_α has not yet been dealt with in any of the previous steps. We then construct by our lemma countably many closed Jordan curves J_r^δ such that $D_\alpha \cap J_r^\delta$ is dense on J_r^δ, $J_r^\delta \cap (\Delta + \sum_{\beta<\delta} P_\beta - D_\alpha) = 0$, and any two points p and q can be separated by some J_r^δ. This completes the δth step. Thus our construction can be carried out through all ordinals of the second number class and will exhaust all the C_α's and D_α's. Consider now the set $A = \Delta + \sum_\beta P_\beta$. We claim that no finite set disconnects it and that any countable dense set totally disconnects it. First of all every cut of the plane intersects our set in an infinite set, thus no finite set can disconnect it. On the other hand let D_α be some countable dense set of A (clearly A is dense in the plane). At some step in our construction we had to deal with D_α and it is clear that $\Delta + \sum_\beta P_\beta - D_\alpha$ is totally disconnected, since if p and q are any two points there exists a J_r separating them and $(\Delta + \sum_{\beta<\alpha} P_\beta - D_\alpha) \cap J_r = 0$. q.e.d.

By a slight modification of Miller's[8] construction of a biconnected set without a dispersion point, we can construct a connected set A dense in an indecomposable continuum, such that there exists an open set O of the plane with $O \cap A$ having power c and if B is any connected subset of A then $A - B$ is nowhere dense in $O \cap A$. Clearly such a set is biconnected and in fact it can be made to have no dispersion point. By a further slight modification of Miller's construction we can construct a biconnected set without a dispersion point such that if $A_1, A_2, \cdots$ is any countable collection of connected subsets of A, then $\bigcup_i (A - A_i) \neq A$ or $\bigcap_i A_i \neq 0$.

I can not decide the question whether there exists a connected set such that the complement of every connected subset of it is nowhere dense in it.

The following problems may be of some interest: Is it true that every connected set contains a connected subset not homeomorphic to it? (Points do not count as connected sets.) Every known connected set (which is subset of a Euclidean space) contains at least three different types of connected subsets. The number three can not be improved as is shown by the arc.

If a topological space does not satisfy any separation axioms the following example communicated to me by S. Eilenberg shows that the above conjecture is not true: Let the space be the integers, the closed sets are the finite sets. Every infinite set is connected and they are clearly all homeomorphic.

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[8] Miller, ibid.

Is it true that every connected set of dimension n contains a connected subset of dimension $n-1$?

Is it true that every connected set of dimension greater than 1 contains 2^c connected subsets?

Perhaps the following theorem which A. Stone and I proved might be of some interest: Let A be a totally disconnected set of power m (in a separable space), then if m is not the sum of countably many smaller cardinals, A can be written as the sum of two separated sets of power m. If m is the sum of $\aleph_0$ smaller cardinals the theorem is in general false.

Proof. Suppose first $m \neq \sum_{k=1}^{\infty} m_k$, $m_k < m$. Denote by A' the set of those points of A for which every neighborhood contains m points of A. Now $A - A'$ has power less than m, since by Lindelöf's theorem $A - A'$ can be covered by countably many open sets each of which contains less than m points, thus $A - A'$ also contains less than m points. Since A is totally disconnected $A' = U + V$, where U and V are open-closed sets. Thus both of them have power m. Let $p \in A - A'$, the distance of p from U be $f(p)$, from V, $\phi(p)$, $p \in U_\alpha$ if $f(p) \leqq \alpha\phi(p)$, $p \in V_\alpha$ if $f(p) \geqq \alpha\phi(p)$. Since the power of $A - A'$ is less than $m \leqq c$ there exists an α such that $U_\alpha \cap V_\alpha = 0$. But then $U + U_\alpha$ and $V + V_\alpha$ are two separated sets of power m whose sum is A; this completes the proof.

If $m = \sum_{k=1}^{\infty} m_k$, $m_k < m$, we define A as follows: A contains m_k points in $(1/k, 1/k+1)$, 0 belongs to A. Clearly A is totally disconnected (its power is less than c, since c is not the sum of m smaller cardinals). Obviously A is not the sum of two separated sets of power m.

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$\alpha \leqq (Hf, f) \leqq \beta$, $(f, f) = 1$). (2) If γ is an arbitrary point in $[\alpha, \beta]$ then there exist two self-adjoint transformations H_1 and H_2 in $\mathfrak{R}\{H\}$ such that the zero spaces of H_1 and H_2 form orthogonal manifolds whose direct sum is the Hilbert space $\mathfrak{H}$ and such that $H = H_1 + H_2$, $H_1H_2 = 0$. The spectrum of H_1 lies essentially in $[\alpha, \gamma]$, that of H_2 in $[\gamma, \beta]$.

Proof of the first point is carried out with the help of an integral due to Hille which allows one to embed certain transformations with real non-negative spectrum in a semi-group.[26] The only transformation we need is $H^{1/2}$ in case $\alpha \geqq 0$. The second difficulty is surmounted by means of formula (8) in the manner which we have indicated.

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[26] See footnote 7, Theorem 14.

THE APRIL MEETING IN NEW YORK

The four hundred third meeting of the American Mathematical Society was held at Columbia University on Friday and Saturday, April 28–29, 1944. The attendance included the following one hundred seventy-nine members of the Society:

R. L. Anderson, R. G. Archibald, K. J. Arnold, L. A. Aroian, S. P. Avann, A. E. Basch, E. G. Begle, Stefan Bergman, Felix Bernstein, Garrett Birkhoff, G. D. Birkhoff, Gertrude Blanch, M. L. Boas, R. P. Boas, H. F. Bohnenblust, C. B. Boyer, H. W. Brinkmann, A. B. Brown, G. W. Brown, E. L. Buell, R. S. Burington, Hobart Bushey, J. H. Bushey, R. H. Cameron, W. B. Campbell, S. S. Chern, Edmund Churchill, R. V. Churchill, M. D. Clement, Nathaniel Coburn, Harvey Cohn, L. M. Comer, T. F. Cope, Richard Courant, A. P. Cowgill, W. H. H. Cowles, H. B. Curry, M. M. Day, Jesse Douglas, Nelson Dunford, W. L. Duren, William H. Durfee, J. E. Eaton, M. C. Eide, J. M. Feld, A. D. Fialkow, R. M. Foster, F. H. Fowler, G. A. Foyle, Bernard Friedman, K. O. Friedrichs, Orrin Frink, R. E. Fullerton, H. P. Geiringer, Abe Gelbart, David Gilbarg, B. P. Gill, Leonard Gillman, Michael Golomb, A. W. Goodman, R. O. Goodman, F. G. Gravalos, Bernard Greenspan, Lewis Greenwald, C. C. Grove, Laura Guggenbühl, E. J. Gumbel, Jacques Hadamard, P. R. Halmos, Olaf Helmer, A. D. Hestenes, M. R. Hestenes, L. S. Hill, Einar Hille, Abraham Hillman, Harold Hotelling, E. M. Hull, Ralph Hull, Witold Hurewicz, L. C. Hutchinson, B. M. Ingersoll, Florence Jacobson, Nathan Jacobson, S. A. Joffe, Fritz John, A. W. Jones, Aida Kalish, Edward Kasner, P. W. Ketchum, S. A. Kiss, J. R. Kline, E. G. Kogbetliantz, E. R. Kolchin, D. M. Krabill, M. E. Ladue, Howard Levene, Howard Levi, D. C. Lewis, Marie Litzinger, L. H. Loomis, E. R. Lorch, A. N. Lowan, N. H. McCoy, L. A. MacColl, J. K. L. MacDonald, Saunders MacLane, H. F. MacNeish, A. J. Maria, D. H. Maria, Carl Markow, W. T. Martin, D. S. Miller, L. W. Miller, E. C. Molina, Richard Morris, D. J. Morrow, F. J. Murray, C. A. Nelson, Otto Neugebauer, P. B. Norman, L. R. Norwood, C. O. Oakley, F. W. Owens, J. C. Oxtoby, E. L. Post, Walter Prenowitz, A. L. Putnam, Hans Rademacher, Mina Rees, Moses Richardson, R. G. D. Richardson, J. F. Ritt, R. K. Ritt, E. K. Ritter, P. C. Rosenbloom, Hans Samelson, Arthur Sard, L. J. Savage, R. H. Scanlan, Henry Scheffé, S. A. Schelkunoff, L. I. Schoenfeld, Alberta Schuettler, I. E. Segal, C. E. Shannon, Max Shiffman, J. A. Shohat, P. A. Smith, Fritz Steinhardt, J. J. Stoker, E. G. Straus, M. M. Sullivan, Fred Supnick, Otto Szász, J. D. Tamarkin, J. H. Taylor, J. M. Thomas, E. M. Torrance, H. F. Tuan, A. W. Tucker, J. W. Tukey, Oswald Veblen, R. M. Walter, W. R. Wasow, G. C. Webber, André Weil, Alexander Weinstein, Louis Weisner, C. P. Wells, George Whaples, A. P. Wheeler, W. L. G. Williams, John Williamson, H. P. Wirth, R. S. Wolfe, Jacob Wolfowitz, Oscar Zariski, M. F. Zucker, Antoni Zygmund.

On Friday evening at 8 o'clock Professor K. O. Friedrichs gave an address entitled *Mathematical aspects of the boundary layer theory*. Professor R. G. D. Richardson presided at this session.

On Saturday there were two addresses: at 11 A.M. *The structure of normed abelian rings* by Professor E. R. Lorch, and at 2 P.M. *Modern algebra and the Riemann hypothesis* by Professor André Weil. Professor W. T. Martin presided at the first of these sessions, and Vice President Einar Hille at the second.

At 3:15 P.M. Saturday there was a session for papers in Algebra, Probability, Statistics and Applied Mathematics, at which Professor N. H. McCoy presided, and another for papers in Analysis and Geometry at which Professor Nelson Dunford presided.

The Council met at 12:30 P.M. on April 29, 1944, in the Long Room of the Faculty Club.

The Secretary announced the election of the following one hundred and twenty-seven persons to membership in the Society (the address given is that of the member at the time of his application):

Professor Louise Adams, High Point College, High Point, N. C.;
Dean Ralph Ellsworth Allison, Northwest Nazarene College, Nampa, Idaho;
Professor Hippocrates George Apostle, University of Rochester;
Dr. Edward George Baker, American Bureau of Shipping, New York, N.Y.;
Mr. Anthony Francis Bartholomay, Brown University;
Mr. Chester Edwin Baston, Hillyer Junior College, Hartford, Conn.;
Mr. Daniel Moore Bates, Philadelphia, Pa.;
Mr. William Edwin Beeman, North Texas State Teachers College, Denton Tex.;
Mr. John Bodnar, Junior College of Connecticut, Bridgeport, Conn.;
Mr. Claude Levi Bordner, State Teachers College, Kutztown, Pa.;
Professor Alger Vernon Boswell, Tennessee Agricultural and Industrial State Teachers College, Nashville, Tenn.;
Professor John Gerald Bowker, Middlebury College, Middlebury, Vt.;
Professor John Grist Brainerd, Moore School of Electrical Engineering, University of Pennsylvania;
Professor Angeline Jane Brandt, Wheaton College, Wheaton, Ill.;
Professor Robert H. Breusch, Amherst College;
Dr. Walter J. Bruns, Hamilton College;
Professor John Harvey Butchart, Grinnell College, Grinnell, Iowa;
Miss Minerva Elda Butz, Cedar Crest College, Allentown, Pa.;
Dr. Ralph Edward Byrne, Jr., Barber-Colman Company, Rockford, Ill.;
Mr. Joseph D. Chapline, Jr., Moore School of Electrical Engineering, University of Pennsylvania;
Mr. Eric H. Clamons, University of Minnesota;
Professor James P. Cole, Louisiana State University;
Captain Clarence Beasley Collier, Jr., United States Military Academy;
Sister Maria Loyola Conlan, College of Mount Saint Vincent, New York, N. Y.;
Mrs. F. Marion Clarke Connell (Mrs. James R.), University of Southern California;
Professor Erben Cook, Jr., University of Connecticut;
Mr. Irvin Douglas Cook, College Park, Md.;
Miss Nancy Elizabeth Cooley, Fairfax Hall Junior College, Waynesboro, Va.;
Mr. V. B. Cooper, South Georgia College, Douglas, Ga.;
Professor Forrest Eugene Craver, Dickinson College, Carlisle, Pa.;
Professor Don Cude, Southwest Texas State Teachers College, San Marcos, Tex.;
Mr. Leland E. Cunningham, Aberdeen, Md.;
Mr. John Moffatt Danskin, Jr., U. S. Naval Reserve, Wright Junior College, Chicago, Ill.;

Dr. Margaret Rae Davis, Greenbrier Junior College, Lewisburg, W. Va.;
Professor Lester C. Dawson, University of Alaska;
Dr. Frances Rousmaniere Dewing, Bennington College;
Sister Agnes Therese Dimond, College of Mount St. Joseph-on-the-Ohio, Mount St. Joseph, Ohio;
Mr. Harold Irving Ewen, Amherst College;
Mr. Edward Joseph Farrell, University of San Francisco;
Mr. David Fisher, Michigan College of Mining and Technology;
Professor Carroll E. Flanagan, State Teachers College, Superior, Wis.;
Mr. Frederick Fraser, St. Petersburg Junior College, St. Petersburg, Fla.;
Mr. Paul Thomas Gilbert, Jr., Utah State Agricultural College;
Mr. Alfred Oscar Ginkel, University of Rochester;
Mr. Eugene Van Ness Goetchius, University of Virginia;
Mr. Adolph W. Goodman, Syracuse University;
Dr. Fausto G. Gravalos, Goodyear Aircraft Company, Akron, Ohio;
Mr. Wallace Clayton Griffith, DePauw University;
Sister Edward Joseph Hackett, St. Mary's College, Holy Cross, Notre Dame, Ind.;
Mr. Franklin Haimo, Reed College;
Mr. Leon Albert Henkin, Kellex Corporation, New York, N. Y.;
Mr. Alvadore John Hinshaw, Central Union High School and Junior College, El Centro, Calif.;
Miss Lois Aileen Hostinsky, University of Illinois;
Reverend Joseph Albert Hratz, St. Ambrose College, Davenport, Iowa;
Mr. Stanley Parlett Hughart, Princeton University;
Dr. John Theodore Johnson, Chicago Teachers College;
Mr. Herbert L. Jones, Hiram College, Hiram, Ohio;
Mr. Winfield Keck, Muhlenberg College, Allentown, Pa.;
Mr. David B. Kirk, Haverford College;
Mr. Kenneth Virgil Knight, American Optical Company, Southbridge, Mass.;
Mr. John C. Koken, Parks Air College, East St. Louis, Ill.;
Mr. Yung-Huai Kuo, California Institute of Technology;
Dr. Paco Axel Lagerstrom, Princeton University;
Professor John Mark LaRue, Central Michigan College of Education, Mount Pleasant, Mich.;
Professor Herbert Leonard Lee, University of Tennessee;
Mr. Kenneth Levenberg, Frankford Arsenal, Philadelphia, Pa.;
Mr. William Judson LeVeque, University of Colorado;
Mr. Neil Lockwood, Michigan College of Mining and Technology;
Mr. Arthur John Lohwater, University of Rochester;
Miss Eleanor Ann Ludwig, Syracuse University;
Professor Clyde Truman McCormick, Fort Hays Kansas State College, Hays, Kan.;
Professor Edith Johnson McKissock, Youngstown College, Youngstown, Ohio;
Mr. Samuel Stuart McNeary, Drexel Institute of Technology;
Mr. Clarence Isaac McSwane, University of Illinois;
Dr. Murray Mannos, University of Delaware;
Mr. Carl Markow, St. Peter's College, Jersey City, N. J.;
Professor Milton Marshall, Brigham Young University;
Mr. Frank Harshman Miller, Kenyon College;
Mr. William H. Mills, Ballistic Research Laboratory, Aberdeen Proving Ground, Md.;
Mr. Arthur Charles Moeller, Denison University, Granville, Ohio;

Mr. William Burton Moye, Georgia Teachers College;
Father Cronan Mullen, Siena College, Loudonville, N. Y.;
Mr. Allen Charles Munster, University of Pennsylvania;
Dr. Václav Myslivec, Czechoslovak Economic Service, New York, N. Y.;
Miss Elizabeth Ann Oliphant, Texas College of Arts and Industries, Kingsville, Tex.;
Professor James Enoch Parker, Knoxville College, Knoxville, Tenn.;
Professor Archie Edgar Patterson, School of Forestry, University of Georgia;
Professor Charles Roberts Pettis, Ohio State University;
Mr. William Augustus Pierce, University of Vermont;
Professor Raymond J. Pitts, Fort Valley State College, Fort Valley, Ga.;
Miss Helen Susanna Pollock, State University of Iowa;
Mr. David Pomeroy, Oregon State College;
Mr. Gordon Raisbeck, Stanford University;
Mr. Meyer Morris Resnikoff, Carleton College, Northfield, Minn.;
Mr. Charles Barton Reynolds, Federal Communications Commission, New York, N. Y.;
Mr. Robert Bruce Rice, Ohio State University;
Ensign Robert Errol Roberson, Naval Research Laboratory, Washington, D. C.;
Miss Helen Murray Roberts, University of Connecticut;
Mr. Gene Fuerst Rose, Glassport, Pa.;
Professor Francis Edward Rothchild, Arkansas Agricultural and Mechanical College, Monticello, Ark.;
Miss Margaret M. Ryan, Syracuse University;
Mr. Herbert Ellis Salzer, Mathematical Tables Project, National Bureau of Standards, New York, N. Y.;
Professor Charles Henry Sampson, Bates College;
Mr. George R. Schaefer, U. S. Naval Observatory, Washington, D. C.;
Sister M. Rosalin Schaeffer, Ursuline College, Louisville, Ky.;
Mr. Lowell I. Schoenfeld, Naval Air Experimental Station, Navy Yard, Philadelphia, Pa.;
Dr. Hans Wilhelm Eduard Schwerdtfeger, University of Adelaide, Adelaide, South Australia;
Lieutenant Commander Henry St. Clair Sharp, United States Coast Guard Academy;
Mr. William Lloyd Shepherd, Southwestern College, Winfield, Kan.;
Professor Heinz Simon, Southern Union College, Wadley, Ala.;
Mr. Frank Brooke Sloss, Westminster College, Fulton, Mo.;
Mr. Maurice Howard Slud, Brown University;
Professor Dudley Eugene South, University of Kentucky;
Miss Elisabeth Stanfield, Bennett College, Greensboro, N. C.;
Mr. Irwin Stoner, University of North Carolina;
Professor Earl Garrod Swafford, Fort Hays Kansas State College, Hays, Kan.;
Mr. John William Swank, Los Angeles, Calif.;
Dr. Robert Catesby Taliaferro, Hamilton College;
Mr. Francisco Vazquez Pueyo, Polytechnic Institute of Puerto Rico;
Mr. Anthony Emilio Ventriglia, Cornell University;
Professor George Frank Waites, Centenary College of Louisiana;
Professor Eleanor Boyd Walters, Delta State Teachers College, Cleveland, Miss.;
Mr. Dana Wallis Whitman, Burlington, Vt.;
Dr. Yung-Chow Wong, University of Pennsylvania;
Mr. Frederick Lewis Wood, State Teachers College, Bridgewater, Mass.;

Mrs. Mary Hart Zink (Mrs. F. J.), Elmhurst College, Elmhurst, Ill.;
Mrs. Miriam F. Zucker (Mrs. S. M.), American Mathematical Society, New York, N.Y.

It was reported that the following had been elected as nominees on the Institutional Memberships of the Institutions indicated:

Brown University: Messrs. Joaquín Basilio Díaz and George Herman Handelman.
Bryn Mawr College: Miss Marie Anna Wurster.
University of California: Miss Louise Hoy Chin, Messrs. E. Allen Davis, Joseph Lawson Hodges, and Siegfried Friedrich Neustadter.
University of California at Los Angeles: Mr. Gordon Klinzman Overholtzer, Mrs. Zivia Wurtele.
University of Chicago: Messrs. Charles F. Brumfiel, Marvin Maynard Lavin, and Charles A. Nichols.
University of Cincinnati: Mr. John Sunapee Stubbe.
Cornell University: Messrs. Robert Lewis Beinert, Charles Hatfield, Jr., and Edward Joseph Scott.
Duke University: Mr. Robert Arthur Clark.
Harvard University: Messrs. E. J. Akutowicz, Philip Davis, L. Fine, Daniel Gorenstein, A. R. Harvey, I. I. Hirschman, Robert Heywood Hoskins, G. Mostow, S. B. Sommerville, James Wolfe, and Joseph Abraham Zilber.
University of Illinois: Miss Mildred Jeannette Brannon, Mr. H. Herbert Fox, Misses Helen Louise Friend, Helen P. Hoffman, and Elaine Vivian Nantkes, Mr. Richard George Stoneham, Professor Earl Reeves Wasserman.
Iowa State College: Mr. Frank Edward Bortle.
Johns Hopkins University: Messrs. Thomas Birdsal Andrews, Jr., Charles Vale Bitterli and Julius Vogel.
Massachusetts Institute of Technology: Messrs. Lawrence Russell Norwood, Walter Pitts, and George Neal Raney.
University of Michigan: Messrs. Pratap Chand and Manuel O. Hizon, Miss Frances Van Every Morfoot.
Northwestern University: Mr. William R. Allen, Miss Alice Christianson, Messrs. B. Kent Dickerson and Martinus Esser, Miss Evelyn Frank.
University of Notre Dame: Rev. Ferdinand L. Brown.
Pennsylvania State College: Mr. John Rankin Kinney.
Stanford University: Miss Kathryn Anne Morgan.
Syracuse University: Professor Howard Whitley Eves.
Texas Technological College: Miss Virginia Elizabeth Bowman.
University of Virginia: Messrs. Raymond Alfred Lytle and Russell Lowell Wine.
University of Washington: Miss Phyllis Cady Johnson, Mr. Max Carl Walske.
Wesleyan University: Messrs. Adolph Grünbaum and Raymond de Shelly.
Western and Southern Life Insurance Company: Mr. William S. Marcaccio.
Williams College: Mr. Edwin Foote Gillette.
University of Wisconsin: Messrs. Louis Harold Kanter, John Thomas Moore, Joseph Rosenthal, Raphael Darrel Wagner, and Robert Livingston Young.
Yale University: Mr. Walter John Klimczak.

It was also reported that Immaculata College, Immaculata, Pennsylvania, had been elected to institutional contributing membership in the Society.

The following appointments by President M. H. Stone were reported: as a committee to study the present method of arranging scientific programs, Professors Saunders MacLane (Chairman), W. T. Martin, C. V. Newsom, Gabor Szegö, and Oswald Veblen; as a Committee on Nomination of Officers and Members of the Council for 1945, Professors Einar Hille (Chairman), G. A. Hedlund, E. P. Lane, C. C. MacDuffee, and Hassler Whitney; as a Committee on Arrangements for the 1944 Summer Meeting at Wellesley College, Professor Marion E. Stark (Chairman), Dr. Katherine E. Hazard, Professors T. R. Hollcroft, B. W. Jones, and Helen G. Russell; as auditors of the Society's accounts for 1944, Professors T. F. Cope and A. E. Meder, Jr.; as representative of the Society at the inauguration of Clark George Kuebler as seventh President of Ripon College (Ripon, Wisconsin) on December 7, 1943, Professor R. E. Langer; as representative of the Society at the inauguration of Russell David Cole as President of Cornell College (Mount Vernon, Iowa) on April 25, 1944, Professor Roscoe Woods; as representative of the Society at the inauguration of the Very Reverend Vincent J. Flynn as President of The College of St. Thomas (St. Paul, Minnesota) on April 27, 1944, Professor R. W. Brink.

The Secretary reported that the following had been chosen as the voting representatives of editorial committees on the Council for the year 1944: Bulletin—Dean Tomlinson Fort; Transactions—Professor A. A. Albert; Colloquium Publications—Professor A. B. Coble; American Journal of Mathematics—Professor G. D. Birkhoff; Mathematical Reviews—Professor O. E. Neugebauer.

The Secretary reported that Professor John von Neumann had accepted the invitation to deliver the Gibbs Lecture at the Annual meeting of 1944; the title of the Lecture will be *The ergodic theorem and statistical mechanics.*

It was reported that the biennial List of Members, usually published as a supplement to the September Bulletin, will be published in November in 1944.

The Secretary reported the receipt of a letter from the London Mathematical Society, expressing appreciation of the fifteen microfilm reading machines, donated by our Society, which have now been distributed to scientific groups in Great Britain.

The Council adopted the following resolution on the death of Professor Thomas Scott Fiske:

> The death on January 10, 1944, of Professor Thomas Scott Fiske, venerable founder of the American Mathematical Society, brought deep sadness to the country's mathematicians.

To the members of the Society, the story of its first years is an absorbing one. In 1888, Professor Fiske, then a young man of twenty-three, saw that the time was ripe for the organization of the mathematicians of America. Through his efforts there was established the New York Mathematical Society which, in the space of three years, grew into a national organization. Fiske held office in the Society as Secretary from 1888 to 1895, as Treasurer from 1898 to 1901 and, finally, as President, 1903 to 1904. In addition, he served for long periods on the editorial staffs of the Bulletin and Transactions.

Fiske's ability as an organizer also found scope in the Department of Mathematics of Columbia University, of which he was a member for fifty years. Himself a scholar of genuine depth, he devoted his energies to the creation of a center of mathematical research at Columbia. He took a sincere interest in the researches of his colleagues and found particular pleasure in encouraging the scientific efforts of the young men serving under him.

Fiske was a striking personality. Distinguished in presence, cordial and frank in nature, he enjoyed the deep regard of his associates. He has left his mark on a period of mathematical history whose color will deepen as the years advance.

On recommendation of the Committee on Places of Meetings, the 1944 Annual Meeting was scheduled for Chicago, Illinois, on November 24–25. The Council also scheduled a far western meeting on November 25 at the University of California at Los Angeles.

It was reported that Professor Willy Feller had been invited to deliver an hour address at the 1944 Annual Meeting.

Professors F. J. Murray and Morgan Ward were reappointed as representatives of the Society on the Editorial Board of the Duke Mathematical Journal.

The Council accepted the resignation of Professor J. W. T. Youngs as Acting Associate Secretary.

The Council voted to recommend to the Board of Trustees that the Transactions of the Society be copyrighted.

The Secretary reported for the War Policy Committee as follows: (1) Problems in connection with mathematicians and Selective Service have occupied the attention of members of the Committee and the Secretary's office has attempted to keep chairmen of departments of mathematics informed of changes in procedure; (2) Efforts are being made to secure funds so that the Society and Association may continue to support the Office of Scientific Personnel in Washington, the Director of which has been useful in representing the mathematicians during these critical times; (3) the Committee on Available Teachers of Collegiate Mathematics is continuing to function as an aid to department chairmen in securing teachers for the service programs; (4) the Subcommittee on War Training Programs has submitted detailed reports on the mathematical aspects of the Army Specialized Training Program and of the Navy College Training Program. The Subcommittee is also studying the work of the Armed Forces Institute.

On recommendation of the committee appointed to consider the possibility of changing the term of office of the President of the Society, the Council voted that no change be made at this time.

Titles and cross references to the abstracts of papers read follow below. Papers whose abstract numbers are followed by the letter *t* were read by title. Papers numbered 1 to 6 were read in the section for Algebra, Probability, Statistics, and Applied Mathematics; papers 7 to 12 in the section for Algebra and Geometry; papers 13 to 28 were read by title. Paper 1 was read by Professor Tukey; paper 9 by Professor Cameron. Dr. Kober was introduced by Professor Einar Hille.

1. Henry Scheffé and J. W. Tukey: *Contributions to the theory of non-parametric estimation.* (Abstract 50-5-164.)

2. John Williamson: *Hadamard's determinant theorem and the sum of four squares.* (Abstract 50-5-119.)

3. Garrett Birkhoff: *Subdirect products in universal algebra.* (Abstract 50-5-106.)

4. J. A. Shohat: *Series expansions for the periodic solution of Van der Pol's equation and its frequency for all values of the parameter.* (Abstract 50-5-156.)

5. P. R. Halmos: *Random alms.* (Abstract 50-5-163.)

6. S. P. Avann: *Relations between join-irreducibles and meet-irreducibles in a modular lattice.* I. (Abstract 50-5-104.)

7. L. H. Loomis: *Abstract congruence and the uniqueness of Haar measure.* (Abstract 50-5-141.)

8. S. S. Chern: *Laplace transforms of a class of higher dimensional varieties in a projective space of n dimensions.* (Abstract 50-5-159.)

9. R. H. Cameron and W. T. Martin: *Transformations of Wiener integrals under a general class of linear transformations.* (Abstract 50-5-130.)

10. Otto Szász: *On uniform convergence of trigonometric series.* (Abstract 50-5-149.)

11. L. I. Schoenfeld: *A transformation formula in the theory of partitions.* (Abstract 50-5-147.)

12. M. M. Day: *On symmetric closed convex curves.* (Abstract 50-5-132.)

13. R. P. Agnew: *Summability of subsequences.* (Abstract 50-5-120-*t*.)

14. Stefan Bergman: *Certain classes of analytic functions of two real variables and their properties.* (Abstract 50-5-123-*t*.)

15. Stefan Bergman: *The determination of some properties of a function satisfying a partial differential equation from its series development.* (Abstract 50-5-124-*t*.)

16. Augusto Bobonis: *A sufficiency theorem for differential systems.* (Abstract 50-5-126-*t*.)

17. H. D. Brunk: *Theorems of composition for Dirichlet series.* Preliminary report. (Abstract 50-5-128-*t*.)

18. C. W. Churchman and Benjamin Epstein: *Estimates of error in parallel experiments.* (Abstract 50-5-162-*t*.)

19. W. B. Fite: *The degree of a linear homogeneous group.* (Abstract 50-5-109-*t*.)

20. Michael Goldberg: *A three-space analog of a plane Kempe linkage.*) (Abstract 50-3-81-*t*.)

21. S. B. Jackson: *Vertices of plane curves.* (Abstract 50-5-160-*t*.)

22. Edward Kasner and John DeCicco: *Isothermal families in general curvilinear coordinates, and loxodromes.* (Abstract 50-7-191-*t*.)

23. Herman Kober: *Approximation by integral functions in the complex domain.* (Abstract 50-5-140-*t*.)

24. A. N. Lowan and H. E. Salzer: *Coefficients for complex interpolation within a square grid.* (Abstract 50-5-142-*t*.)

25. G. W. Mackey: *On infinite dimensional linear spaces.* (Abstract 50-5-143-*t*.)

26. A. M. Peiser: *On the average sum of the real roots of a random algebraic equation.* (Abstract 50-5-144-*t*.)

27. G. T. Whyburn: *Interior mappings into the circle.* (Abstract 50-7-197-*t*.)

28. Oscar Zariski: *The theorem of Bertini concerning the variable singular points of a linear system of varieties.* (Abstract 50-7-194-*t*.)

J. R. KLINE,
Secretary

THE APRIL MEETING IN CHICAGO

The four hundred fourth meeting of the American Mathematical Society was held at the Museum of Science and Industry, Chicago, Illinois, on Friday and Saturday, April 28–29, 1944. The attendance was about one hundred twenty-five, including the following one hundred eleven members of the Society:

A. A. Albert, Warren Ambrose, E. W. Anderson, Reinhold Baer, Walter Bartky, P. O. Bell, S. F. Bibb, K. E. Bisshopp, B. H. Bissinger, G. A. Bliss, D. G. Bourgin, A. J. Brandt, J. L. Brenner, R. H. Bruck, Herbert Busemann, L. E. Bush, F. A. Butter, R. E. Byrne, Abraham Charnes, L. W. Cohen, Nancy Cole, Max Coral, B. K. Dickerson, Samuel Eilenberg, H. P. Evans, C. J. Everett, H. S. Everett, L. R. Ford, J. S. Frame, Evelyn Frank, J. H. Giese, J. W. Givens, R. A. Good, Cornelius Gouwens, L. M. Graves, L. W. Griffiths, V. G. Grove, R. W. Hamming, W. L. Hart, M. L. Hartung, C. T. Hazard, O. C. Hazlett, M. H. Heins, E. D. Hellinger, Fritz Herzog, J. F. Heyda, E. H. C. Hildebrandt, T. H. Hildebrandt, J. D. Hill, D. L. Holl, A. S. Householder, H. K. Hughes, M. H. Ingraham, Karl Johannes, G. K. Kalisch, Wilfred Kaplan, William Karush, D. E. Kearney, Fred Kiokemeister, W. C. Krathwohl, R. E. Langer, E. H. Larguier, A. L. Lewis, M. I. Logsdon, A. T. Lonseth, W. C. McDaniel, Janet MacDonald, C. C. MacDuffee, Morris Marden, J. R. Mayor, C. W. Mendel, Karl Menger, H. J. Miser, C. W. Moran, E. J. Moulton, A. L. Nelson, C. J. Nesbitt, Ivan Niven, Isaac Opatowski, J. W. Porter, Tibor Rado, R. B. Rasmusen, Maxwell Reade, W. T. Reid, Haim Reingold, M. D. P. Rochford, R. G. Sanger, O. F. G. Schilling, G. E. Schweigert, M. E. Shanks, I. S. Sokolnikoff, Abraham Spitzbart, N. E. Steenrod, C. F. Stephens, B. M. Stewart, A. H. Stone, R. L. Swain, J. L. Synge, R. M. Thrall, M. K. Toft, Henry Van Engen, Bernard Vinograde, S. E. Warschawski, J. V. Wehausen, G. W. Whitehead, L. R. Wilcox, J. E. Wilkins, M. A. Wurster, G. S. Young, J. W. T. Youngs, M. H. Zink.

Sessions for the reading of contributed papers were held on Friday morning, with Professor J. S. Frame presiding, and Saturday morning with Professor J. L. Synge presiding. On Friday afternoon, Professor J. L. Synge gave an invited address on *The problem of Saint Venant for a cylinder with free sides*, and Professor I. S. Sokolnikoff gave an invited address on *Torsion and flexure of compound prisms*. Professor Walter Bartky presided at these lectures.

All sessions were held in the West lecture hall in the Museum of Science and Industry.

Titles and cross references for the abstracts of papers read at the meeting follow below: Papers 1 to 8 were read Friday morning, papers 9 to 17 on Saturday morning, papers 18 to 38, whose abstract numbers are followed by the letter *t* were read by title. Paper 2 was read by Dr. Bissinger and paper 5 by Dr. Kaplan. Ida Roettinger was introduced by Professor R. V. Churchill.

1. J. E. Wilkins: *The contact of a cubic surface with a ruled surface.* (Abstract 50-5-161.)

2. B. H. Bissinger and Fritz Herzog: *An extension of some previous results on generalized continued fractions.* (Abstract 50-5-125.)

3. S. E. Warschawski: *On conformal mapping of nearly circular regions.* (Abstract 50-5-151.)

4. Maxwell Reade: *On functions of class PL.* (Abstract 50-5-145.)

5. Wilfred Kaplan and Max Dresden: *The mechanism of the condensation of gases.* (Abstract 50-5-152.)

6. Morris Marden: *Axisymmetric harmonic vectors.* (Abstract 50-5-153.)

7. H. K. Hughes: *The asymptotic expansions of entire functions defined by Maclaurin series.* (Abstract 50-5-138.)

8. Tibor Rado: *On the Geöcze area of Frechet surfaces.* Preliminary report. (Abstract 50-7-182.)

9. R. H. Bruck: *Quasigroups with the inverse property.* I. Preliminary report. (Abstract 50-5-107.)

10. G. K. Kalisch: *Generalized Hilbert spaces over fields with a non-Archimedean valuation.* (Abstract 50-5-113.)

11. G. S. Young: *Concerning spaces in which every arc has two sides.* (Abstract 50-5-170.)

12. Samuel Eilenberg: *Skew-invariant cohomology groups.* Prelininary report. (Abstract 50-5-166.)

13. N. E. Steenrod: *The classification of sphere bundles.* (Abstract 50-5-168.)

14. P. O. Bell: *A study of surfaces by means of a system of differential equations of the first order.* (Abstract 50-5-158.)

15. R. M. Thrall: *On modular representations induced by ordinary representations.* (Abstract 50-5-117.)

16. Bernard Vinograde: *On properties of completely primary rings.* (Abstract 50-5-118.)

17. Warren Ambrose: *A variation of a theorem of I.E. Segal on the group ring of a compact group.* (Abstract 50-7-172-*t.*)

18. Leon Alaoglu and Paul Erdös: *On highly composite and similar numbers.* (Abstract 50-5-103-*t.*)

19. Reinhold Baer: *Groups without proper isomorphic quotient groups.* (Abstract 50-5-105-*t.*)

20. Reinhold Baer: *Crossed isomorphisms.* (Abstract 50-7-173-*t.*)

21. Reinhold Baer: *The fundamental theorems of elementary geometry. An axiomatic analysis.* (Abstract 50-5-157-*t.*)

22. E. F. Beckenbach and Maxwell Reade: *Further results on mean-values and harmonic polynomials.* (Abstract 50-5-121-*t.*)

23. E. F. Beckenbach and Maxwell Reade: *Regular solids and harmonic polynomials.* (Abstract 50-5-122-*t*.)

24. R. H. Bruck: *Simple quasigroups.* (Abstract 50-5-108-*t*.)

25. H. D. Brunk: *Transformations of Dirichlet series.* Preliminary report. (Abstract 50-5-129-*t*.)

26. M. M. Day: *Cluster points of subsequences.* (Abstract 50-5-131-*t*.)

27. M. M. Day: *Inner-product spaces.* (Abstract 50-5-133-*t*.)

28. J. D. Hill: *Cesàro summability of sequences of* 0's *and* 1's. (Abstract 50-5-136-*t*.)

29. J. D. Hill: *Nörlund methods of summability that include the Cesàro methods of all positive orders.* (Abstract 50-5-137-*t*.)

30. G. B. Huff: *The completion of a theorem of Kantor.* (Abstract 50-3-93-*t*.)

31. Albert Newhouse: *On finite extending groups.* (Abstract 50-5-115-*t*.)

32. Maxwell Reade: *Two applications of generalized Laplaceans.* (Abstract 50-5-146-*t*.)

33. Ida Roettinger: *Certain integral transformations, with applications to boundary value problems.* (Abstract 50-5-155-*t*.)

34. M. F. Smiley: *An application of lattice theory to quasigroups.* (Abstract 50-5-116-*t*.)

35. C. F. Stephens: *Nonlinear difference equations analytic in a parameter.* (Abstract 50-5-148-*t*.)

36. F. A. Valentine: *Contractions in non-Euclidean spaces.* (Abstract 50-5-150-*t*.)

37. G. S. Young: *A generalization of Moore's theorem on simple triods.* (Abstract 50-5-169-*t*.)

38. G. S. Young: *On continua whose links are non-intersecting.* (Abstract 50-5-171-*t*.)

J. W. T. Youngs,
Acting Associate Secretary

THE APRIL MEETING IN BERKELEY

The four hundred fifth meeting of the American Mathematical Society was held at the University of California on Saturday, April 29, 1944. The attendance was about fifty-five, including the following thirty-eight members of the Society:

H. L. Alder, H. M. Bacon, E. W. Barankin, B. A. Bernstein, W. Z. Chien, L. H. Chin, C. H. Dix, Herbert Federer, A. L. Foster, S. M. Hallam, O. H. Hamilton, J. G. Herriot, Alfred Horn, D. H. Hyers, E. L. Lehmann, D. H. Lehmer, S. H. Levy, J. V. Lewis, C. C. Lin, A. D. Michal, A. P. Morse, S. F. Neustadter, A. R. Noble, George Polya, J. B. Robinson, R. M. Robinson, R. A. Rosenbaum, A. C. Schaeffer, S. A. Schaaf, D. C. Spencer, Pauline Sperry, L. H. Swinford, Gabor Szegö, Alfred Tarski, R. K. Wakerling, L. F. Walton, A. R. Williams, František Wolf.

After a brief opening session for contributed papers at which Professor B. A. Bernstein presided, the meeting in the morning was devoted to the invited one hour addresses at which Professor Gabor Szegö presided. By invitation of the Program Committee, Professor D. H. Hyers of the University of Southern California and the California Institute of Technology spoke on *Linear topological spaces* and Professor R. M. Robinson of the University of California on *Hadamard's three circles theorem.*

The Symposium on Applied Mathematics was held in the afternoon with Professor George Polya presiding. The program consisted of two addresses: *Hydrodynamical stability*, by Dr. C. C. Lin, and *The intrinsic theory of elastic plates and shells*, by Dr. W. Z. Chien, both of the California Institute of Technology.

Titles and cross references to the abstracts of the papers read follow below. Papers whose abstract numbers are followed by the letter *t* were read by title. Papers 1 to 3 were read in the morning session while papers 4 to 11 were read by title. Paper 3 was read by Professor Foster. Mr. Reed was introduced by Professor Michal.

1. J. G. Herriot: *Nörlund summability of multiple Fourier series.* (Abstract 50-5-135.)

2. Herbert Federer: *The Gauss-Green theorem.* (Abstract 50-5-134.)

3. A. L. Foster and B. A. Bernstein: *A dual-symmetric definition of field.* (Abstract 50-5-110.)

4. E. F. Beckenbach and R. H. Bing: *On generalized convex functions.* (Abstract 50-5-165-*t*.)

5. H. D. Brunk: *Dirichlet series meromorphic in a half-plane.* Preliminary report. (Abstract 50-5-127-*t*.)

6. R. C. James: *Linear functionals and orthogonality in normed linear spaces.* (Abstract 50-5-167-*t*.)

7. R. D. James: *Power series expansions for doubly periodic functions of the second kind.* Preliminary report. (Abstract 50-5-111-*t*.)

8. S. A. Jennings: *A note on chain conditions in nilpotent rings and groups.* (Abstract 50-5-112-*t*.)

9. A. D. Michal: *Physical models of some curved differential-geometric metric spaces of infinite dimensions.* II. *Vibrations of elastic beams and other elastic media as studies in geodesics.* Preliminary report. (Abstract 50-5-154-*t*.)

10. Knox Millsaps: *On powers of third and fourth order matrices.* (Abstract 50-5-114-*t*.)

11. I. S. Reed: *On the solution of a general transform.* (Abstract 50-7-181-*t*.)

A. D. MICHAL,
Associate Secretary

OSKAR BOLZA—IN MEMORIAM

Oskar Bolza came to the United States from Germany in 1888, and returned in 1910 to make his home in Freiburg im Baden during the later years of his life. He was an influential member of our American mathematical community for twenty-two years, including those significant ones during which the American Mathematical Society was founded and had its early growth. He was one of the founders of the Chicago Section of the Society, was a member of the Council of the Society in 1900–1902 and vice president in 1904. He was beloved by his students and associates. He was born on May 12, 1857 in Bergzabern, Germany, and word reached this country in the autumn of 1942 through the American Red Cross that he died peacefully on July 5, 1942, presumably in Freiburg. The following pages are devoted to a sketch of his life and scientific activities. They have unusual significance for our American mathematical community.

Bolza's father was in the judicial service in Germany and the family dwelt in various places in southern Germany during Bolza's childhood years. In 1873 they made Freiburg their permanent home, and the ties which bound Bolza to that city were thereafter always very strong. His mother was a daughter of Friedrich Koenig, the inventor of a rapid printing press in the early part of the nineteenth century. The manufacture of printing presses proved to be profitable through the years, and so far as is known Bolza's life was free from financial worries.

Even in his early years Bolza was an industrious and independent student. His first strong interests seemed to be the beginning of a career in languages and comparative philology. But in 1873 he attended an academy in Neuchatel and later the Gymnasium in Freiburg, in both of which he had excellent instructors in mathematics. There for the first time he became enamoured of the subject, and mathematics remained his principal interest during all but a few years of the rest of his life.

In the autumn of 1875 Bolza went to Berlin to begin his university student career. At that time it was expected that he would ultimately succeed to a share in the management of the family printing press factory. But his father wanted him to have a liberal as well as a technical education, and for that reason he first attended both the University and the Gewerbeakademie in Berlin. It soon became clear that this double program was impractical, and that Bolza's interests were likely to lie much more in pure science than in business. He at first turned to physics as a possible principal life activity, and during the years 1876–1878 he studied in that field under Kirchoff and Helmholz in Berlin, Quinke in Heidelberg, and Kundt in Strassburg. He even undertook to write a doctor's dissertation in physics under Kundt. But experimental work had always been burdensome to him and in the autumn of 1878 he decided finally, and certainly fortunately in view of his later attainments, to make pure mathematics his specialty.

The next three years Bolza pursued his interest in pure mathematics under Christoffel and Reye in Strassburg, Weierstrass in Berlin, Schwarz in Göttingen, and Weierstrass again. He was not pressed financially and had the approval of his father in his effort to prepare himself for a university career. But in the autumn of 1880 he found himself still far from his doctor's degree and already in his eleventh university semester. He determined therefore to prepare himself for the Staatsexamen which in Germany was a prerequisite for Gymnasium teaching though not for a university career. It was three years before his examination was passed and his year of practice teaching in the Gymnasium at Freiburg completed. He did well in this undertaking, and was successful and greatly interested in his teaching work.

For two years, 1883–1885, he worked privately on his doctor's dissertation in Freiburg. The subject which he had chosen was the determination of hyperelliptic integrals which can be reduced to elliptic integrals by a transformation of the third degree. He found the solution of his problem in 1885 just as a more elegant one by Goursat appeared in the Comptes Rendus. Nothing daunted Bolza attempted and found it easy to solve the corresponding problem for transformations of the fourth degree. His solution was published promptly in the Berichten der Freiburger Naturforschenden Gesellschaft. It was his first published paper.

In 1885 and 1886 Bolza studied again at Berlin under Fuchs and Kronecker. He extended his results on the transformation problem and after some correspondence with Schwarz and Klein in Göttingen his paper was accepted by the latter as a dissertation for the doctor's degree. Bolza's results were undoubtedly of great interest to Klein since they were closely related to some which Klein had himself recently published. In June 1886 Bolza passed his examination and was awarded the doctor's degree at the age of 29.

The next year, 1886–1887, Bolza and his intimate student friend Heinrich Maschke had a kind of private seminar with Klein in Göttingen. The three met once a week at Klein's house and planned and reported on research projects of common interest. It was a great experience for the younger men, but it had an unexpected effect upon

Bolza. Klein had an altogether exceptional genius and breadth of view in mathematics, and his personality was justifiably confident and overpowering. Bolza was a modest person, and as a result of his association with Klein he experienced a loss of confidence in himself which was completely unjustified, as later years have well shown.

It was at this stage in his career that Bolza debated seriously with himself what his future might turn out to be. As a student in Berlin he had made two exceptionally intimate young friends, the mathematician Heinrich Maschke mentioned above and the physicist Franz Schulze-Berge. Both of them had taken positions as gymnasium instructors but had found the teaching exceedingly heavy with little opportunity for scientific work. In the spring of 1887 Schulze-Berge decided to try his fortune in the United States, and upon his arrival here he promptly found a place in the experimental laboratory of Thomas Edison in New Jersey. This experience and the persuasiveness of two American mathematical students in Göttingen, M. W. Haskell and F. N. Cole, decided Bolza to follow the example of his friend, and in the spring of 1888 he landed in Hoboken.

Not long after his arrival he was greatly encouraged to find a place as "Reader in Mathematics" at Johns Hopkins University beginning in January 1889. Before the college year was out he was appointed "Associate in Mathematics" in the newly founded Clark University of Worcester, Massachusetts. The appointment began in October 1889 when the University first opened. Clark University had been founded as a graduate school with funds which soon proved to be insufficient. The faculty became restive and dissatisfied, and in 1892 when the new University of Chicago was opened, President Harper was able to persuade a number of them to move to his still newer institution. Bolza was one of these. He was made an associate professor, beginning January 1, 1893, with the understanding that he should be advanced to a full professorship after one year. He had also persuaded President Harper and E. H. Moore, the youthful head of the department of mathematics, to appoint Maschke to an associate professorship, Maschke having meanwhile also come to the United States.

Bolza was highly successful in his teaching and research during the eighteen years which he spent at the University of Chicago. The trio, Moore, Bolza, Maschke, had a profoundly stimulating influence on each other and on the continually numerous group of graduate students who went to Chicago to listen to them, a group now widely distributed among the faculties of our American universities. Among notable mathematical events which took place during this period were the International Mathematical Congress at the World's Fair in Chicago in 1893 and the immediately following inspiring Colloquium Lectures by Klein at Northwestern University, the transformation of the New York Mathematical Society into the American Mathematical Society, the foundation of the Chicago Section and other sections of the Society, and the foundation of the Transactions of the American Mathematical Society in 1900. In all of these activities Moore, Bolza, and Maschke took a prominent part. At the third colloquium of the Society in Ithaca in August 1901, Bolza lectured in beautiful fashion on the calculus of variations, a subject then much in vogue because of the then recent appearance of Kneser's book in that field.

In March of 1908 Maschke died and one of the strongest ties which bound Bolza to Chicago was thereby broken. During his sojourn in Chicago Bolza had frequently refreshed his relations with his relatives and friends in Germany by journeys to Freiburg during his vacation periods. Furthermore he had the feeling that he should make way for some of the large group of young and able American mathematicians then becoming established in increasing numbers in American mathematical departments. Soon after Maschke's death, therefore, Bolza decided to return to Freiburg to spend his later years in his original home environment. The mathematical department at the University of Chicago released him with great reluctance and with the title Nonresident Professor of Mathematics which he held until his death.

In Freiburg Bolza was appointed Ordentlicher Honorar professor at the University there, at the instigation of his friend, Lüroth, and for several years he continued his mathematical lectures and research with great success. In the summer quarter of 1913 he lectured again at the University of Chicago to the great satisfaction of his friends and hearers. But the war of 1914–1918 had a disastrous effect upon Bolza's mathematical research. He continued his lectures until 1926 but from as early as 1917, under the influence of several very good friends, his interest turned strongly to religious psychology and to Sanskrit which he studied for the purpose of reading at first hand literature concerning the religious systems of India. The result of these activities was a book entitled *Glaubenlose Religion* published in 1930 under the pseudonym F. H. Marneck.

After an interruption of three years Bolza resumed his mathematical lectures in 1929 at the University of Freiburg before the largest classes he had ever had. In one case ninety students listened to his lectures on differential equations. But the flood of students ebbed away there as elsewhere during succeeding years, and in 1933 at the age of 76 Bolza finally gave up lecturing and turned again to his psychological studies. Then a very interesting thing happened to him. Professor J. H. MacDonald, one of his Ph.D. students in 1901, sojourned for two weeks in Freiburg, and the two of them took long walks together during which they discussed again the subject which had long before engaged their interest, the transformation of hyperelliptic into elliptic integrals by transformations of specified degrees. Bolza was inspired to attempt anew some research in this field of his earliest mathematical activity and he succeeded in completing the solution of a problem which had long interested him much. His results were published in the Mathematische Annalen in 1935 in the paper numbered 57 in the bibliography below. In 1938 Bolza published one further study in religious psychology (no. 62 in the bibliography). It was his last published paper.

We know, however, that Bolza's enthusiasm for mathematics continued to the end. In a letter to Professor Arnold Dresden, written in 1939 at the age of 82, Bolza tells of his interest in studying the foundations of geometry with his friend and colleague in Freiburg, the geometer Heffter. According to Bolza's own statement his intention was to try to fill a yawning gap in his earlier mathematical training.

Since 1939 we have had little news from Bolza until the announcement in the autumn of 1942 of his death. But the impressions of his able and kindly personality which he has left behind, and the records of his inspiring lectures and research, are a part of our American mathematical traditions which will be with us always.

In the pages to follow in this biography some indications will be given of the scope and character of Bolza's scientific work.

THE SCIENTIFIC WORK OF OSKAR BOLZA

The scope and emphasis of Bolza's research are indicated roughly by the following table which lists for each of his principal fields the papers which he published and the years during which he was active in that field. The numbers refer to the bibliography at the end of this paper.

I. Elliptic and hyperelliptic integrals and functions: 1–6, 13–15, 17–20, 22, 23, 25, 60; 1885–1901, 1936.
II. Miscellaneous: 7–12, 16, 21, 34, 40, 41, 58, 61, 62; 1888–1938.
III. Calculus of variations: 24, 26–33, 35–39, 42–53, 56, 57, 59; 1901–1933.
IV. Integral equations and general analysis: 46, 54, 55; 1910–1914.

In Group I Bolza studied first the problem of determining hyperelliptic integrals which are reducible to elliptic integrals by transformations especially of the third and fourth degrees. It was his first research interest, the field of his dissertation sponsored by Klein in Göttingen, a problem often studied by leading mathematicians in the latter part of last century. Beginning with his sojourn under Klein in Göttingen in 1886, and continuing during something more than a decade following his move to the United States, Bolza published eleven papers concerning hyperelliptic Θ-, σ-, and $\wp$-functions and related subjects. In these he emphasized the elliptic theory and reformulated it in numerous instances to show how it appears as a special case of the hyperelliptic theory, and he carried over to the more general case many theorems and formulas well established in connection with the theory of elliptic functions. He was an able follower of his great teachers, Weierstrass and Klein, in this field.

When Bolza first went to Johns Hopkins in 1889 he was asked by the noted astronomer, Simon Newcomb, then head of the department of mathematics, to lecture on the theory of substitution groups and its applications to algebraic equations. In 1891 the lectures were published by invitation in the paper numbered 8 below, and in neighboring years Bolza wrote three further papers on the subject. One was a review of Cole's translation of Netto's *Theory of substitutions*, a book which aroused much interest in this country. The review was most thorough and scholarly, quite characteristic of Bolza, and much more comprehensive and critical than most reports of this sort.

Under the title "Miscellaneous" above I have grouped two papers on linear differential equations of the second order, Bolza's two publications concerning religious psychology, and other papers. But I wish to mention especially his autobiography *Aus meinem Leben* published privately in 1936. I had some time before urged him to write an account of his life and interests and was overjoyed when the result reached me. It is the source of much of the data in the sketch of his life presented here, and many readers besides his own students and colleagues will find great encouragement and interest in it.

By far the greatest of Bolza's interests was the calculus of variations, as indicated by the twenty-nine publications under III above. He was a student in the famous course of Weierstrass on the calculus of variations at the University of Berlin in 1879, but did not pursue the subject actively further until 1901 when he was invited to give a series of colloquium lectures at the summer meeting of the American Mathematical Society in Ithaca. He chose the calculus of variations as his subject. It was beautifully presented. The lectures much amplified, published in 1904 in the book numbered 33 in the bibliography below, contained the contributions to that date, especially of Weierstrass, Kneser, and Hilbert, for problems of the calculus of variations in the plane in both non-parametric and parametric form, enriched by the able presentation of Bolza himself. This was only his beginning. He published in 1909 his *Vorlesungen über Variationsrechnung*, numbered 44 below, which is now a classic, indispensable to every scholar in the field, and much wider in its scope than his earlier book. The later volume contains besides the theory of problems in the plane, the theory of the problem of Lagrange with fixed end points, and an introduction to the problem of the calculus of variations for double integrals.

One of Bolza's important contributions was his extension and application of existence theorems for implicit functions and for solutions of differential equations. Before his time most theorems concerning implicit functions defined by equations of the form

$$f_i(x_1, \cdots, x_m; y_1, \cdots, y_n) = 0 \qquad (i = 1, \cdots, n)$$

were based upon the hypothesis that there is a single solution $(x; y) = (a; b)$ of these equations, and the purpose of the theorems was to show that for x in a neighborhood of a there exists a uniquely defined system of solutions

$$y_i(x_1, \cdots, x_m) \qquad (i = 1, \cdots, n)$$

near the point $y = b$. Bolza assumed a set P of solutions $(x; y) = (a; b)$, instead of a single one, and proved the existence of solutions defined over a neighborhood of the projection X of P in x-space. Similarly for a system of differential equations

$$f_i(x, y_1, \cdots, y_n, y_1', \cdots, y_n') = 0 \qquad (i = 1, \cdots, n)$$

he assumed the existence of a solution arc $y_i(x)$ $(x_1 \leqq x \leqq x_2;$ $i = 1, \cdots, n)$ and proved that it could be imbedded in an n-parameter family of neighboring solutions. These theorems or their equivalents, as Bolza showed and as other students of the subject now well recognize, are essential in many instances to a rigorous theory of the calculus of variations.

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In these expressions the symbol y stands for the set $(y_1, \cdots, y_n)$, and similarly for y'. Bolza deduced first necessary conditions for a minimum and showed how the problems of Lagrange and Mayer are unified under his theory. He distinguishes with great clarity between so-called normal and abnormal minimizing arcs in cases much more general than had been considered before. In the second of the two papers he studies problems for which some of the side conditions written above are inequalities in place of equalities. It is impossible to describe or even mention here all of the contributions which Bolza made to the calculus of variations. He added liberally to the theory in the plane and to the problem of Lagrange, and improved them in many important respects. His methods are models which have greatly influenced many workers in the field.

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It would require a book to describe all of the research which Bolza published, even in the calculus of variations alone, and I shall content myself with the brief indications in the paragraphs which precede. His work was always meticulously precise, and was usually accompanied by a most scholarly and appreciative analysis of publications in the same neighborhood which preceded his own. Wherever his interest was aroused he has left valuable and indelible impressions in the field of mathematical research.

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13. *On Weierstrass' system of hyperelliptic integrals of the first and second kind*, Mathematical Papers read at the International Congress held in conjunction with the World's Columbian Exposition, Chicago, 1893, pp. 1–12.

turned to physics as a possible principal life activity, and during the years 1876–1878 he studied in that field under Kirchoff and Helmholz in Berlin, Quinke in Heidelberg, and Kundt in Strassburg. He even undertook to write a doctor's dissertation in physics under Kundt. But experimental work had always been burdensome to him and in the autumn of 1878 he decided finally, and certainly fortunately in view of his later attainments, to make pure mathematics his specialty.

The next three years Bolza pursued his interest in pure mathematics under Christoffel and Reye in Strassburg, Weierstrass in Berlin, Schwarz in Göttingen, and Weierstrass again. He was not pressed financially and had the approval of his father in his effort to prepare himself for a university career. But in the autumn of 1880 he found himself still far from his doctor's degree and already in his eleventh university semester. He determined therefore to prepare himself for the Staatsexamen which in Germany was a prerequisite for Gymnasium teaching though not for a university career. It was three years before his examination was passed and his year of practice teaching in the Gymnasium at Freiburg completed. He did well in this undertaking, and was successful and greatly interested in his teaching work.

For two years, 1883–1885, he worked privately on his doctor's dissertation in Freiburg. The subject which he had chosen was the determination of hyperelliptic integrals which can be reduced to elliptic integrals by a transformation of the third degree. He found the solution of his problem in 1885 just as a more elegant one by Goursat appeared in the Comptes Rendus. Nothing daunted Bolza attempted and found it easy to solve the corresponding problem for transformations of the fourth degree. His solution was published promptly in the Berichten der Freiburger Naturforschenden Gesellschaft. It was his first published paper.

In 1885 and 1886 Bolza studied again at Berlin under Fuchs and Kronecker. He extended his results on the transformation problem and after some correspondence with Schwarz and Klein in Göttingen his paper was accepted by the latter as a dissertation for the doctor's degree. Bolza's results were undoubtedly of great interest to Klein since they were closely related to some which Klein had himself recently published. In June 1886 Bolza passed his examination and was awarded the doctor's degree at the age of 29.

The next year, 1886–1887, Bolza and his intimate student friend Heinrich Maschke had a kind of private seminar with Klein in Göttingen. The three met once a week at Klein's house and planned and reported on research projects of common interest. It was a great experience for the younger men, but it had an unexpected effect upon

Bolza. Klein had an altogether exceptional genius and breadth of view in mathematics, and his personality was justifiably confident and overpowering. Bolza was a modest person, and as a result of his association with Klein he experienced a loss of confidence in himself which was completely unjustified, as later years have well shown.

It was at this stage in his career that Bolza debated seriously with himself what his future might turn out to be. As a student in Berlin he had made two exceptionally intimate young friends, the mathematician Heinrich Maschke mentioned above and the physicist Franz Schulze-Berge. Both of them had taken positions as gymnasium instructors but had found the teaching exceedingly heavy with little opportunity for scientific work. In the spring of 1887 Schulze-Berge decided to try his fortune in the United States, and upon his arrival here he promptly found a place in the experimental laboratory of Thomas Edison in New Jersey. This experience and the persuasiveness of two American mathematical students in Göttingen, M. W. Haskell and F. N. Cole, decided Bolza to follow the example of his friend, and in the spring of 1888 he landed in Hoboken.

Not long after his arrival he was greatly encouraged to find a place as "Reader in Mathematics" at Johns Hopkins University beginning in January 1889. Before the college year was out he was appointed "Associate in Mathematics" in the newly founded Clark University of Worcester, Massachusetts. The appointment began in October 1889 when the University first opened. Clark University had been founded as a graduate school with funds which soon proved to be insufficient. The faculty became restive and dissatisfied, and in 1892 when the new University of Chicago was opened, President Harper was able to persuade a number of them to move to his still newer institution. Bolza was one of these. He was made an associate professor, beginning January 1, 1893, with the understanding that he should be advanced to a full professorship after one year. He had also persuaded President Harper and E. H. Moore, the youthful head of the department of mathematics, to appoint Maschke to an associate professorship, Maschke having meanwhile also come to the United States.

Bolza was highly successful in his teaching and research during the eighteen years which he spent at the University of Chicago. The trio, Moore, Bolza, Maschke, had a profoundly stimulating influence on each other and on the continually numerous group of graduate students who went to Chicago to listen to them, a group now widely distributed among the faculties of our American universities. Among notable mathematical events which took place during this period were the International Mathematical Congress at the World's Fair in Chicago in 1893 and the immediately following inspiring Colloquium Lectures by Klein at Northwestern University, the transformation of the New York Mathematical Society into the American Mathematical Society, the foundation of the Chicago Section and other sections of the Society, and the foundation of the Transactions of the American Mathematical Society in 1900. In all of these activities Moore, Bolza, and Maschke took a prominent part. At the third colloquium of the Society in Ithaca in August 1901, Bolza lectured in beautiful fashion on the calculus of variations, a subject then much in vogue because of the then recent appearance of Kneser's book in that field.

In March of 1908 Maschke died and one of the strongest ties which bound Bolza to Chicago was thereby broken. During his sojourn in Chicago Bolza had frequently refreshed his relations with his relatives and friends in Germany by journeys to Freiburg during his vacation periods. Furthermore he had the feeling that he should make way for some of the large group of young and able American mathematicians then becoming established in increasing numbers in American mathematical departments. Soon after Maschke's death, therefore, Bolza decided to return to Freiburg to spend his later years in his original home environment. The mathematical department at the University of Chicago released him with great reluctance and with the title Nonresident Professor of Mathematics which he held until his death.

In Freiburg Bolza was appointed Ordentlicher Honorar professor at the University there, at the instigation of his friend, Lüroth, and for several years he continued his mathematical lectures and research with great success. In the summer quarter of 1913 he lectured again at the University of Chicago to the great satisfaction of his friends and hearers. But the war of 1914–1918 had a disastrous effect upon Bolza's mathematical research. He continued his lectures until 1926 but from as early as 1917, under the influence of several very good friends, his interest turned strongly to religious psychology and to Sanskrit which he studied for the purpose of reading at first hand literature concerning the religious systems of India. The result of these activities was a book entitled *Glaubenlose Religion* published in 1930 under the pseudonym F. H. Marneck.

After an interruption of three years Bolza resumed his mathematical lectures in 1929 at the University of Freiburg before the largest classes he had ever had. In one case ninety students listened to his lectures on differential equations. But the flood of students ebbed away there as elsewhere during succeeding years, and in 1933 at the age of 76 Bolza finally gave up lecturing and turned again to his psychological studies. Then a very interesting thing happened to him. Professor J. H. MacDonald, one of his Ph.D. students in 1901, sojourned for two weeks in Freiburg, and the two of them took long walks together during which they discussed again the subject which had long before engaged their interest, the transformation of hyperelliptic into elliptic integrals by transformations of specified degrees. Bolza was inspired to attempt anew some research in this field of his earliest mathematical activity and he succeeded in completing the solution of a problem which had long interested him much. His results were published in the Mathematische Annalen in 1935 in the paper numbered 57 in the bibliography below. In 1938 Bolza published one further study in religious psychology (no. 62 in the bibliography). It was his last published paper.

We know, however, that Bolza's enthusiasm for mathematics continued to the end. In a letter to Professor Arnold Dresden, written in 1939 at the age of 82, Bolza tells of his interest in studying the foundations of geometry with his friend and colleague in Freiburg, the geometer Heffter. According to Bolza's own statement his intention was to try to fill a yawning gap in his earlier mathematical training.

Since 1939 we have had little news from Bolza until the announcement in the autumn of 1942 of his death. But the impressions of his able and kindly personality which he has left behind, and the records of his inspiring lectures and research, are a part of our American mathematical traditions which will be with us always.

In the pages to follow in this biography some indications will be given of the scope and character of Bolza's scientific work.

THE SCIENTIFIC WORK OF OSKAR BOLZA

The scope and emphasis of Bolza's research are indicated roughly by the following table which lists for each of his principal fields the papers which he published and the years during which he was active in that field. The numbers refer to the bibliography at the end of this paper.

I. Elliptic and hyperelliptic integrals and functions: 1–6, 13–15, 17–20, 22, 23, 25, 60; 1885–1901, 1936.
II. Miscellaneous: 7–12, 16, 21, 34, 40, 41, 58, 61, 62; 1888–1938.
III. Calculus of variations: 24, 26–33, 35–39, 42–53, 56, 57, 59; 1901–1933.
IV. Integral equations and general analysis: 46, 54, 55; 1910–1914.

In Group I Bolza studied first the problem of determining hyperelliptic integrals which are reducible to elliptic integrals by transformations especially of the third and fourth degrees. It was his first research interest, the field of his dissertation sponsored by Klein in Göttingen, a problem often studied by leading mathematicians in the latter part of last century. Beginning with his sojourn under Klein in Göttingen in 1886, and continuing during something more than a decade following his move to the United States, Bolza published eleven papers concerning hyperelliptic Θ-, σ-, and $\wp$-functions and related subjects. In these he emphasized the elliptic theory and reformulated it in numerous instances to show how it appears as a special case of the hyperelliptic theory, and he carried over to the more general case many theorems and formulas well established in connection with the theory of elliptic functions. He was an able follower of his great teachers, Weierstrass and Klein, in this field.

When Bolza first went to Johns Hopkins in 1889 he was asked by the noted astronomer, Simon Newcomb, then head of the department of mathematics, to lecture on the theory of substitution groups and its applications to algebraic equations. In 1891 the lectures were published by invitation in the paper numbered 8 below, and in neighboring years Bolza wrote three further papers on the subject. One was a review of Cole's translation of Netto's *Theory of substitutions*, a book which aroused much interest in this country. The review was most thorough and scholarly, quite characteristic of Bolza, and much more comprehensive and critical than most reports of this sort.

Under the title "Miscellaneous" above I have grouped two papers on linear differential equations of the second order, Bolza's two publications concerning religious psychology, and other papers. But I wish to mention especially his autobiography *Aus meinem Leben* published privately in 1936. I had some time before urged him to write an account of his life and interests and was overjoyed when the result reached me. It is the source of much of the data in the sketch of his life presented here, and many readers besides his own students and colleagues will find great encouragement and interest in it.

By far the greatest of Bolza's interests was the calculus of variations, as indicated by the twenty-nine publications under III above. He was a student in the famous course of Weierstrass on the calculus of variations at the University of Berlin in 1879, but did not pursue the subject actively further until 1901 when he was invited to give a series of colloquium lectures at the summer meeting of the American Mathematical Society in Ithaca. He chose the calculus of variations as his subject. It was beautifully presented. The lectures much amplified, published in 1904 in the book numbered 33 in the bibliography below, contained the contributions to that date, especially of Weierstrass, Kneser, and Hilbert, for problems of the calculus of variations in the plane in both non-parametric and parametric form, enriched by the able presentation of Bolza himself. This was only his beginning. He published in 1909 his *Vorlesungen über Variationsrechnung*, numbered 44 below, which is now a classic, indispensable to every scholar in the field, and much wider in its scope than his earlier book. The later volume contains besides the theory of problems in the plane, the theory of the problem of Lagrange with fixed end points, and an introduction to the problem of the calculus of variations for double integrals.

One of Bolza's important contributions was his extension and application of existence theorems for implicit functions and for solutions of differential equations. Before his time most theorems concerning implicit functions defined by equations of the form

$$f_i(x_1, \cdots, x_m; y_1, \cdots, y_n) = 0 \qquad (i = 1, \cdots, n)$$

were based upon the hypothesis that there is a single solution $(x; y) = (a; b)$ of these equations, and the purpose of the theorems was to show that for x in a neighborhood of a there exists a uniquely defined system of solutions

$$y_i(x_1, \cdots, x_m) \qquad (i = 1, \cdots, n)$$

near the point $y = b$. Bolza assumed a set P of solutions $(x; y) = (a; b)$, instead of a single one, and proved the existence of solutions defined over a neighborhood of the projection X of P in x-space. Similarly for a system of differential equations

$$f_i(x, y_1, \cdots, y_n, y_1', \cdots, y_n') = 0 \qquad (i = 1, \cdots, n)$$

he assumed the existence of a solution arc $y_i(x)$ $(x_1 \leqq x \leqq x_2;$ $i = 1, \cdots, n)$ and proved that it could be imbedded in an n-parameter family of neighboring solutions. These theorems or their equivalents, as Bolza showed and as other students of the subject now well recognize, are essential in many instances to a rigorous theory of the calculus of variations.

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50. *Über die Integrabilitätsbedingungen beim Bestehen von Nebenbedingungen*, Math. Ann. vol. 71 (1911) pp. 533–547.

51. *Bemerkungun zu Newton's Beweis seines Satzes über den Rotationskörper kleinsten Widerstandes*, Bibliotheca Mathematica (3) vol. 13 (1913) pp. 146–149.

52. *Über zwei Eulersche Aufgaben aus der Variationsrechnung*, Annali di Matematica Pura ed Applicata vol. 20 (1913) pp. 245–255.

53. *Über den "Anormalen Fall" beim Lagrangeschen und Meyerschen Problem mit gemischten Bedingungen und variabeln Endpunkten*, Math. Ann. vol. 74 (1913) pp. 430–446.

54. *Lectures on integral equations*. A hand written record by W. V. Lovitt of a course given by Bolza at the University of Chicago, Summer Quarter 1913. A revision of these notes appears in Lovitt, *Linear integral equations*, 1924, xiv+253 pp.

55. *Einführung in E. H. Moore's "General Analysis" und deren Anwendung auf die Verallgemeinerung der Theorie der linearen Integralgleichungen*, Jber. Deutschen Math. Verein vol. 23 (1914) pp. 248–303.

56. *Über Variationsprobleme mit Ungleichungen als Nebenbedingungen*, Hermann Amandus Schwarz zu seinem funfzigjährigen Doktorjubiläum am 6. August 1914, pp. 1–18.

57. *Gauss und die Variationsrechnung*, see Gauss' Werke, vol. 10, Abtheilung 2, 5 Abhandlung (1922), 95 pp.

58. *Glaubenlose Religion*, Verlag von Ernst Reinhardt in München (1931) 197 pp., written by Bolza under the pseudonym F. H. Marneck.

59. *Vorlesungen über Variationsrechnung*, 2d edition, 1933, ix+705+13 pp.

60. *Der singuläre Fall der Reduction hyperelliptische Integrale erster Ordnung auf elliptische durch Transformation dritten Grades*, Math. Ann. vol. 111 (1935) pp. 477–500.

61. *Aus meinem Leben*, 1936, 45 pp.

62. *Meister Eckehart als Mystiker*, eine religions psychologische Studie, Verlag von H. Reinhardt in München, 1938, 39 pp.

G. A. Bliss

BOOK REVIEW

Vector and tensor analysis. By H. V. Craig. New York and London, McGraw-Hill, 1943. 14+434 pp. $3.50.

During the last decade, there has been considerable emphasis on the presentation of subjects in textbooks from the axiomatic viewpoint. This approach implies carefully worded definitions, axioms, and theorems. Further, in this point of view, the stress is placed on the analytical and logical rather than the geometrical and physical aspects of a subject. The type of presentation associated with this view has furnished interesting and valuable textbooks in such introductory subjects as college algebra and calculus. Previously, no such presentation had been attempted for a senior-graduate level text in vector and tensor analysis. Craig has admirably presented vector and tensor analysis in the light of this analytical-logical viewpoint.

In writing such a text on vector and tensor analysis, a subject whose origins and developments are closely connected with geometry and physics, an author faces many problems. Perhaps one of the most difficult of these problems is concerned with the author's treatment of differentials. From the logical viewpoint, differentials are non-essential tools since all their functions may be performed by derivatives. However, differentials act in two very important roles in vector and tensor analysis. First, the literature, both past and present, of the subject abounds in the use of differential notation. Second, differentials are useful to both the geometer and the physicist in interpreting his results. From these interpretations, many new results have been obtained, often by proofs which are incorrect from the modern point of view. However, rigor and concise thinking are fundamental to the author's approach to the subject. Hence, he has chosen to omit the treatment of differentials. Another important problem is the treatment of coordinate transformations and invariants. Very few introductory texts to vector and tensor analysis furnish an adequate treatment of this important topic. The principal difficulty is that a proper presentation of this topic requires that the student possess a knowledge of determinants and linear transformations. Because of the present author's analytical approach, he is able to offer an excellent account of this subject. In fact, the concept of invariance dominates the greater part of the book.

The book is divided into four parts; Part A deals with advanced calculus; Part B with elementary vector analysis; Part C with tensors and extensors; Part D with applications.

Part A is concerned with those parts of advanced calculus which are frequently used in the remainder of the text. Chapter I points out some common errors in reasoning and offers suggestions on reading to various types of students. The real number system, continuity, differentiability and integrability of one function of one real variable are carefully treated in the next chapter. In chapter III, the author considers the analogous properties of a function of two or more real variables. Implicit functions, the distinction between iterated and multiple integrals and the relation of these integrals are discussed. Geometry is introduced into the text in the next chapter via a discussion of the notion of parameterized arcs. In particular, Theorem (A 15.1) should be mentioned. This interesting result is neglected in most texts on advanced calculus. The chapter closes with a treatment of some topics in the calculus of variations. In particular, the author discusses the Euler equations in order to provide a base for a later consideration of the theory of geodesics. An excellent account of the e-systems and δ-systems, determinants and coordinate transformations, brings Part A to a close. Although the group concept is completely developed in the treatment of coordinate transformations, unfortunately, the specific term "group" is nowhere defined nor used. The summation convention is introduced in Part A and the role of the Jacobian in integral transformations is clearly presented. Further, the problems in this part and the following parts of the book are well chosen and serve to supplement the theorems of the text proper.

Part B deals principally with the standard topics of vector analysis. The presentation differs considerably from that found in most texts in that: (1) definitions, axioms, theorems are clearly stated; (2) considerable stress is placed upon invariants. In chapter VI, the properties of vectors are defined, the scalar and vector products are introduced as invariants, and the triple scalar and vector products are discussed. It should be noted that with the aid of the e-system, the author offers an easy proof of the usually difficult triple vector expansion. Chapter VII contains an axiomatic approach to n-dimensional vector spaces and linear manifolds. Following the modern approach to geometry, the author, next, introduces a metric in his n-space. An interesting exposition of the Schmidt orthogonalization process for positive definite and indefinite metrics concludes the chapter. Chapter VIII contains a discussion of the elements of the differential geometry of curves and surfaces, the gradient, divergence, and curl, and an introduction to tensor and extensor transformation theory. In concluding Part B, the author discusses the important integral transformations of vector analysis—the Gauss, Stokes, and Green theorems. These topics—particularly Stokes' theorem—are well treated. This section contains only one troublesome typographical error: on page 159, the expression $(\gamma-\omega\times R)$ should read $\gamma-(\omega\times R)$. Further, the term "unitary orthogonal" on page 174 appears to be misapplied. In the literature, this term is reserved for vectors which are functions of a complex variable.

Part C deals with the theory of weighed tensors and extensors. Since the author has made important research contributions to extensor theory, his presentation of this subject should prove valuable to potential research workers in this field. Chapter X deals with the algebra of weighed tensors and extensors—the addition, multiplication, and contraction laws. The next chapter presents the differential calculus of tensors and extensors. Fundamental extensors, the fundamental tensors, and the Christoffel symbols are introduced in terms of derivatives of the radius vector in a Euclidean n-space. By a contraction of these fundamental extensors, the author obtains the ordinary covariant derivative. This procedure gives the combined theory of extensors and tensors considerable unity. A treatment of geodesics, geodesic coordinates, and the Riemann-Christoffel tensor concludes this part of the text. There is one minor point in notation which may cause some confusion. The author uses a pair of brackets around any two indices to indicate their antisymmetric nature. However, in a few instances, brackets are inserted about three indices without adequate explanations. Aside from this omission, part *C* is clearly written and constitutes a very good introduction to the theory of tensors and extensors.

Part D applies the material developed in parts B and C to classical dynamics and relativity. The first chapter in this section discusses such topics as moving coordinate systems, kinetic and potential energy of a particle and a system of particles, and the mechanics of continuous media. An interesting feature in the treatment of this last topic is the introduction of the Newtonian (mass constant) form and special relativity (mass variable) form of the equations of motion. Chapter XIII furnishes a discussion of the basis of special relativity, the Lorentz-Einstein transformation, applications of this transformation to velocity, acceleration, force, and mass, and finally a discussion of the energy-momentum tensor. The final chapter gives an introduction to General Relativity. In particular, the computation of the Schwarzchild line element is given in considerable detail.

This book presents vector and tensor analysis from an interesting point of view. It should provide a valuable addition to mathematical literature.

N. Coburn

NOTES

Lectures on matrices by J. H. M. Wedderburn, volume 17 of the American Mathematical Society Colloquium Publications, is again available. A limited number of copies have been prepared photographically from the original edition. This reprint has an expanded bibliography but otherwise contains exactly the material of the original edition. The price is unchanged, $3.35, with the usual 25 per cent discount to members of the Society.

The Association for Symbolic Logic has elected Dr. Alfred Tarski of the University of California as president and Professor C. A. Baylis of Brown University as vice president. Dr. C. G. Hempel of Queens College, Professor Karl Menger of the University of Notre Dame, and Professor Saunders MacLane of Harvard University have been elected members of the Executive Committee.

Dr. J. H. C. Whitehead of Balliol College, Oxford, has been elected a fellow of the Royal Society.

Professor Saunders MacLane of Harvard University has been elected a member of the American Academy of Arts and Sciences.

Professor G. D. Birkhoff and Harlow Shapley of Harvard University have been elected honorary members of the Mexican Mathematical Society. Professor Birkhoff has also been elected a corresponding member of the Mexican National Academy of Sciences.

Assistant Professor A. E. Taylor of the University of California at Los Angeles has been elected a corresponding member of the National Academy of Sciences of Lima.

Professor André Weil of Lehigh University has been awarded a Guggenheim fellowship. He will prepare a book on the foundations of algebraic geometry.

Dr. Warren Weaver of the Rockefeller Foundation, Professor E. P. Wigner of Princeton University, and Professor E. T. Whittaker of the University of Edinburgh have been elected members of the American Philosophical Society.

Lord Bertrand Russell has been elected a fellow of Trinity College, Cambridge.

Professor Alan Hazeltine of Stevens Institute of Technology has resigned.

Associate Professor B. W. Jones of Cornell University has been promoted to a professorship.

Associate Professor L. L. Lowenstein of Alfred University, Alfred, New York, has been promoted to a professorship.

Associate Professor L. F. Ollmann of Hofstra College, has been promoted to a professorship.

Assistant Professor G. L. Walker of the University of Delaware has been appointed to an acting assistant professorship at Cornell University.

Dr. C. P. Wells of Michigan State College has been promoted to an assistant professorship.

Mrs. A. W. Goodman has been appointed to an instructorship at Vassar College, and Dr. Murray Mannos to an instructorship at the University of Notre Dame.

Professor Emeritus A. E. Jolliffe of the University of London died March 17, 1944 at the age of seventy-three years.

Mr. Arne Fisher, mathematician with the Western Union Telegraph Company in New York, died April 8, 1944 at the age of fifty-seven years.

Professor H. C. Hicks of Carnegie Institute of Technology died April 14, 1944. He had been a member of the Society since 1923.

Dr. Clyde Wolfe of the Radiation Laboratory of the University of California died March 25, 1944.

ABSTRACTS OF PAPERS

SUBMITTED FOR PRESENTATION TO THE SOCIETY

The following papers have been submitted to the Secretary and the Associate Secretaries of the Society for presentation at meetings of the Society. They are numbered serially throughout this volume. Cross references to them in the reports of the meetings will give the number of this volume, the number of this issue, and the serial number of the abstract.

ALGEBRA AND THEORY OF NUMBERS

172. Warren Ambrose: *A variation of a theorem of I. E. Segal on the group ring of a compact group.*

It is shown that the L_2-ring of a compact group is a direct sum of full matrix algebras. This is done using the special properties of the L_2-ring, and its ideal theory, without reference to the classical theory of group representations, though the proof parallels, in part, the proof of the Peter-Weyl theorem. (Received April 14, 1944.)

173. Reinhold Baer: *Crossed isomorphisms.*

The 1:1 correspondence f, mapping the elements in the group G upon the elements in the group H, is termed a crossed isomorphism of G upon H, if there exists to every element x in G an endomorphism $e(x)$ of G such that $(u^{e(v)}v)^f = u^f v^f$ for u, v in G. The position of crossed isomorphisms between ordinary isomorphisms and 1:1 correspondences is discussed and their relation to projectivities of G upon H is investigated. If the endomorphism $e(x)$ of G maps every subgroup of G upon itself, then f is termed an integral crossed isomorphism. All the groups that are integral crossed isomorphic maps of abelian groups are determined. (Received April 28, 1944.)

174. R. J. Duffin and R. S. Pate: *Structure elements of quasigroups.* II.

This paper continues the study of the concepts of ***rank*** and ***left associativity*** (Duke Math. J. vol. 10 (1943) p. 743). A ***left set*** is a set L of elements such that $(Lx)y = L(xy)$ for every pair (x, y) of elements of Q. The ***middle rank*** of Q is the number of permutations of the columns of the multiplication table of Q which are simultaneously permutations of the rows. The ranks of Q divide its order. A Q of middle rank r contains a sub-Q of order r whose elements consist of all the middle associative elements $\{m\}$, that is, $x(my) = (xm)y$. These considerations furnish a practical test to determine whether a quasigroup is a group because the rank of a group is equal to the order. The meet, join and product of left sets are left sets. The set of elements $\{a\}$ such that $La = L$ is both a left set and a sub-Q. Unlike group theory, quotient Q's may be defined even when there are no sub-Q's. The lattice of quotient Q's is modular. The existence of a quotient quasigroup is equivalent to the existence of a set C satisfying certain associative and commutative laws. Permutation of rows, columns or symbols of the multiplication table leaves both rank and the lattice of weak quotients invariant. (Received May 31, 1944.)

175. R. J. Duffin and R. S. Pate: *Structure elements of quasigroups. III.*

This paper is a study of the three groups generated by the right and left substitutions of a quasigroup; G_r, G_l, and G. Group theoretic theorems concerning systems of imprimitivity are found to be useful in this connection. A quotient expansion of Q is equivalent to a system of imprimitivity of G. The lattice of quotients is isomorphic to a certain (synthetic) lattice of intransitive normal subgroups of G. If K is the subgroup corresponding to the quotient Q', then the G of Q' is the abstract group G/K. The Jordan-Hölder and Schrierer-Zassenhaus developments follow from these considerations. A left coset expansion of Q is equivalent to a system of imprimitivity in G_r. The lattice of left coset expansions is isomorphic to a certain lattice of subgroups of G_r. These results are related to abstract groups by employing a method of R. Baer (Trans. Amer. Math. Soc. vol. 46 (1939) pp. 110–141). Special types of quotient quasigroups are classified by the isotopies sending them into simpler types. A general feature noted is that the associative properties of the quasigroup are often reflected as commutative properties of the groups. (Received May 31, 1944.)

176. O. F. G. Schilling: *Automorphisms of fields of formal power series.*

Let $F=\Omega\{t\}$ be a field of power series of one variable over the field Ω. This note deals with the structure of the automorphism group of F/Ω. It is shown how the concepts of the Hilbert theory can be carried over so as to provide universal ramification groups for the totality of subfields $\Omega\subset L\subset F$ over which F is normal. (Received May 31, 1944.)

177. O. F. G. Schilling: *Noncommutative valuations.*

Let D be a division algebra and V a valuation on D with values in a simply ordered 1-group. It is shown that the ideal theory in D with respect to V can be developed as in the commutative case provided proper safeguards are observed. It turns out that the division algebras in question must be transcendental over their centers. Certain algebras may be considered as crossed extensions of division algebras by 1-groups. (Received May 31, 1944.)

178. S. M. Ulam and C. J. Everett: *On ordered groups.*

An o-group G is an l-group (G. Birkhoff, *Lattice ordered groups*, Ann. of Math. vol. 43 (1942)) with the lattice property weakened to that of Moore-Smith. An o-group is embeddable in a complete ordered group if and only if it is integrally closed. If the commutator group of an integrally closed o-group is in its center, it is commutative. The conjecture of the "suggestion" on p. 329, loc. cit., is thus proved. Counterexamples are given for Problems 1, 2, loc. cit. The group of functions $ax+A$, a, A real, $a>0$, under composition admits no integrally closed order. Every l-group is embeddable in a sequence complete group in the sense of o-convergence. The results of Everett, *Sequence completion of lattice moduls*, Duke Math. J. vol. 11 (1944), are extended to noncommutative groups. Various orders in the free group of two generators establish curious properties of the group of topological transformations of the line into itself. (Received April 25, 1944.)

ANALYSIS

179. R. P. Agnew: *A family of bounded sequences summable M.*

A simple example is given of a family of bounded sequences, summable M, which is not a separable subset of the space B of bounded sequences. (Received May 2, 1944.)

180. R. P. Agnew: *Convergence fields of methods of summability.*

Let A be a multiplicative matrix method of summatibility with multiplier ρ. If $\rho=0$, each sequence x_n which oscillates sufficiently slowly is summable A to 0. If $\rho\neq 0$ and x_n is summable A to L, then the sequence $(x_n-L/\rho)\xi_n$ is summable A to 0 if the sequence ξ_n oscillates sufficiently slowly. (Received May 16, 1944.)

181. I. S. Reed: *On the solution of a general transform.*

The purpose of this paper is to give a brief extension to the solution of a Watson transform with an unsymmetrical kernel. Use has been made of the work that has been done by Hardy, Watson, Titchmarsh, Goodspeed, and others. (Received April 26, 1944.)

182. Tibor Rado: *On the Geöcze area of Fréchet surfaces.* Preliminary report.

The Geöcze area of a surface S is defined in terms of projections upon the coordinate planes. Perfecting previous theories originating with the work of Geöcze, Reichelderfer (Trans. Amer. Math. Soc. vol. 53 (1943) pp. 251–291) introduced and studied an *essential area*. In his definition (loc. cit. p. 274) we replace *simple Jordan regions* by *domains of any connectivity*, and obtain an area to be denoted by $G(S)$, where S stands for a Fréchet surface of the type of the sphere or the disc. In this paper we develop a comprehensive theory of $G(S)$ as a foundation for the theory of the area. One of the results states that if the Lebesgue area $L(S)$ is finite, then $G(S)=L(S)$. Essential use is made of methods and results developed in recent years by Reichelderfer, Morrey, Youngs, and the author. (Received April 19, 1944.)

183. A. R. Schweitzer: *On functional equations with solutions containing arbitrary functions.* VI.

The author constructs equations in iterative compositions satisfied by specific functions of variables and derived from conditions of invariance of iterative compositions of these functions under substitution groups on the variables of the latter. For example, functional equations satisfied by arbitrary functions of the type $\gamma(x_1+x_2+\cdots+x_n)$ are obtained by assuming that the function $\beta\{\alpha(x_1+x_2+\cdots+x_m)+\alpha(y_1+y_2+\cdots+y_m)+\cdots+\alpha(t_1+t_2+\cdots+t_m)\}$ corresponds to a set of imprimitive systems of a given imprimitive substitution group G_S on the variables $x_i, y_i, \cdots, t_i$ $(i=1, 2, \cdots, m)$. These equations express invariance (in the sense indicated) under G_S of the corresponding functional composition $\phi\{f(x_1, x_2, \cdots, x_m), f(y_1, y_2, \cdots, y_m), \cdots, f(t_1, t_2, \cdots, t_m)\}$. A set of functional equations of the preceding type corresponds to any abstract group G with subgroup H, since G can be represented as a (regular) substitution group G_S simply isomorphic to G and such that G_S is imprimitive with set of imprimitive systems corresponding to H. Functional equations are also constructed when the arbitrary function β has as arguments functions whose variables are not necessarily equal in number. (Received May 29, 1944.)

184. A. R. Schweitzer: *On functional equations representing abstract groups.*

Let $f(x_1, x_2, \cdots, x_{n+1}) = a_1x_1 + a_2x_2 + \cdots + a_{n+1}x_{n+1}$ and assume $a_i \cdot (a_1x_1 + a_2x_2 + \cdots + a_{n+1}x_{n+1}) = (a_ia_1)x_1 + (a_ia_2)x_2 + \cdots + (a_ia_{n+1})x_{n+1}$; then the equation $f(u_1, u_2, \cdots, u_{n+1}) = f(L_1, L_2, \cdots, L_{n+1})$, where $u_i = f(x_{i1}, x_{i2}, \cdots, x_{i\,n+1})$ and L_i are certain formal sums of the x's, represents a class of multiplication tables for the a's. Group representation by a single functional equation follows if the associated multiplication table defines an abstract group G (with a_1 as the identical element). In this case the formal sums $L_1, L_2, \cdots, L_{n+1}$ are, in effect, derived from L_1 by applying to the variables of L_1 substitutions of a regular group simply isomorphic to G. The preceding representation is of formal additive, that is, special type. Group representation of general type is obtained by subjecting the variables x_{ij} in the composition $f(u_1, u_2, \cdots, u_{n+1})$ to conditions of invariance of the latter suggested by the invariance of the formal sums serving as arguments of the function $f(L_1, L_2, \cdots, L_{n+1})$. An analogous representation of linear algebras is discussed. (Received May 29, 1944.)

185. A. R. Schweitzer: *Functional relations valid in the domains of abstract groups and Grassmann's space analysis.*

The function of group elements $f(x_1, x_2, \cdots, x_{n-1}) = x_1 \cdot x_2^{-1} \cdot x_3, \cdots, x_{n+1}^{\pm 1}$ when n is odd satisfies the author's postulates generalizing the group concept to $(n+1)$-ary composition (abstract 50-3-73). When n is even, the preceding function satisfies the postulates: 1. $f(u_1, u_2, \cdots, u_{n+1}) = f(u, t_1, t_2, \cdots, t_n)$ where $u_i = f(x_i, t_1, t_2, \cdots, t_n)$ and $u = f(x_1, x_2, \cdots, x_{n+1})$. 2. Given $f(x_1, x_2, \cdots, x_{n-1})$ there exists $\phi(x_1, x_2, \cdots, x_{n+1})$ such that $f\{\phi(x, t_1, t_2, \cdots, t_n), t_1, t_2, \cdots, t_n\} = x$ and $\phi\{f(x, t_1, t_2, \cdots, t_n), t_1, t_2, \cdots, t_n\} = x$. 3. $f(x, x, \cdots, x) = x$. 4. The set S of elements x_i is closed under the compositions f and ϕ. When $n=1$ the postulates are satisfied by $f(x_1, x_2) = x_2 \cdot x_1 \cdot x_2^{-1}$ and $f(x_1, x_2) = x_2 \cdot x_1^{-1} \cdot x_2$. A solution containing an arbitrary function is $f(x_1, x_2, \cdots, x_{n+1}) = x_1 \cdot x_{n+1}^{-1} \cdot \alpha(w_2, w_3, \cdots, w_n) \cdot x_{n+1}$ where $w_i = x_i \cdot x_{n+1}^{-1}$. In Grassmann's space the postulates are satisfied if $f(x_1, x_2, \cdots, x_{n+1}) = a_1x_1 + a_2x_2 + \cdots + a_{n+1}x_{n+1}$ where the x_i are points and the a_i belong to a field with sum equal to unity. In the above postulates the functions f and ϕ are dual to one another; if $f\{f(x, t_1, t_2, \cdots, t_n), t_1, t_2, \cdots, t_n\} = x$ then $f = \phi$. Also f is distributive over ϕ. For $n=1$ the restricted associative properties, $f\{x_1, f(x_2, x_1)\} = f\{f(x_1, x_2), x_1\}$ and $f\{x_1, \phi(x_2, x_1)\} = \phi\{f(x_1, x_2), x_1\}$ hold. (Received May 29, 1944.)

186. W. J. Thron: *A family of simple convergence regions for continued fractions.*

In this paper a new two parameter family of simple convergence regions for continued fractions (1) $1 + K(-c_n^2/1)$ is obtained. It is further shown that the regions cannot be improved except possibly by the removal of an epsilon. A simpler though less inclusive convergence criterion is provided by the following corollary of the main theorem: the continued fraction (1) converges if for all $n \geqq 1$ its elements $c_n = re^{i\theta}$ satisfy the conditions (a) $r \leqq (d - \epsilon + 1/2) \cos \alpha \sec (\theta - \alpha)$, $-\pi/2 + \alpha < \theta \leqq \beta_1$, (b) $r \leqq (d - 1/2) \cos \alpha \sec (\theta - \alpha)$, $0 \leqq \theta < \pi/2 + \alpha$, (c) $r = 0$ otherwise, (d) $\sum |b_n| = \infty$, where $b_1 = 1/c_1^2$, $b_n = 1/c_n^2 b_{n-1}$. Here $-\pi/2 < \alpha < \pi/2$, $0 \leqq d \leqq 1/2$, $0 < \epsilon < d/2$ and $\cot \beta_1 = -(\tan \alpha + ((d+1/2)/d)^{1/2} \sec \alpha)$. For $d = 0$ this result reduces to the well known parabola theorem, an isolated result of Paydon (abstract 47-11-473) is also included in this theorem. (Received June 1, 1944.)

APPLIED MATHEMATICS

187. A. N. Lowan and H. E. Salzer: *Formulas for complex interpolation.*

Whenever an analytic function is tabulated for arguments z_ν located at equidistant points along any straight line in the z-plane, the application of the Lagrange-Hermite interpolation formula, which approximates the function by a complex polynomial of degree n passing through any $n+1$ points, leads to the sum from $\nu=-[n/2]$ to $\nu=[(n+1)/2]$ of $f(z)\sim\sum A_\nu^{(n)}(p)f(z_\nu)$ where $A_\nu^{(n)}(P)$ are polynomials in $P=(z-z_0)/h$ and h is the complex tabular interval and where $[x]$ stands for the largest integer in x. Since in general z is not on the straight line containing the point z_ν, P is complex $(=p+iq)$. Expressions for the real and imaginary parts of $A_\nu^{(n)}(P)$ as functions of p and q were obtained for the cases of three, four, five and six point interpolation and arranged in a form especially suited for computational purposes. These expressions are particularly applicable to the cases of functions tabulated along rays through the origin, as for instance: the table of the Bessel functions $J_0(z)$ and $J_1(z)$ computed by the Mathematical Tables Project (Columbia Press, 1943), the forthcoming table of $Y_0(z)$ and $Y_1(z)$, H. T. Davis' table of $1/\Gamma(z)$ and Kennelly's tables of complex circular and hyperbolic functions for Cartesian and polar arguments. (Received April 12, 1944.)

188. H. E. Salzer: *Coefficients for mid-interval numerical integration with central differences.*

Coefficients in the formula for numerical integration from mid-interval to mid-interval using central differences were calculated to the extent where one can employ central differences up to the forty-ninth order. (Previous calculations have gone only as far as coefficients of the seventh central difference.) The coefficients K_{2s} occur in the following formula, which is also known as the first Gaussian summation-formula: $(1/h)\int_{a-h/2}^{a+(n-1/2)h} f(x)dx=[f(a)+f(a+h)+\cdots+f(a+(n-2)h)+f(a+(n-1)h)]+\sum_{s=1}^{m-1}K_{2s}[\delta^{2s-1}f(a+(n-1/2)h)-\delta^{2s-1}f(a-h/2)]+nh^{2m}K_{2m}f^{(2m)}(\xi)$. Due to the extreme rapidity with which the coefficients K_{2s} decrease with increasing s, the calculated table can be used with high accuracy in either integration or summation even though successive differences might not show the slightest tendency to decrease for the given interval. The quantities K_2 to K_{20} are given exactly and K_{22} to K_{50} are given to 18 decimals, accurate to within 0.6 units in the 18th place. These coefficients were checked by two cumulative recursion formulas, by differencing of the ratios K_{2s}/K_{2s+2}, and by a numerical example. (Received May 13, 1944.)

189. H. E. Salzer: *Table of coefficients for differences in terms of the derivatives.*

A table was prepared which lists the exact values of the coefficients $B_{m,s}$ for $m=1, 2, \cdots, 20$ and $s=m, \cdots, 20$ in the formula of Markoff which expresses the mth advancing difference in terms of the derivatives according to the equation $\Delta_h^m f(a)=\sum_{s=m}^{n-1}B_{m,s}h^sD^sf(a)+B_{m,n}h^nD^nf(\eta)$ where h denotes the tabular interval. The quantity $B_{m,s}$ in Milne-Thomson's notation is equal to the $(s-m)$th Bernoulli number of order $-m$, divided by $(s-m)!$ and in Jordan's notation $B_{m,s}$ is equal to $m!/s!$ multiplied by $\mathfrak{S}_s^m$ where $\mathfrak{S}_s^m$ is the Stirling number of the second kind. The coefficients $B_{m,s}$ were calculated by first obtaining the Stirling numbers $\mathfrak{S}_s^m$ (using their well known recursion relationships, and checking by a neat formula given in Jordan's *Calculus of finite differences*) and then multiplying by $m!/s!$. The numbers $B_{m,s}$ are expressed in lowest terms. (Received April 20, 1944.)

190. Seymour Sherman; J. DiPaola, and H. F. Frissel: *Routh's discrimant, flutter, and ground resonance.* Preliminary report.

Routh's criterion for the stability of the solutions of system of linear differential equations with constant coefficients is extended to cover cases arising in airplane flutter and helicopter ground resonance calculations. With this new tool, the stability of the flutter "polynomial" at a given reduced frequency for more than two degrees of freedom can be determined in one-fifth of the time hitherto required. (Received May 29, 1944.)

GEOMETRY

191. Edward Kasner and John DeCicco: *Isothermal families in general curvilinear coordinates, and loxodromes.*

If the square of the linear element of a surface Σ is given in isothermal coordinates (x, y) by $ds^2=E(x, y)(dx^2+dy^2)$, then the family of curves $g(x, y)=$const. on Σ is isothermal if and only if $(\partial^2/\partial x^2+\partial^2/\partial y^2)$ arc tan $g_y/g_x=0$. In the present paper, the authors obtain the necessary and sufficient condition that $g(x, y)=$const. represent an isothermal family when (x, y) are general curvilinear coordinates. This gives a large extension of Lie's theorem. The condition is simpler when the parametric curves form an orthogonal net. As an application, the condition is obtained that $g(x, y)=$const. represent an isothermal family upon the Cartesian surface $z=f(x, y)$. Finally the condition is found that the level curves of the surface be an isothermal family. This is applied to the mapping of loxodromes, showing that they can be represented by straight lines for a sphere (Mercator) and spheroid (Lambert), but not for an ellipsoid of three unequal axes. Use is made of Kasner's theorem in Math. Ann. (1904). (Received April 20, 1944.)

192. Abraham Seidenberg: *Valuation ideals in polynomial rings.*

A constructive study of the valuation ideals in a polynomial ring $\mathfrak{D}=K[x, y]$ in two indeterminates, where K is an algebraically closed (ground-) field, is made. Let $\mathfrak{q}_1, \mathfrak{q}_2, \cdots$ be the Jordan sequence of v-ideals belonging to a valuation B of Σ/K, where Σ is the quotient field of $\mathfrak{D}$, and let $\mathfrak{q}_{i_j}$ be the jth ideal such that $v(\mathfrak{q}_{i_j})$ is not in the additive group generated by $v(\mathfrak{q}_1), \cdots, v(\mathfrak{q}_{i_j-1})$. A tool corresponding to the Puiseux series expansion for a valuation, which is available if K is of characteristic 0 but not in general, is found in introducing certain polynomials f_{i_j} such that $v(f_{i_j})=v(\mathfrak{q}_{i_j})$. Considerations are reduced to valuations of rational rank 2. If B is of rational rank 2, place $v(\mathfrak{q}_1)=1$ and let τ be the least irrational value assumed by elements of $\mathfrak{D}$. The description of the v-ideals in $\mathfrak{D}$ for B is intimately connected with the approximants and quasiapproximants to a certain integral multiple of τ. In particular, a simple 0-dimensional v-ideal $\mathfrak{q}_i$ is characterized in terms of the values of $\mathfrak{q}_i$ and $\mathfrak{q}_{i+1}$. This characterization yields a proof that the transform of a simple v-ideal under a quadratic transformation is simple. If $\mathfrak{q}_i$ is not simple, an explicit factorization of $\mathfrak{q}_i$ in terms of the mentioned values is given. (Received May 22, 1944.)

193. A. H. WHEELER: *One-sided polyhedra from the five regular solids.*

If planes are passed through a polyhedron in such a way as to permit removing parts and leaving cells which are joined along their edges, and adjoining cells are bounded by one or more planes which are common to the two cells, then one-sided polyhedra can be derived. In this paper the method is applied to the five regular solids and several forms are shown constructed of plastic material. The intersecting planes may be so disposed as to cut away the vertices, the edges in whole or in part, and parts of the faces of the original solids. In particular, in the case of the regular icosahedron, a part of a regular dodecahedron of the second species may be removed from the interior of the solid leaving a group of cells distributed along the edges of the original solid, and the cells can be entered or traversed in more than one way. Two types of one-sided octahedra are shown. (Received May 26, 1944.)

194. Oscar Zariski: *The theorem of Bertini concerning the variable singular points of a linear system of varieties.*

If the ground field k of an algebraic irreducible r-dimensional variety V/k is extended by the adjunction of indeterminates $u_1, u_2, \cdots, u_m$, we denote by V/K the extended variety over $K=k(u)$ which has the same general point (ξ) as V/k. Each subvariety W/k of V/k has similarly a unique extension W/K on V/K, and each subvariety W^*/K of V/K has a unique contraction W/k on V/k. Given $m+1$ linearly independent polynomials $f_i(\xi)$, $i=0, 1, \cdots, m$, there is a unique $(r-1)$-dimensional irreducible subvariety F^*/K of V/K whose general point (η) satisfies the equation $f_0(\eta)+u_1f_1(\eta)+\cdots+u_mf_m(\eta)=0$. The main result is as follows: (1) W/k is a base variety of the linear system $|F|$ defined on V/k by $f_0+\lambda_1f_1+\cdots+\lambda_mf_m=0$, if and only if W/K is on F^*/K; (2) if W^*/K is a singular subvariety of F^*/K, then the contraction W/k is either singular for V/k or is a base variety of $|F|$. The theorem of Bertini *in its classical formulation* ("the variable singular points of $|F|$ are either singular for V/k, or lie on the base locus of $|F|$") is equivalent to (2), if k is of characteristic zero, and is false if k is of characteristic $p\neq 0$. (Received April 6, 1944.)

Topology

195. W. H. Gottschalk: *Orbit-closure decompositions and almost periodic properties.*

Let X be a metric space with metric ρ, let $f(X)\subset X$ be a continuous mapping, and let $h(X)=X$ be a homeomorphism. For $x\in X$, the set $\sum_{n=0}^{+\infty}f^n(x)$ is called the *semi-orbit* of x under f and the set $n\sum_{-\infty}^{+\infty}h^n(x)$ is called the *orbit* of x under h. The mapping f is said to be *pointwise almost periodic* provided that if $x\in X$, then to each $\epsilon>0$ there corresponds a positive integer N with the property that in every set of N consecutive positive integers appears an integer n so that $\rho(x, f^n(x))<\epsilon$. The mapping f is said to be *uniformly pointwise almost periodic* provided that to each $\epsilon>0$ there corresponds a positive integer N such that if $x\in X$, then in every set of N consecutive positive integers appears an integer n so that $\rho(x, f^n(x))<\epsilon$. The following theorems are proved: In order that the mapping f (homeomorphism h) give a semi-orbit-closure (orbit-closure) decomposition of X it is sufficient that f (h) be pointwise almost periodic; and in case X is locally compact (compact), this condition is also necessary. In order that the mapping f (homeomorphism h) give a continuous semi-orbit-closure (orbit-closure) decomposition of X it is sufficient that f (h) be uniformly pointwise almost periodic; and in case X is compact, this condition is also necessary. (Received May 15, 1944.)

196. Paul A. White: *On additive properties of compact sets.*

A property p^i concerned with i-dimensional Vietoris cycles of a compact set is called *additive* with respect to i if the following theorem is true. If the compact sets A and B have p^i and $A\cdot B$ has p^{i-1}, then $A\cdot B$ has p^i. It is shown that local i-connectedness and simple i-connectedness are two additive properties. Finally it is shown that the property $P(n, j)$ of being a compact, simply i-connected ($i=0, 1, \cdots, m-2$) lc^{n-2} embedded in E_n, having only simply-j-connected ($j\leqq n-2$) generalized closed manifolds as boundaries of its complementary domains, is additive with respect to n and j simultaneously. (Received April 10, 1944.)

197. G. T. Whyburn: *Interior mappings into the circle.*

In this paper it is shown that any continuous transformation of a locally connected cyclic continuum M into the circle S is homotopic to an interior mapping of M into S. If M is not assumed to be cyclic, it is shown that any mapping $f(M)=S$ is homotopic to a mapping $\phi r(M)=S$ where $r(M)=M'$ is the monotone retraction of M onto the smallest A-set M' in M containing all cyclic elements on which f is essential and where $\phi(M')=S$ is interior. (Received April 14, 1944.)

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THE EXTREMALS OF TWO INVARIANT INTEGRALS

M. L. MacQUEEN

1. Introduction. In a recent paper, Wilkins shows [1, p. 175],[1] among other things, that at a point of an analytic surface in ordinary projective space a general canonical line of the first kind may be characterized in terms of the cusp-axes of the two families of hypergeodesics which are the extremals of a certain pair of invariant integrals.

In this note we introduce two invariant integrals which do not differ essentially from those employed by Wilkins. At each point of the surface, a quadric cone called the *cusp-axis cone* and its dual called the *flex-ray conic* are defined in terms of the cusp-axes and flex-rays, respectively, of the hypergeodesics which are the extremals of these integrals. The equations of these loci are derived and some of their properties are briefly studied.

If the four homogeneous projective coordinates x of a variable point on an analytic non-ruled surface S in ordinary space are given as analytic functions of two independent variables u, v, and if the parametric net on S is the asymptotic net, then the functions x are solutions of a system of differential equations which may be assumed to be reduced to Fubini's canonical form

$$x_{uu} = px + \theta_u x_u + \beta x_v, \quad x_{vv} = qx + \gamma x_u + \theta_v x_v \quad (\theta = \log \beta\gamma). \tag{1.1}$$

It will be recalled that two lines $l_1(a, b)$, $l_2(a, b)$ are reciprocal lines [2, p. 150] at a point x of a surface if the line $l_1(a, b)$ joins the point x and the point y defined by

$$y = -ax_u - bx_v + x_{uv} \tag{1.2}$$

and the line $l_2(a, b)$ joins the points ρ, σ defined by placing

$$\rho = x_u - bx, \qquad \sigma = x_v - ax, \tag{1.3}$$

where a, b are scalar functions of u, v.

Moreover, two reciprocal lines $l_1(a, b)$, $l_2(a, b)$ are canonical lines of the first and second kind respectively in case

$$a = -k\psi, \qquad b = -k\phi, \tag{1.4}$$

where k is a constant and

$$\phi = (\log \beta\gamma^2)_u, \qquad \psi = (\log \beta^2\gamma)_v.$$

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[1] Numbers in brackets refer to the references listed at the end of the paper.

It will be assumed that $\phi\psi\neq 0$, so that the surface under consideration is not a coincidence surface. Canonical lines of the first kind lie in the canonical plane whose local equation is

$$\phi x_2 - \psi x_3 = 0, \tag{1.5}$$

when referred to the tetrahedron of reference x, x_u, x_v, x_{uv}. The canonical plane intersects the tangent plane $x_4=0$ in the first canonical tangent t_1 at the point x of the surface. Canonical lines of the second kind lie in the tangent plane and pass through the canonical point

$$(0, \psi, -\phi, 0). \tag{1.6}$$

The second canonical tangent t_2, which joins the point x to the canonical point, has the equations

$$\phi x_2 + \psi x_3 = 0, \qquad x_4 = 0. \tag{1.7}$$

2. **The extremals of two invariant integrals.** The differential equation of the extremals of the invariant integral

$$\int_{u_0}^{u} \beta^{(2-n)/3}\gamma^{(1+n)/3}v'^{n}du \qquad (n \neq 0), \tag{2.1}$$

where n is a constant, is found to be

$$v'' = B_1 v' + C_1 v'^2, \tag{2.2}$$

in which accents indicate total differentiation with respect to u and the coefficients B_1 and C_1 are defined by

$$3(1-n)B_1 = (\log \beta^{2-n}\gamma^{1+n})_u, \quad -3nC_1 = (\log \beta^{2-n}\gamma^{1+n})_v. \tag{2.3}$$

These extremals are hypergeodesics whose cusp-axis is the line $l_1(a_1, b_1)$ for which

$$\begin{aligned} a_1 &= ((2n-1)/6n)\psi = -k_1\psi, \\ b_1 &= ((2n-1)/6(n-1))\phi = -k_2\phi. \end{aligned} \tag{2.4}$$

Similarly, the differential equation of the extremals of the invariant integral

$$\int_{u_0}^{u} \beta^{(1+n)/3}\gamma^{(2-n)/3}v'^{1-n}du \qquad (n \neq 1) \tag{2.5}$$

is given by

$$v'' = B_2 v' + C_2 v'^2, \tag{2.6}$$

where

$$3nB_2 = (\log \beta^{1+n}\gamma^{2-n})_u, \qquad 3(n-1)C_2 = (\log \beta^{1+n}\gamma^{2-n})_v. \tag{2.7}$$

Therefore these extremals are hypergeodesics whose cusp-axis is the line $l_1(a_2, b_2)$ for which

$$a_2 = ((2n - 1)/6(n - 1))\psi = - k_2\psi, \qquad b_2 = ((2n - 1)/6n)\phi = - k_1\phi. \tag{2.8}$$

These results can be used to obtain the following theorem, a part of which is similar to Theorem 4.1 in the paper of Wilkins [1, p. 175].

THEOREM 2.1. *The extremals of the invariant integrals* (2.1), (2.5) *are hypergeodesics whose cusp-axes determine the plane*

$$\phi x_2 + \psi x_3 - ((1 - 2n)^2/6n(1 - n))\phi\psi x_4 = 0. \tag{2.9}$$

This plane intersects the tangent plane $x_4 = 0$ *in the second canonical tangent and intersects the canonical plane in the canonical line* $l_1(k)$ *for which*

$$k = (1 - 2n)^2/12n(1 - n). \tag{2.10}$$

Moreover, the plane which is determined by the cusp-axis of the hypergeodesics (2.2) *and the v-tangent at the point x intersects the plane which is determined by the cusp-axis of the hypergeodesics* (2.6) *and the u-tangent at the point x in the canonical line* $l_1(k)$ *for which*

$$k = (1 - 2n)/6n. \tag{2.11}$$

Finally, the cusp-axis of the hypergeodesics (2.2) *and the u-tangent at the point x determine a plane which intersects the plane determined by the cusp-axis of the hypergeodesics* (2.6) *and the v-tangent in the canonical line* $l_1(k)$ *for which*

$$k = (1 - 2n)/6(n - 1). \tag{2.12}$$

Each of the canonical lines thus determined may, by a proper selection of the constant n, be made to become any desired line of the first canonical pencil, except the first axis of Čech.

It may be remarked that, in case $n = 2$, the two cusp-axes are the scroll directrices of Sullivan. Application of the theorem leads to the canonical line $l_1(k)$ for which $k = -3/8$, the first edge of Green, and the first directrix of Wilczynski.

3. **The cusp-axis cone.** We now propose to find the locus of the cusp-axis of the extremals of each of the invariant integrals (2.1), (2.5), when n varies. For this purpose the cusp-axis of the extremals

of the integral (2.1) can be regarded as determined by the planes whose equations are

$$x_3 - k_2\phi x_4 = 0, \qquad x_2 - k_1\psi x_4 = 0 \tag{3.1}$$

and the cusp-axis of the extremals of the integral (2.5) by the planes

$$x_3 - k_1\phi x_4 = 0, \qquad x_2 - k_2\psi x_4 = 0, \tag{3.2}$$

where k_1 and k_2 are defined by

$$k_1 = (1 - 2n)/6n, \qquad k_2 = (1 - 2n)/6(n - 1). \tag{3.3}$$

The locus of the cusp-axis (3.1), when n varies, is found by eliminating n from equations (3.1) to be a quadric cone which will be called the *cusp-axis cone* at the point x. The same cone is obtained as the locus of the cusp-axis (3.2). This result can be stated in the following way:

THEOREM 3.1. *At a point of a surface, the locus of the cusp-axes of the extremals of the invariant integrals* (2.1), (2.5), *when n varies, is the cusp-axis cone whose equation is*

$$6x_2x_3 + \phi x_2x_4 + \psi x_3x_4 = 0. \tag{3.4}$$

The vertex of this cone is, of course, the point x. It is evident that this cone is intersected by the tangent plane at the point x of the surface in the asymptotic tangents through the point. Moreover, the polar line of the tangent plane $x_4 = 0$ with respect to this cone is the canonical line $l_1(k)$ for which $k = -1/6$, namely, the first principal line of Fubini and Čech. The canonical plane intersects the cone (3.4) in the projective normal and in the first axis of Čech. The polar plane of the second canonical tangent with respect to the cone is the canonical plane.

It is easy to establish the truth of the following statement:

THEOREM 3.2. *The cusp-axes of the extremals* (2.2), (2.6) *determine the plane* (2.9) *the polar line of which with respect to the cusp-axis cone is a canonical line $l_1(k)$ for which*

$$k = -(1 - 2n)^2/6(2n^2 - 2n + 1). \tag{3.5}$$

In particular, let us consider the plane determined by the two scroll directrices of Sullivan. The polar line of this plane with respect to the cusp-axis cone is the canonical line $l_1(k)$ for which $k = -3/10$.

The cusp-axis cone and any quadric of Darboux

$$x_2x_3 - x_1x_4 + k_4x_4^2 = 0 \qquad (k_4 \text{ arbitrary}) \tag{3.6}$$

intersect, besides in the asymptotic tangents at the point x, also in a residual conic which lies in the plane whose equation is

$$x_1 + \phi x_2/6 + \psi x_3/6 - k_4 x_4 = 0. \tag{3.7}$$

This plane cuts the tangent plane at the point x of the surface in the canonical line $l_2(k)$ for which $k = -1/6$, namely, the reciprocal of the first principal line of Fubini and Čech.

Furthermore, the cusp-axis cone and a principal quadric of Lane [3]

$$x_2 x_3 + x_4(-3x_1 - \phi x_2/2 - \psi x_3/2 + k_4 x_4) = 0 \quad (k_4 \text{ arbitrary}) \tag{3.8}$$

intersect, besides in the asymptotic tangents, also in a conic which lies in the plane

$$x_1 + 2\phi x_2/9 + 2\psi x_3/9 - k_4 x_4/3 = 0. \tag{3.9}$$

This plane intersects the tangent plane $x_4 = 0$ in the canonical line $l_2(k)$ for which $k = -2/9$.

4. The flex-ray conic. Let us consider the envelope of the flex-rays of the extremals of the integrals (2.1), (2.5). The flex-ray of the family of hypergeodesics (2.2) is the reciprocal of the cusp-axis (3.1) and has the equations $x_4 = 0$ and

$$x_1 + ((2n - 1)/6(n - 1))\phi x_2 + ((2n - 1)/6n)\psi x_3 = 0. \tag{4.1}$$

We find that the envelope of this line when n varies is a conic. The same conic is obtained as the envelope of the flex-rays of the hypergeodesics (2.6). This conic will be called the *flex-ray conic* of the two families of hypergeodesics (2.2), (2.6). The equations of the flex-ray conic can easily be obtained by setting equal to zero the discriminant of equation (4.1) regarded as a quadratic equation in n. The result is contained in the following theorem:

THEOREM 4.1. *At a point of a surface, the envelope of the flex-rays of the extremals of the two invariant integrals* (2.1), (2.5), *when n varies, is the flex-ray conic whose equations are $x_4 = 0$ and*

$$x_1(x_1 + \phi x_2/3 + \psi x_3/3) + (\phi x_2 - \psi x_3)^2/36 = 0. \tag{4.2}$$

This conic lies in the tangent plane of the surface at the point x and touches the parametric tangents at the point x in the points

$$(-\phi/6, 1, 0, 0), \qquad (-\psi/6, 0, 1, 0). \tag{4.3}$$

The line joining these two points is the reciprocal of the first principal line of Fubini and Čech. This line may therefore be characterized by the property that it is the polar line of the point x with respect to the flex-ray conic.

The first canonical tangent intersects the flex-ray conic in the points whose local coordinates are

$$(-2\phi\psi/3, \psi, \phi, 0), \qquad (0, \psi, \phi, 0). \tag{4.4}$$

The second axis of Čech is tangent to the flex-ray conic at the first of the points (4.4), and the reciprocal of the projective normal is tangent to the conic at the second of the points (4.4). Thus we find that the canonical point is the pole of the first canonical tangent with respect to the flex-ray conic.

It is easy to verify the following statement:

THEOREM 4.2. *The extremals of the two invariant integrals* (2.1), (2.5) *are hypergeodesics whose flex-rays pass through the point*

$$((1-2n)^2/6n(1-n))\phi\psi, \psi, \phi, 0 \tag{4.5}$$

which lies on the first canonical tangent. The harmonic conjugate of the first canonical tangent with respect to the two flex-rays through the point (4.5) *is the canonical line* $l_2(k)$ *for which* k *is defined by* (2.10).

The following result is an immediate consequence of the fact that the two flex-rays which pass through the point (4.5) are tangent to the flex-ray conic.

THEOREM 4.3. *The polar line of the point* (4.5) *with respect to the flex-ray conic is the canonical line* $l_2(k)$ *for which* k *is defined by* (3.5).

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PROOF OF A CONJECTURE OF P. ERDÖS ON THE DERIVATIVE OF A POLYNOMIAL

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Introduction. We start out from the following consequence of S. Bernstein's well known theorem on trigonometric polynomials. Let $p_n(z)$ be a polynomial of degree n for which $|p_n(z)| \leqq 1$ holds as $|z| \leqq 1$; then $|p_n'(z)| \leqq n$ as $|z| \leqq 1$ with $|p_n'(z)| = n$ if and only if $p_n(z) = az^n$, $|a| = 1$.

Some time ago P. Erdös conjectured that *if* $|p_n(z)| \leqq 1$ *holds as* $|z| \leqq 1$ *and* $p_n(z)$ *has no roots inside the unit circle, then* $|p_n'(z)| \leqq n/2$ as $|z| \leqq 1$. In the present note we give a proof of this conjecture.

Preliminaries. Let us introduce the following notation which shall be used throughout this paper:

$$p_n(z) = c\prod_{\nu=1}^{n}(z - z_\nu), \qquad c \neq 0;$$

$$q_n(z) = \bar{c}\prod_{\nu=1}^{n}(1 - z\bar{z}_\nu) = z^n\bar{p}_n(z^{-1}).$$

Then for $|z| = 1$ we have $|p_n(z)| = |q_n(z)|$.

LEMMA I. *If* $p_n(z)$ *has no roots inside the unit circle, that is* $|z_\nu| \geqq 1$, *the polynomial* $p_n(z) + \epsilon q_n(z)$, $|\epsilon| = 1$, *will have all its roots on the unit circle.*[1]

LEMMA II. *If* $p_n(z)$ *has no roots inside the unit circle,* $|z_\nu| \geqq 1$, *we have* $|p_n'(z)| \leqq |q_n'(z)|$ *as* $|z| = 1$.

Let $z \neq z_\nu$; using the abbreviation $z^{-1}z_\nu = A_\nu$ we find

$$|p_n'(z)/p_n(z)| = \left|\sum_{\nu=1}^{n}(z - z_\nu)^{-1}\right| = \left|\sum_{\nu=1}^{n}(1 - A_\nu)^{-1}\right|,$$

$$|q_n'(z)/q_n(z)| = \left|\sum_{\nu=1}^{n}(z - \bar{z}_\nu^{-1})^{-1}\right| = \left|\sum_{\nu=1}^{n}A_\nu(1 - A_\nu)^{-1}\right|.$$

Since $|A_\nu| \geqq 1$, $A_\nu \neq 1$, we obtain

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[1] G. Pólya and G. Szegö, *Aufgaben und Lehrsätze aus der Analysis*, Berlin, 1925, vol. 1, p. 88, Problem 26.

$$(1)\quad \begin{gathered} \Re(1-A_\nu)^{-1} \leqq 1/2, \qquad \Re\sum(1-A_\nu)^{-1} \leqq n/2, \\ \left|\Re\sum(1-A_\nu)^{-1}\right| \leqq \left|\Re\sum(1-A_\nu)^{-1} - n\right| = \left|\Re\sum A_\nu(1-A_\nu)^{-1}\right|. \end{gathered}$$

On the other hand

$$(2)\qquad \left|\Im\sum(1-A_\nu)^{-1}\right| = \left|\Im\sum(1-A_\nu)^{-1}A_\nu\right|.$$

(1) and (2) prove Lemma II.

Proof of the special case, when all the roots are on the unit circle. The first proof for this special case was supplied by Professor Szegö. His argument is as follows.

Let $|z_\nu|=1$ and $\max|p_n'(z)| = |p_n'(z_0)|$, $|z_0|=1$.

Case I. $p_n(z_0)\neq 0$. Since $p_n'(z_0e^{i\theta})\bar{p}_n'(\bar{z}_0e^{-i\theta})$ is maximum for $\theta=0$, we find that $z_0p_n''(z_0)/p_n'(z_0)$ is real. Now

$$zp_n'(z)/p_n(z) = \sum_{\nu=1}^{n} z/(z-z_\nu),$$

$$z^2p_n''(z)/p_n(z) = \left(\sum_{\nu=1}^{n} z/(z-z_\nu)\right)^2 - \sum_{\nu=1}^{n}(z/(z-z_\nu))^2,$$

$$zp_n''/p_n'(z) = \sum_{\nu=1}^{n} z/(z-z_\nu) - \sum_{\nu=1}^{n}(z/(z-z_\nu))^2 \Big/ \sum_{\nu=1}^{n} z/(z-z_\nu).$$

Let $z_0/(z_0-z_\nu) = 1/2+it_\nu$; then

$$z_0\frac{p_n''(z_0)}{p_n'(z_0)} = n/2 + i\sum t_\nu - \frac{\sum(1/4-t_\nu^2)+i\sum t_\nu}{n/2+i\sum t_\nu}$$

so that

$$(3)\qquad \sum t_\nu = \frac{(n/2)\sum t_\nu - \sum t_\nu\sum(1/4-t_\nu^2)}{(n/2)^2+(\sum t_\nu)^2}$$

and

$$(4)\qquad z_0\frac{p_n''(z_0)}{p_n'(z_0)} = n/2 - \frac{(n/2)\sum(1/4-t_\nu^2)+(\sum t_\nu)^2}{(n/2)^3+(\sum t_\nu)^2}.$$

Case Ia. $\sum t_\nu = 0$,

$$z_0p_n'(z_0)/p_n(z_0) = n/2 + i\sum t_\nu = n/2, \quad |p_n'(z_0)| = (n/2)|p_n(z_0)| \leqq n/2.$$

Case Ib. $\sum t_\nu \neq 0$ so that from (3),

$$(n/2)^2 + (\sum t_\nu)^2 = n/2 - \sum(1/4-t_\nu^2).$$

Hence from (4),

$$z_0 \frac{p_n''(z_0)}{p_n'(z_0)} = n/2 - \frac{(n/2)^2 - (n/2)^3 - (n/2)(\sum t_\nu)^2 + (\sum t_\nu)^2}{(n/2)^2 + (\sum t_\nu)^2}$$

$$= n/2 + n/2 - 1 = n - 1;$$

$$|p_n'(z_0)| = (n-1)\max|p_n'(z)|.$$

According to S. Bernstein's theorem this is possible only when $|p_n'(z)| = \text{const}\, z^{n-1}$ or $p_n(z) = \text{const}\,(z^n + c)$, $|c| = 1$.

Case II. $p_n(z_0) = 0$. Since $p_n'(z_0) \neq 0$, z_0 is a simple root of $p_n(z)$, say $z_0 = z_\mu$. Now

$$z_0 \frac{p_n'(z_0)}{p_n'(z_0)} = \lim_{z \to z_\mu} z \frac{p_n''(z)}{p_n'(z)} = \lim_{z \to z_\mu} \frac{(\sum z/(z - z_\nu))^2 - \sum (z/(z - z_\nu))^2}{\sum z/(z - z_\nu)}$$

$$= 2 \sum_{\nu \neq \mu} z_\mu/(z_\mu - z_\nu) = n - 1 + 2i \sum_{\nu \neq \mu} t_\nu$$

which is real, consequently equal to $n-1$; see case Ib.

Second proof of the previous special case. A much simpler proof is due to Professor G. Pólya. Let $c = |c|e^{i\gamma}$, $z_\nu = e^{i\theta_\nu}$; then

$$h(\theta) = i^n e^{-i\gamma} \prod_{\nu=1}^{n} (e^{-i\theta_\nu/2}) \cdot e^{-in\theta} p_n(e^{2i\theta})$$

$$= |c| \prod_{\nu=1}^{n} [ie^{i(\theta - \theta_\nu/2)} - ie^{-i(\theta - \theta_\nu/2)}]$$

is a trigonometric polynomial of degree n with real coefficients. Hence according to a sharper form of S. Bernstein's theorem[2]

$$(5) \qquad \begin{aligned} &(h'(\theta))^2 + n^2(h(\theta))^2 \leqq n^2, \qquad |h'(\theta) + inh(\theta)| \leqq n, \\ &|dp_n(e^{2i\theta})/d\theta| \leqq n, \qquad 2|p_n'(z)| \leqq n, \qquad |z| = 1. \end{aligned}$$

Proof in the general case. Let $p_n(z)$ be a polynomial of degree n, $|p_n(z)| \leqq 1$ for $|z| \leqq 1$, $p_n(z) \neq 0$ for $|z| < 1$. We define $Q_n(z) = (p_n(z) + \epsilon q_n(z))/2$, and determine ϵ so that $|\epsilon| = 1$, $\epsilon q_n'(z)/p_n'(z)$ should be real and positive at $z = z_0$, $|z_0| = 1$, where $|p_n'(z)|$ reaches its maximum.

Then $|Q_n(z)| \leqq 1$, $|z| \leqq 1$.

Also, by Lemma I, $Q_n(z)$ has all its roots on the unit circle.

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[2] Inequality (5) can be found in J. G. van der Corput and G. Schaake, *Ungleichungen für Polynome und trigonometrische Polynome*, Compositio Math. vol. 2 (1935) p. 337; it follows from an older and sharper theorem of G. Szegö, *Über einen Satz des Herrn Serge Bernstein*, Schriften der Königsberger Gelehrten Gesellschaft vol. 22 (1928) p. 69.

We have, by Lemma II,

$$\begin{aligned}\max_{|z|=1} |Q_n'(z)| \geqq |Q_n'(z_0)| &= |p_n'(z_0) + \epsilon q_n'(z_0)|/2 \\ &= |q_n'(z_0)|/2 + |p_n'(z_0)|/2 \\ &\geqq |p_n'(z_0)| = \max_{|z|=1} |p_n'(z)|.\end{aligned}$$

But $\max |Q_n'(z)| \leqq n/2$, therefore the same follows for $p_n(z)$. This establishes the conjecture of P. Erdös. The bound is obviously the best possible, as the example $(1+z^n)/2$ shows.

Remark. Returning to the case when all the roots z_ν of $p_n(z)$ are on the unit circle, it is interesting to observe that in this particular case $\max |p_n'(z)| = (n/2) \max |p_n(z)| = n/2$, $|z| \leqq 1$. Indeed, let us write $|p_n(z')| = \max |p_n(z)|$, $|z| \leqq 1$; then $\max |p_n'(z)| \geqq |p_n'(z')| = |p_n(z')| \, |\sum_{\nu=1}^{n}(z-z_\nu)^{-1}| \geqq |p_n(z')| \, |\sum_{\nu=1}^{n}\Re(1-\bar{z}'z')^{-1}| = (n/2)|p_n(z')| = (n/2) \max |p_n(z)| = n/2$.

Another inequality for the derivative of a polynomial. Replacing the "Tchebychev deviation" by the "Bessel deviation," we obtain the following analog of the inequality formulated at the beginning of this paper.

Let $p_n(z)$ be a polynomial of degree n; then[3]

$$(6) \qquad \left\{(2\pi)^{-1}\int_{-\pi}^{\pi} |p_n'(e^{i\theta})|^2 d\theta\right\}^{1/2} \leqq n\left\{(2\pi)^{-1}\int_{-\pi}^{\pi} |p_n(e^{i\theta})|^2 d\theta\right\}^{1/2}.$$

Setting $p_n(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ and using Parseval's formula, this inequality can be written in the form

$$\sum_{\nu=1}^{n} |a_\nu|^2 \nu^2 \leqq n^2 \sum_{\nu=0}^{n} |a_\nu|^2,$$

which is obvious.

We prove now the following analog of P. Erdös's conjecture.

If $|p_n(z)| \leqq 1$ holds as $|z| \leqq 1$, and $p_n(z) \neq 0$ for $|z| < 1$, then in (6) *the factor n can be replaced by $2^{-1/2}n$. This factor is the best possible.*

By Lemma II the following inequality holds:

$$(7) \qquad (2\pi)^{-1}\int_{-\pi}^{\pi} |p_n'(e^{i\theta})|^2 d\theta \leqq (2\pi)^{-1}\int_{-\pi}^{\pi} |q_n'(e^{i\theta})|^2 d\theta.$$

Now, $q_n(z) = \sum_{\nu=0}^{n} \bar{a}_{n-\nu} z^\nu$, hence by (7)

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[3] See A. Zygmund, *A remark on conjugate functions*, Proc. London Math. Soc. (2) vol. 34 (1932) pp. 392–400, esp. pp. 394–396.

$$\pi^{-1}\int_{-\pi}^{\pi}|p_n'(e^{i\theta})|^2d\theta \leqq (2\pi)^{-1}\int_{-\pi}^{\pi}|q_n'(e^{i\theta})|^2d\theta + (2\pi)^{-1}\int_{-\pi}^{\pi}|p_n'(e^{i\theta})|^2d\theta$$

$$= \sum_{\nu=0}^{n}(\nu^2+(n-\nu)^2)\,|\alpha_\nu|^2.$$

The greatest of the numbers $\nu^2+(n-\nu)^2$, $\nu=1, \cdots, n$, is n^2. Therefore

$$\pi^{-1}\int_{-\pi}^{\pi}|p_n'(e^{i\theta})|^2d\theta \leqq n^2\sum_{\nu=0}^{n}|\alpha_\nu|^2, \tag{8}$$

which was the assertion.

For $p_n(z)=(z^n+1)/2$ the sign of equality holds in (8).

NEW YORK CITY

A NEW FORMULA FOR INVERSE INTERPOLATION

HERBERT E. SALZER

This paper is devoted to the derivation of a formula for inverse interpolation in a table of equally spaced arguments. The resulting formula (5) is more concise and convenient than those in existence. It involves neither differences nor polynomial coefficients other than small powers. In use it will be found much simpler and quicker than those given by Davis, Aitken, Steffensen and Milne-Thomson. In a sense, it is the analogue of the Lagrangian formula for direct interpolation without differences (that is, in terms of the tabular entries only) if the usual expression (right member of (1) below) is rearranged in terms of powers of the argument p.

Lagrange's general interpolation formula is

$$f(x) = \sum_{\nu=0}^{\nu=k}\frac{P_\nu(x)}{P_\nu(a_\nu)}f(a_\nu), \quad \text{where} \quad P_\nu(x) = \frac{1}{(x-a_\nu)}\prod_{i=0}^{i=k}(x-a_i).$$

For equally spaced arguments at interval h, after suitable relabelling of the arguments a_i, Lagrange's formula becomes

$$f_p = \sum_{i=-[(n-1)/2]}^{i=[n/2]} L_i^{(n)}(p)f_i, \tag{1}$$

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where f_x denotes $f(a_0+xh)$, $[x]$ denotes the largest integer in x, and $L_i^{(n)}(p)$ are well known polynomials in p of the $(n-1)$th degree. Making use of the relation

$$\sum_{i=-[(n-1)/2]}^{i=[n/2]} L_i^{(n)}(p) \equiv 1, \tag{2}$$

we can rewrite (1) as

$$\frac{p}{f_p-f_0} = 1 \Big/ \sum_{i=-[(n-1)/2]}^{i=[n/2]} \frac{L_i^{(n)}(p)}{p}(f_i-f_0). \tag{3}$$

Since $L_0^{(n)}(p)/p$ is not present in the summation, the denominator of the right-hand side of (3) is still a polynomial in p.

Now we employ an expansion due to Bürmann which states that if $\psi(z)=(z-a)/(\phi(z)-b)$, then

$$f(z) = f(a) + \sum_{m=1}^{k-1} \frac{[\phi(z)-b]^m}{m!}\,\frac{d^{m-1}}{da^{m-1}}\left[f'(a)\{\psi(a)\}^m\right] + R_k.$$

Consider the equation

$$\frac{p-a}{f_p-f_0} = 1 \Big/ \sum_{i=-[(n-1)/2]}^{i=[n/2]} \frac{L_i^{(n)}(p)}{p}(f_i-f_0). \tag{3'}$$

Letting

$$z = p,\quad \phi(p) = f_p,\quad b = f_0,\quad \psi(p) = 1 \Big/ \sum_{i=-[(n-1)/2]}^{i=[n/2]} \frac{L_i^{(n)}(p)}{p}(f_i-f_0)$$

and $f(p)=p$, we obtain, by Bürmann's expansion,

$$\begin{aligned} p &= a + \sum_{m=1}^{k-1} \frac{(f_p-f_0)^m}{m!} \\ &\qquad \frac{d^{m-1}}{da^{m-1}}\left(1 \Big/ \sum_{i=-[(n-1)/2]}^{i=[n/2]} \frac{L_i^{(n)}(a)}{a}(f_i-f_0)\right)^m + R_k'. \end{aligned} \tag{4'}$$

To obtain p as an explicit function of f_p and the f_i's according to its implicit definition in (1) or (3), we take the limit of (4′) as $a\to 0$ and arrive at the following expansion:

$$\begin{aligned} p &= \sum_{m=1}^{k-1} \frac{(f_p-f_0)^m}{m!} \\ &\qquad \left[\frac{d^{m-1}}{da^{m-1}}\left(1 \Big/ \sum_{i=-[(n-1)/2]}^{i=[n/2]} \frac{L_i^{(n)}(a)}{a}(f_i-f_0)\right)^m\right]_{a=0} + R_k''. \end{aligned} \tag{4}$$

Corresponding to the n-point formulas for direct interpolation, n ranging from 3 to 7, we define quantities r, s, t, u, v and w thus:

3-point

$$r = \frac{2(f_p - f_0)}{f_1 - f_{-1}},$$

$$s = \frac{f_1 - 2f_0 + f_{-1}}{f_1 - f_{-1}},$$

$$t = u = v = w = 0.$$

4-point

$$r = \frac{6(f_p - f_0)}{-f_2 + 6f_1 - 3f_0 - 2f_{-1}},$$

$$s = \frac{3(f_1 - 2f_0 + f_{-1})}{-f_2 + 6f_1 - 3f_0 - 2f_{-1}},$$

$$t = \frac{f_2 - 3f_1 + 3f_0 - f_{-1}}{-f_2 + 6f_1 - 3f_0 - 2f_{-1}},$$

$$u = v = w = 0.$$

5-point

$$r = \frac{24(f_p - f_0)}{-2f_2 + 16f_1 - 16f_{-1} + 2f_{-2}},$$

$$s = \frac{-f_2 + 16f_1 - 30f_0 + 16f_{-1} - f_{-2}}{-2f_2 + 16f_1 - 16f_{-1} + 2f_{-2}},$$

$$t = \frac{2(f_2 - 2f_1 + 2f_{-1} - f_{-2})}{-2f_2 + 16f_1 - 16f_{-1} + 2f_{-2}},$$

$$u = \frac{f_2 - 4f_1 + 6f_0 - 4f_{-1} + f_{-2}}{-2f_2 + 16f_1 - 16f_{-1} + 2f_{-2}},$$

$$v = w = 0.$$

6-point

$$r = \frac{120(f_p - f_0)}{4f_3 - 30f_2 + 120f_1 - 40f_0 - 60f_{-1} + 6f_{-2}},$$

$$s = \frac{5(-f_2 + 16f_1 - 30f_0 + 16f_{-1} - f_{-2})}{4f_3 - 30f_2 + 120f_1 - 40f_0 - 60f_{-1} + 6f_{-2}},$$

$$t = \frac{5(-f_3 + 7f_2 - 14f_1 + 10f_0 - f_{-1} - f_{-2})}{4f_3 - 30f_2 + 120f_1 - 40f_0 - 60f_{-1} + 6f_{-2}},$$

$$u = \frac{5(f_2 - 4f_1 + 6f_0 - 4f_{-1} + f_{-2})}{4f_3 - 30f_2 + 120f_1 - 40f_0 - 60f_{-1} + 6f_{-2}},$$

$$v = \frac{f_3 - 5f_2 + 10f_1 - 10f_0 + 5f_{-1} - f_{-2}}{4f_3 - 30f_2 + 120f_1 - 40f_0 - 60f_{-1} + 6f_{-2}},$$

$$w = 0.$$

7-point

$$r = \frac{720(f_p - f_0)}{12f_3 - 108f_2 + 540f_1 - 540f_{-1} + 108f_{-2} - 12f_{-3}},$$

$$s = \frac{2(2f_3 - 27f_2 + 270f_1 - 490f_0 + 270f_{-1} - 27f_{-2} + 2f_{-3})}{12f_3 - 108f_2 + 540f_1 - 540f_{-1} + 108f_{-2} - 12f_{-3}},$$

$$t = \frac{15(-f_3 + 8f_2 - 13f_1 + 13f_{-1} - 8f_{-2} + f_{-3})}{12f_3 - 108f_2 + 540f_1 - 540f_{-1} + 108f_{-2} - 12f_{-3}},$$

$$u = \frac{5(-f_3 + 12f_2 - 39f_1 + 56f_0 - 39f_{-1} + 12f_{-2} - f_{-3})}{12f_3 - 108f_2 + 540f_1 - 540f_{-1} + 108f_{-2} - 12f_{-3}},$$

$$v = \frac{3(f_3 - 4f_2 + 5f_1 - 5f_{-1} + 4f_{-2} - f_{-3})}{12f_3 - 108f_2 + 540f_1 - 540f_{-1} + 108f_{-2} - 12f_{-3}},$$

$$w = \frac{f_3 - 6f_2 + 15f_1 - 20f_0 + 15f_{-1} - 6f_{-2} + f_{-3}}{12f_3 - 108f_2 + 540f_1 - 540f_{-1} + 108f_{-2} - 12f_{-3}}.$$

Then in every case (4) assumes the form

$$
\begin{aligned}
p = r - r^2 s + r^3(2s^2 - t) + r^4(-5s^3 + 5st - u) \\
+ r^5(14s^4 - 21s^2 t + 3t^2 + 6su - v) \\
+ r^6(-42s^5 + 84s^3 t - 28st^2 - 28s^2 u \\
+ 7tu + 7sv - w) + \cdots .
\end{aligned} \tag{5}
$$

When (5) is employed for interpolation for a number of values within the same interval, the only variable is r^k.

From (5) which gives $p(r)$, it is immediately apparent that for numerical integration of the inverse function we obtain

$$
\int_{f_0}^{f_p} x(f_t)d(f_t) = (f_p - f_0)(a_0 + (h/r) \int p(r)dr), \tag{6}
$$

while for the mth derivative of the inverse function we have

$$
x^{(m)}(f_p) = (hr^m/(f_p - f_0)^m)p^{(m)}(r). \tag{7}
$$

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ALMOST ORTHOGONAL SERIES

RICHARD BELLMAN

1. **Almost orthogonal series.** Let us consider an infinite sequence $\{\phi_n(x)\}$, $n=1, 2, \cdots$, of complex-valued functions of the real variable x, of class $L^2(a, b)$, normalized so that $\int_a^b|\phi_n(x)|^2dx=1$ for all n. Assume further that the sequence satisfies the following condition

$$\sum_{m,n}|a_{mn}|^2<\infty, \tag{1}$$

where $a_{mn}=\int_a^b\phi_m\bar{\phi}_n dx$ $(m\neq n;\ n,\ m=1,\ 2,\ \cdots)$, $a_{mn}=0$, $m=n$.

We wish to show that under the above conditions we have a Bessel inequality and an analogue of the Riesz-Fisher theorem.

THEOREM 1 (BESSEL'S INEQUALITY). *Under the above conditions, let $f(x)$ be a real-valued function belonging to $L^2(a, b)$, and $b_n=\int_a^b f\bar{\phi}_n dx$, then*

$$\sum_1^\infty|b_k|^2\leqq\int_a^b|f|^2dx\left[1+\left(\sum_{m,n}|a_{mn}|^2\right)^{1/2}\right].$$

We have

$$\sum_1^n|b_k|^2=\int_a^b f\left[\sum_1^n \bar{b}_k\bar{\phi}_k\right]dx.$$

Using Schwartz's inequality, this becomes

$$\begin{aligned}\sum_1^n|b_k|^2&\leqq\left[\int_a^b|f|^2dx\right]^{1/2}\left[\int_a^b\left[\sum_1^n b_k\phi_k\right]\left[\sum_1^n \bar{b}_k\bar{\phi}_k\right]dx\right]^{1/2}\\&\leqq\left[\int_a^b|f|^2dx\right]^{1/2}\left[\sum_1^n|b_k|^2+\sum_{1,1,k\neq l}^{n,n}b_kb_la_{kl}\right]^{1/2}\\&\leqq\left[\int_a^b|f|^2dx\right]^{1/2}\left[\sum_1^n|b_k|^2\right.\\&\qquad\left.+\left\{\sum_{1,1}^{n,n}|b_k|^2|b_l|^2\right\}^{1/2}\left\{\sum_{1,1}^{n,n}|a_{kl}|^2\right\}^{1/2}\right]^{1/2}\\&\leqq\left[\int_a^b|f|^2dx\right]^{1/2}\left[\sum_1^n|b_k|^2\right]^{1/2}\\&\qquad\cdot\left[1+\left(\sum_{m,n}|a_{kl}|^2\right)^{1/2}\right]^{1/2}.\end{aligned}$$

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Hence, simplifying,

$$\sum_1^n |b_k|^2 \leqq \left(\int_a^b |f|^2 dx\right)\left[1 + \left(\sum_{m,n} |a_{kl}|^2\right)^{1/2}\right],$$

and since this is true for all n, the result is obtained.

THEOREM 2 (ANALOGUE OF RIESZ-FISHER). *Under the above conditions, if* $\sum_1^\infty |b_n|^2 < \infty$, *there exists a function* $f(x) \subset L^2(a, b)$ *such that*

$$\sum_1^\infty \left| b_k - \int_a^b f\bar{\phi}_k dx \right|^2 \leqq \left(\sum_1^\infty |b_k|^2\right)\left(\sum_{k,l} |a_{kl}|^2\right),$$

and therefore $\lim_{n\to\infty} (b_n - \int_a^b f\bar{\phi}_n dx) = 0$.

Let $s_n = \sum_1^n b_k\phi_k$, then

$$\begin{aligned}\int_a^b & |s_m - s_n|^2 dx \\ &= \int_a^b \left[\sum_{n+1}^m b_k\phi_k\right]\left[\sum_{n+1}^m \bar{b}_k\bar{\phi}_k\right] dx \\ &= \sum_{n+1}^m |b_k|^2 + \sum_{n+1 \leqq k,\, l \leqq m} b_k\bar{b}_l a_{kl} \\ &\leqq \sum_{n+1}^m |b_k|^2 + \left(\sum_{n+1}^m |b_k|^2 |b_l|^2\right)^{1/2}\left(\sum_{m,n} |a_{mn}|^2\right)^{1/2} \\ &\leqq \sum_{n+1}^m |b_k|^2 \left(1 + \left(\sum_{m,n} |a_{mn}|^2\right)^{1/2}\right)\end{aligned}$$

(using Schwartz's inequality). Hence, since $\sum b_n^2 < \infty$, s_n converges in mean to a function $f(x) \subset L^2(a, b)$, that is,

$$\lim_{n\to\infty} \int_a^b |f(x) - s_n(x)|^2 dx = 0.$$

We have, $k < n$,

$$b_k - \int_a^b f\bar{\phi}_k dx = b_k - \int_a^b s_n\bar{\phi}_k dx + \int_a^b (s_n - f)\bar{\phi}_k dx,$$

$$\begin{aligned}\int_a^b |s_n - f||\phi_k| dx &\leqq \left(\int_a^b |f - s_n|^2 dx\right)^{1/2}\left(\int_a^b |\phi_k|^2 dx\right)^{1/2} \\ &= \left(\int_a^b |f - s_n|^2 dx\right)^{1/2},\end{aligned}$$

$$b_k - \int_a^b s_n\bar{\phi}_k dx = \sum_{j=1}^{n} b_j a_{kj},$$

$$\left| b_k - \int_a^b s_n\bar{\phi}_k dx \right| \leqq \sum_1^n | b_j a_{kj} |$$

$$\leqq \left(\sum_1^n | b_j |^2 \right)^{1/2} \left(\sum_{j=1}^n | a_{kj} |^2 \right)^{1/2}.$$

Hence,

$$\left| b_k - \int_a^b f\bar{\phi}_k dx \right| \leqq \left(\sum_1^n | b_j |^2 \right)^{1/2} \left(\sum_{j=1}^n | a_{kj} |^2 \right)^{1/2} + \left(\int_a^b | f - s_n |^2 dx \right)^{1/2}.$$

Let $n \to \infty$, this becomes

$$\left| b_k - \int_a^b f\bar{\phi}_k dx \right| \leqq \left(\sum_1^\infty | b_j |^2 \right)^{1/2} \left(\sum_{j=1}^\infty | a_{kj} |^2 \right)^{1/2}.$$

Squaring both sides, and summing over k,

$$\sum_k \left| b_k - \int_a^b f\bar{\phi}_k dx \right|^2 \leqq \left(\sum_1^\infty | b_j |^2 \right) \left(\sum_{k,j} | a_{kj} |^2 \right),$$

which is the result in question.

2. **An "almost" moment problem.** Let us consider the sequence of functions $\{ e^{i\lambda_n t}/(b-a)^{1/2} \}$, the λ_n being real and distinct, over a finite interval (a, b). Then we have the following theorem.

THEOREM 3. *If* $\sum_{k \neq l} 1/(\lambda_k - \lambda_l)^2 < \infty$, *and* $\sum_n | b_n |^2 < \infty$, *there exists a function* $f(t) \subset L^2(a, b)$, *such that*

$$\sum_n \left| b_n - \frac{1}{(b-a)^{1/2}} \int_a^b f(t) e^{-i\lambda_n t} dt \right|^2 < \infty.$$

We have

$$\left| \int_a^b e^{i\lambda_l t} e^{-i\lambda_k t} dt \right| \leqq \frac{2}{| \lambda_k - \lambda_l |}, \qquad k \neq l.$$

Therefore, in view of the hypothesis, this is a corollary of Theorem 2.

PRINCETON UNIVERSITY

A KNOWN PROBLEM OF GEOMETRY AND ITS CASES OF INDETERMINATION

J. HADAMARD

A recent note[1] induced me to resume the classic problem (P_0) and its natural generalization:

(P_0) *To construct a square, the sides of which go through four given points.*

(P) *To construct a quadrilateral* $Q(XYZT)$ *similar*[2] *to a given quadrilateral* q *with its sides* XY, YZ, $\cdots$, *respectively passing through four given points* K, L, M, N.

The solution of (P_0) is classic: we immediately get a locus for each vertex, namely the circumference having KL, LM and so on for its diameter. Moreover, the diagonal XZ of the required square must again cut the half circumference of diameter NK at its middle point α and, similarly, the half circumference of diameter LM at its middle point γ. This generally determines it, whence the required construction directly results.

The solution extends automatically to (P):

(i) We know a circular locus for each vertex of the unknown quadrilateral.

(ii) We also know the two points α, γ where the diagonal XZ intersects the loci of the vertices X, Z.

This generally allows us to draw the required quadrilateral and:

(iii) The quadrilateral Q constructed in that way is actually similar to q and, therefore, *satisfies all the conditions of the problem.*

The verification of the latter fact offers no difficulty; but it must be insisted on, for it is quite essential for what we are going to say. Indeed, instead of reasoning on the diagonal XZ, we could operate in the same way on the other diagonal YT, of which, similarly, we know two points β, δ, respectively belonging to the circumferences which are the loci of the vertices Y and T; and it results from (iii) that this second construction necessarily gives the same result as the first one.

But, moreover, a singular case may occur, namely that in which

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[1] Mathematicae Notae. Boletin del Instituto Matematica vol. 3 (1943) p. 155.

[2] In the above-mentioned note, it is investigated whether the given points are on the sides themselves or on their prolongations, a question which we shall not examine.

On the other hand, for us, every similitude is meant to be a direct one. This is the reason why, from our point of view, the problem generally admits of *one* solution, while in the article in Mathematicae Notae it is considered as having two of them.

the problem is ***indeterminate***:[3] this happens when α and γ coincide. Now, if so, the same circumstance must appear when operating on the diagonal YT so that:

(iv) *If α coincides with γ, then β also coincides with δ.*

It is that remarkable circumstance which has seemed to me worth studying and, more generally, induced me to consider the relations between the points α, β, γ, δ.

I. In the special case of problem (P_0), everything can be settled by the properties and, especially, the rules of composition of rotations. Let us choose as unit angle the right angle in the direct sense (the retrograde right angle being denoted by -1). The triangles αNK, and so on, being right-angled and isoceles, we have (the perimeter of our square being assumed to be described in the direct sense)

$$(1)\quad (a)\ K = \alpha_1 N,\quad (b)\ L = \beta_1 K,\quad (c)\ M = \gamma_1 L,\quad (d)\ N = \delta_1 M,$$

this meaning that K results from N by a direct rotation of a right angle around α, and so on. Therefore, we also have

$$(2)\qquad L = \beta_1\alpha_1 N,$$

where the rotation $\omega_2 = \beta_1\alpha_1$ is through the angle 180°, so that its center ω is the midpoint of LN. Moreover, the half-turn $\delta_1\gamma_1$ changes N into L and, therefore, is nothing else than $\beta_1\alpha_1$.

Now, the rules of composition of rotations show that ω is the vertex of a right-angled isosceles triangle, the hypotenuse of which is $\alpha\beta$, and similarly for $\gamma\delta$. We are here taking into account the senses of the angles,

$$(3)\qquad \beta = \omega_{-1}\alpha,\qquad \delta = \omega_{-1}\gamma.$$

This gives the answers to our questions:

(i) As the retrograde rotation ω_{-1}, which is through a right angle, changes α into β and γ into δ, we see that *$\beta\delta$ is perpendicular to $\alpha\gamma$ and equal to $\alpha\gamma$.*

(ii) The first part of this conclusion is evident on account of the fact that our problem is possible, as $\alpha\beta$ and $\gamma\delta$ are segments of the two diagonals of the square. But the question of senses deserves consideration. $\beta\delta$ is deduced from $\alpha\gamma$ by a rotation of -1, while bd is deduced from ac by a rotation of $+1$. Therefore, if we inquire whether $\alpha\gamma$ is of the same sense as ac, and whether $\beta\delta$ is of the same sense as bd,

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[3] A single case of that kind has been mentioned, as far as we know: in the Nouvelles Annales de Mathématiques (see footnote 4), Laisant notices that indetermination takes place for a square circumscribed to a square.

the two questions necessarily admit of contrary answers. If the diagram is such that K, L, M, N lie on the sides of the square themselves[2] (and not on their continuations) and α, β, γ, δ on the diagonals themselves, we can say that if the segments $a\alpha$, $c\gamma$ overlap, the segments $b\beta$, $d\delta$ do not, and conversely.

(iii) The equality $\beta\delta=\alpha\gamma$ shows us why δ must coincide with β if γ coincides with α.

(iv) *The condition for that is that the two lines LN, KM be equal and perpendicular to each other* (in other words, that the parallelogram whose vertices are the midpoints of the sides KL, LM, and so on, be a square), as the rotation α_1 must change N into K and L into M.

Similarly, both products $\gamma_1\beta_1$ and $\alpha_1\delta_1$ give a rotation of a half-turn around the midpoint π of KM, so that

$$\gamma = \pi_{-1}\beta, \qquad \alpha = \pi_{-1}\delta. \tag{3a}$$

(v) The comparison of (3a) with (3) shows that the half-turn $\pi_{-1}\omega_{-1}$ takes place around the midpoint C of $\alpha\gamma$ and the half-turn $\omega_{-1}\pi_{-1}$ around the midpoint D of $\beta\delta$.

There are, therefore, two right-angled isosceles triangles having CD for their common hypotenuse and ω, π for their vertices, located on both sides of CD. *The midpoints of KM, LN and the midpoints, C, D of $\alpha\gamma$, $\beta\delta$ are the vertices of a square.*

II. In order to treat the case where the given quadrilateral is no longer a square, but has an arbitrary shape, we must find the laws of composition not only of rotations, but also, more generally, of similitudes in the plane.[4]

When we shall speak of similitudes, it will be implicitly understood, if not otherwise specified, that we mean direct ones, in which not only the magnitudes but also the senses of the angles shall be preserved. In such a similitude, any line MN in the second figure F is in a constant ratio r and makes a constant angle ϕ with the corresponding one M_0N_0 in the first figure F_0; this can be expressed algebraically by saying that the two complex numbers which represent MN and M_0N_0 are in the constant complex ratio $\rho=re^{i\phi}$.

If that ratio is equal to 1 (that is $r=1$ and $\phi=0$), the similitude reduces to a translation. In any other case, there is one and only one point α—which we shall call the center of similitude—such that the lines joining it to any point M_0 in F_0 and to its homologue M in F are represented by imaginaries the ratio of which is ρ. Thus, the triangle $\alpha M_0 M$ will be similar to a fixed triangle oa_0a. Such transformations can be denoted by $\alpha_{(\phi,r)}$ (this differs from our former notation for rotations only by the introduction of the homothety ratio r, the angle ϕ being still measured with the direct right angle as unit) or by $\alpha_{(oa_0a)}$, the relation

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[4] A study of plane similitudes has been developed by Giusto Bellavitis, under the name of *Theory of equipollences.* It has been expounded in French in the Nouvelles Annales de Mathématiques vol. 8 (1869) (by Houël) and in a complete translation of Bellavitis' memoir in the same periodical, vol. 12 (1873), by Laisant. But the location of the new center in a product of two similitudes is not examined by Bellavitis.

$$M = \alpha_{(oa_0a)}M_0$$

meaning that the triangle $\alpha M_0 M$ is directly similar to oa_0a; oa_0a can be called the characteristic triangle of the transformation.

Now we have to represent the product[4] of two such similitudes S_0, S_1 with centers α, β and complex ratios $\rho_0=r_0e^{i\phi_0}$, $\rho_1=r_1e^{i\phi_1}$.

If the two centers coincide at α, the answer is simple: the product-similitude will have again α as its center, with a rotation angle $\phi_0+\phi_1$ and a similitude ratio r_0r_1. If we draw the two characteristic triangles so that they have one side oa in common, the resultant characteristic triangle will consist of the two remaining sides oa_0, oa_1, and the connecting line a_0a_1.

The latter relation subsists in any case; but when the two centers do not coincide, we must find the center of the product-similitude S_1S_0.

We shall refer this general case to the first one by considering, as an intermediary, the composition of an arbitrary similitude $S_0=\alpha_{(oa_0a)}$ with a translation. The latter will not alter the rotation angle ϕ_0 nor the similitude ratio r_0, but there will be a new center α' such that, M_0 being any point of the original figure, M its transform by the given similitude, and M' the transform of M by the translation, the triangle $\alpha' M_0 M'$ will be similar to the characteristic triangle oa_0a of S_0. Now, we immediately see that the segment $\alpha_0\alpha'$ connecting this new center with the original one α can be deduced from M_0M', that is from the given translation, by a similitude the center of which is M_0 and the characteristic triangle the same as above, only with a different order of the vertices. Conversely, if a first similitude is given, another similitude with the same rotation angle and the same homothety ratio, but another center, can be considered as the product of the first one and a translation, the direction and magnitude of which are in the relation just obtained to the direction and magnitude of the line connecting the two centers.

We master the general case by simply combining the two above results. In order to obtain the product of the similitude $S_0=\alpha_{(oa_0a)}$ followed by the similitude $S_1=\beta_{(oaa_1)}$ (the two characteristic triangles being located so as to have the common side oa), we begin by introducing the intermediary operation S' equipollent with S_1 (that is, having the same characteristic triangle) and homocentric with S_0. The product $S'S_0$ is $\alpha_{(oa_0a_1)}$.

S_1 is deduced from S' and, therefore, S_1S_0 from $S'S_0$ by composition with a translation T, the direction and magnitude of which are determined by submitting $\alpha\beta$ to a rotation whose angle is equal, in magnitude and sign, to $\angle oaa_1$, and changing its size in the ratio aa_1/ao.

Now, let ω be the center of the product-similitude. The translation T is the same one which corresponds to changing $S'S_0$ into the equipollent operation S_1S_0. Therefore, the imaginary quantity which represents the segment $\alpha\omega$ will be given by the relation

$$(\alpha\omega) = \lambda(T),$$

where the complex coefficient λ is

$$\lambda = (a_0o)/(a_0a_1) \tag{4}$$

(every segment within parentheses denoting the corresponding complex quantity). In a purely geometrical and real language, the segment $\alpha\omega$ will have the magnitude $|\lambda|\,|T|$, with

$$|\lambda| = \overline{a_0o}/\overline{a_1a_1}, \tag{4a}$$

and the direction making with T the angle

$$\psi = \angle oa_0a_1. \tag{4b}$$

Now, the form of the quantity (4) (taking into account the expression of T, a complex cross ratio) or, which is equivalent, the formulae (4a), (4b) suggest a rather unexpected, but unavoidable, connection with inversion. Let $\mathbf{a}_0$, $\mathbf{a}$, $\mathbf{a}_1$ be the inverse points of a_0, a, a_1 with respect to o as pole: *the triangle $\alpha\omega\beta$ will be inversely similar to $\mathbf{a}_1\mathbf{a}\mathbf{a}_0$.* It can be deduced from a triangle constructed on $\alpha\beta$ and directly similar to $\mathbf{a}_0\mathbf{a}\mathbf{a}_1$ by a symmetry with respect to the axis of $\alpha\beta$.

III. The case of a lozenge can be investigated, with the exception of one single result, with the help of rotations only. Let 2θ and $2\theta' = 2 - 2\theta$ be the angles of the lozenge, still measured in right angles; the points K, L, M, N will be deduced from N, K, L, M by successive rotations around α, β, γ, δ, the angles of which will be alternately 2θ and $2\theta'$. Therefore, formulae (1a)–(1d) are to be replaced by

$$K = \alpha_{2\theta}N, \qquad L = \beta_{2\theta'}K, \qquad M = \gamma_{2\theta}L, \qquad N = \delta_{2\theta'}M.$$

As the two angles of the lozenge are supplementary, we shall have

$$L = \beta_{2\theta'}\alpha_{2\theta}N = \omega_2N, \qquad N = \delta_{2\theta'}\gamma_{2\theta}L = \omega_2L,$$

ω being again the midpoint of LN and, therefore,

$$\beta = \omega_{(-1,r)}\alpha, \qquad \delta = \omega_{(-1,r)}\gamma,$$

so that the segments $\alpha\gamma$, $\beta\delta$ must have perpendicular directions (with the same circumstance as above concerning senses) and a ratio between them equal to $r = \tan\theta = \overline{xs}/\overline{yt}$.

The condition for indetermination is that the lines KM, LN be equal to each other and intersect at an angle equal to one of the angles of the lozenge.

The only essential difference from the above case is that the triangles $\omega\alpha\beta$, $\omega\gamma\delta$, $\pi\beta\gamma$, $\pi\delta\alpha$ are no longer isosceles. We must introduce the similitude Σ obtained by the rotation -1 around ω, combined with a homothety of ratio r, with respect to the same point as well as the analogous operation Σ', one of which changes α into β and γ into δ, the other changing β into γ and δ into α. It results therefrom that the midpoints C, D of $\alpha\gamma$, $\beta\delta$ and the midpoints ω, π of KM, LN are the vertices of a quadrilateral (a so-called "rhomboid") with two right angles, which has CD as an axis of symmetry, the latter dividing it into two right-angled triangles similar to those into which the lozenge is divided by both its diagonals, while each of the triangles $C\omega\pi$, $D\omega\pi$ is similar to one of the halves into which the lozenge is divided by a single diagonal.

IV. Let us now take the general case of an arbitrary quadrilateral q. The successive similitudes which transform N into K, and so on, have their centers at α, β, γ, δ; their characteristic triangles can be obtained in the following way. Let[5] the diagonals xs, yt of q intersect at o: if, from o, we draw ok, ol, om, on perpendicular to the sides of q, onk, okl, and so on, will be the characteristic triangles.

We see that if these four similitudes were homocentric, their product would be the identical transformation. If they had different centers, the product would, as a rule, become a translation. But, in our case, there is at least one point which finally remains unchanged, namely N: therefore, this translation must be zero and *the product of the four similitudes*

$$S = \alpha_{(onk)}, \qquad S' = \beta_{(okl)}, \qquad S'' = \gamma_{(olm)}, \qquad S''' = \delta_{(omn)}$$

must again be the identical transformation.

The operation $S'''S''$ is the same as the operation $S^{-1}S'^{-1}$: it changes L into N and its characteristic triangle is oln. We can find its center, and even in two different ways, by our rule of composition. This introduces the inverse points $\mathbf{k}$, $\mathbf{l}$, $\mathbf{m}$, $\mathbf{n}$ of k, l, m, n with respect to a circle having its center at o: in other words, the poles of the sides of q with respect to that circle. The connecting lines $\mathbf{kl}$, $\mathbf{lm}$, and so on, being the polar lines of the vertices x, y, z, t, *our four inverse points are the vertices of a parallelogram.*[6]

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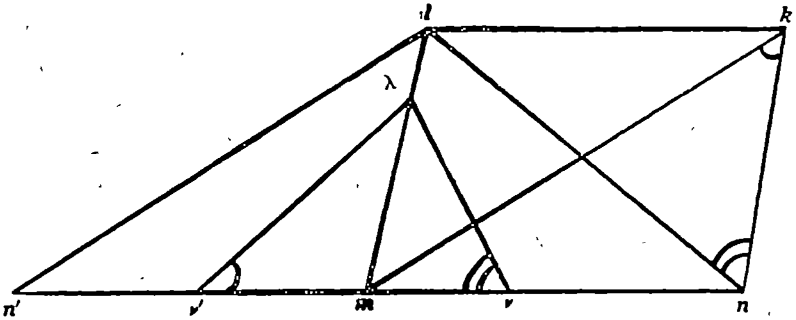
[5] An exceptional case, which ought to be examined separately, would occur if the diagonals of q were parallel, a circumstance which is possible if q, instead of being a proper quadrilateral, were a self-crossing four-sided closed line.

The triangle $\mathbf{kln}$ is equal to $\mathbf{mnl}$; the two triangles $\omega\alpha\beta$, $\omega\gamma\delta$ are directly similar to each other and inversely similar to the triangle $\mathbf{kln}$. Therefore, the segments $\alpha\gamma$, $\beta\delta$ are in the ratio $\mathbf{kl}/\mathbf{kn}$, which is equal to $\mathbf{xz}/\mathbf{yt}$ (denoting by $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$, $\mathbf{t}$ the inverse points of x, y, z, t) $=(xz/yt)\cdot(oy\cdot ot/ox\cdot oz)$.

As to the angle between these segments, it is, as is evident a priori, the angle between the diagonals of q. We can see, at least when q is assumed to be convex,[7] that the question of senses is to be answered as previously: for, in that case, the angle $\angle\mathbf{nkl}$ is supplementary to $\angle x\dot{o}y$.

The product SS''' is the same as $S'^{-1}S''^{-1}$: its characteristic triangle is $\mathbf{okm}$; its center π is such that each of the triangles $\pi\beta\gamma$, $\pi\delta\alpha$ is inversely similar to $\mathbf{nkm}$.



The product of the similitude $\Sigma=\omega_{(\mathbf{mnl})}$ followed by $\Sigma'=\pi_{(\mathbf{nkm})}$ transforms α into γ, and the product of the same factors with reversal of the order changes β into δ. Each of them is a half-turn, as is seen by assembling the triangle $\mathbf{mnl}$ with a triangle $\mathbf{mln}'$ equal to $\mathbf{nkm}$, so that their centers will be the midpoints C, D of $\alpha\gamma$, $\beta\delta$.

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[6] In the case of a lozenge, $\mathbf{klmn}$ would be a rectangle, which is decomposed by both its diagonals into four triangles, similar two by two to one-half of the lozenge.

[7] This condition is not satisfied in the problem which is the object of the article in Mathematicae Notae, in which q is formed by four consecutive sides of a regular polygon or their prolongations: such a quadrilateral is, of course, concave, except for a square or a regular pentagon. At least, matters are such if the sides are taken in their original order, which is not necessary (see footnote 9).

The relations of these centers with ω, π are given by our rule of composition; having transferred the triangle nkm into a position mln' adjacent to mln along ml (see the figure), the triangle $C\omega\pi$ must be inversely similar to the triangle $\lambda\nu\nu'$ the vertices of which are the inverse points of l, n, n' with respect to m, that is, as is easily seen, to the triangle formed by kn and the two diagonals of the parallelogram $klmn$. We shall see, in the same way, that the midpoint D of $\beta\delta$ forms with ω, π a triangle inversely similar to the one the sides of which are kl and the two diagonals, so that the points C, D, ω, π are on the same circle and divide it harmonically. The point I of intersection of the diagonals XZ, YT also belongs to that circumference.[8]

When q is itself a parallelogram, the condition for the coincidence of α with γ can be easily found and proves to be the same as that which implies the coincidence of β with δ; for in this case the similitude S'' will be the same as S, whence it immediately follows that the segments KM, LN must intersect at the same angle and be in the same ratio as the sides of the given parallelogram; in other words, the midpoints of KL, LM, and so on, must be the vertices of a parallelogram similar to q.

In the general case the question is more complicated. The fact that $klmn$ is a parallelogram is again essentially needed for its solution. Let okl, omn be two triangles assumed, for simplicity's sake, to be entirely exterior to each other except for the common vertex o. Let us consider the replacement of the second triangle omn by any (directly) similar one om_1n_1 having still a vertex at o, the other triangle okl remaining fixed. Among the transformations of this kind, there is precisely one such that the inverse points of the four vertices with respect to the pole o, taken in their due order, form a parallelogram; when this condition is satisfied, we can say that we have the reduced form of the diagram.

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[8] That I, C, D, ω, π are concyclic can be seen in a more direct way: for any circle through ω, I divides $\alpha\gamma$ and $\beta\delta$ similarly—an elementary consequence of the direct similitude of $\omega\alpha\beta$, $\omega\gamma\delta$; and so does any circle through π and I, but with an exchange between the roles of β and δ. It is clear that both correspondences agree when and only when the points of division are the midpoints.

In the case alluded to in footnote 5, the triangles onk, okl, olm, omn can be obtained by considering any segment ox parallel to the common direction of the diagonals of q and drawing from x parallels to the four sides of the polygonal line; ok, ol, om, on will be perpendiculars dropped from o on the lines thus drawn. The inverse points k, l, m, n will all lie on one straight line perpendicular to ox, so that ω will be the intersection between $\alpha\beta$ and $\gamma\delta$, π the intersection of $\beta\gamma$ and $\delta\alpha$. Then $CD\omega\pi$ will be a straight line, divided harmonically by the two diagonals. As to the ratio $\beta\delta/\alpha\gamma = kl/kn$, it is easy to see that it will be equal to xz/yt.

In the contrary case, let the original diagram ***oklmn*** be arbitrary. Applying a transformation of the aforesaid kind which changes m,n into m_1, n_1, let us consider the two intermediate triangles olm, onk, which are changed into olm_1, on_1k. On the basis lm_1, we can construct a triangle lm_1o_1 directly similar to lmo and, on the basis n_1k, the triangle n_1ko_1' directly similar to nko; in general, the two new vertices o_1, o_1' will be different. They will coincide if the original diagram is a reduced one.[9]

New York City

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[9] The problem would require further investigation because there is no reason not to consider the *complete* quadrilateral formed by the sides of q and, therefore, we ought to apply the present considerations to three diagonals instead of two.

NOTE ON CONVEX SPHERICAL CURVES

L. A. SANTALÓ

1. Introduction. The formula

$$L = \int_0^\pi \alpha d\tau \tag{1}$$

for plane convex curves in which L is the length and α the breadth according to the direction τ is well known [2, p. 65].[1]

The principal object of the present note is to obtain the formula (8) which generalizes (1) to convex curves on the sphere of unit radius and to deduce from this some consequences.

2. Principal formula. Let us consider the sphere of unit radius. A closed curve on the sphere is said to be *convex* when it cannot be cut by a great circle in more than two points. It is well known that a convex curve divides the surface of the sphere into two parts, one of which is always wholly contained in a hemisphere; that is, there is always a great circle which has the whole convex curve on the same side. When we say the area of a convex curve K we understand the area of that part of the surface of the sphere which is bounded by K and is smaller than or equal to a hemisphere.

Let K be a convex curve on the sphere of unit radius of length L and area F ($L \leq 2\pi$, $F \leq 2\pi$). The great circles which have only one common point or include a complete segment common with the curve

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[1] Numbers in brackets refer to the references cited at the end of the paper.

K are called "great circles of support" of K. In each point of K for which there exists a tangent great circle the great circle of support coincides with this.

Let τ be the angle between a fixed and a variable great circle of support and s the arc of K. If ρ_g is the radius of geodesic curvature of K the following formulas are known [4, p. 712],

$$\rho_g = ds/d\tau, \qquad \int_L ds/\rho_g = \int_K d\tau = 2\pi - F. \tag{2}$$

Let C be a great circle which cuts K. We suppose C "oriented" and to each orientation we make correspond one of the points P_c of the two in which the diameter perpendicular to C cuts the sphere. If we consider all the great circles which cut K (each one of these counted twice to correspond with the two orientations) the area covered by the points P_c has the value $2L$ [4, p. 709]; that is to say, if dP_c represents the element of area of the sphere of unit radius corresponding to the point P_c,

$$\int_{C\cdot K\neq 0} dP_c = 2L. \tag{3}$$

The convex curve K being fixed, the oriented great circle C or, what is the same, the point P_c, can be determined by the point A of K in which the great circle orthogonal to a circle of support of K is also orthogonal to C, and by the distance a from A to C (Fig. 1). If τ is

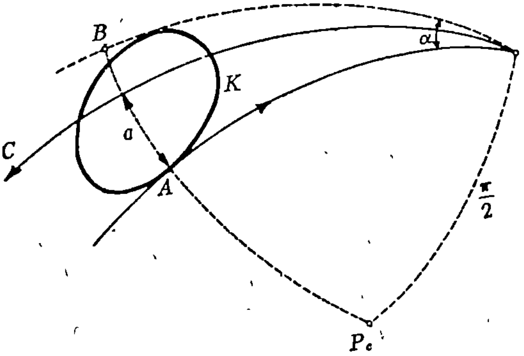

FIG. 1

the angle which determines the great circle of support of K in the point A, the coordinates to determine P_c will be a, τ. We must then express the element of area P_c as a function of the coordinates a, τ.

Suppose first that a remains fixed and τ passes to $\tau+d\tau$. The point P_o will describe an arc $P_oP'_o=d\sigma$, the value of which is (Fig. 2)

$$d\sigma = \sin(\pi/2 - a + \phi)d\psi = \cos(a-\phi)d\psi, \tag{4}$$

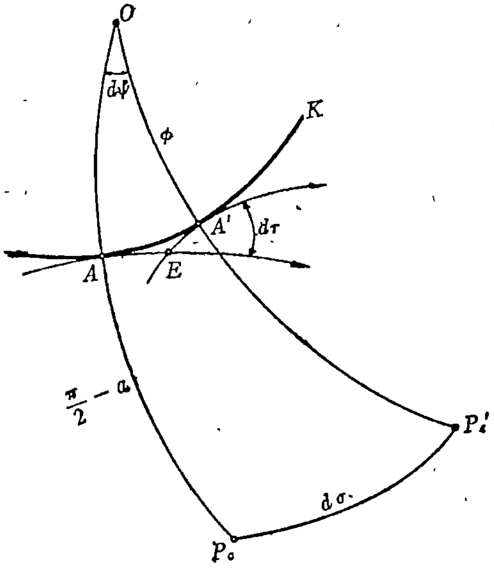

FIG. 2

$\phi=AO$ being the radius of spherical curvature of K at the point A. As a function of the radius of geodesic curvature it is

$$\tan\phi = \rho_g = ds/d\tau. \tag{5}$$

The element $d\psi=AOA'$ is the angle between the great circles orthogonal to K in A and A'. The area of the quadrilateral $AOA'E$ has the value $(\pi-d\tau)+d\psi+\pi-2\pi=d\psi-d\tau$. But save for infinitesimals of secondary order the same area is also equal to the area of the circular sector AOA' which has the value $(1-\cos\phi)d\psi$. In consequence

$$d\tau = \cos\phi d\psi. \tag{6}$$

If τ remains fixed and a passes to $a+da$, the point P_o describes an arc da upon the great circle P_oA. Therefore, taking into account (4) and (6) the element of area dP_o will be expressed

$$dP_o = d\sigma da = (\cos(a-\phi)/\cos\phi)da d\tau,$$

that is to say, in accordance with (5),

$$dP_c = \cos a da d\tau + \sin a da ds. \tag{7}$$

Let us substitute this expression in (3). For each value of s (or τ) the arc a can vary from 0 to the breadth α of K corresponding to the point s (or direction τ). Therefore

$$\int_{C\cdot K\neq 0} dP_c = \int_K d\tau \int_0^\alpha \cos a da + \int_L ds \int_0^\alpha \sin a da,$$

or, in accordance with (3),

$$L = \int_K \sin \alpha d\tau - \int_L \cos \alpha ds. \tag{8}$$

This is the principal formula we wished to obtain; it is valid for any convex curve K on the sphere of unit radius. The breadth α corresponding to a point A of K is equal to the arc AB (Fig. 1) of the great circle orthogonal to K at the point A comprehended between A and the next point of intersection with another great circle of support of K also orthogonal to AB.

3. **Dual formula.** By duality [4, p. 710] to the breadth α corresponds the arc $\pi-h$, h being the arc AE (Fig. 3) of the great circle

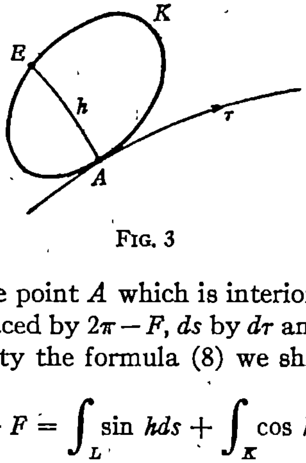

FIG. 3

orthogonal to K at the point A which is interior to K. Moreover the length L must be replaced by $2\pi-F$, ds by $d\tau$ and $d\tau$ by ds. Therefore transforming by duality the formula (8) we shall obtain

$$2\pi - F = \int_L \sin h ds + \int_K \cos h d\tau. \tag{9}$$

4. **Consequences.** (a) Let β be the minimum breadth of K, that is to say the minimum value of α, and δ the maximum value of α; δ is equal to the "diameter" of K. Let us suppose that $\beta \leqq \alpha \leqq \delta \leqq \pi/2$. Then $\sin \delta \geqq \sin \alpha$, $\cos \delta \leqq \cos \alpha$, and from (8) and (2) we deduce

$$L \leqq (2\pi - F) \sin \delta - L \cos \delta$$

from which we have

$$L/(2\pi - F) \leqq \tan (\delta/2). \tag{10}$$

Similarly, $\sin \alpha \geqq \sin \beta$, $\cos \alpha \leqq \cos \beta$, and consequently

$$L/(2\pi - F) \geqq \tan (\beta/2). \tag{11}$$

Therefore: *On the sphere of unit radius for any convex curve K of minimum breadth β and diameter $\delta \leqq \pi/2$, the inequalities* (10) *and* (11) *are verified.*

(b). In (10) and (11) there is equality only when $\alpha = \beta = \delta = \text{const.}$, that is to say for the cases of *convex curves of constant breadth.* These curves of constant breadth on the sphere have been studied by Blaschke [1]. For such curves it is deduced from (8) that

$$L = (2\pi - F) \tan (\alpha/2). \tag{12}$$

This formula, obtained also by Blaschke [1], appears here as a particular case of our general formula (8).

(c). It is well known that isoperimetric inequality $L^2 + (2\pi - F)^2 \geqq 4\pi^2$ holds for any convex curve K on the unit sphere and that the equality is only true for the circles [4, p. 718]. This isoperimetric inequality and (10), (11) give us

$$F \leqq 2\pi(1 - \cos (\delta/2)), \qquad L \geqq 2\pi \sin (\beta/2). \tag{13}$$

Consequently: *upon the sphere, given the diameter $\delta \leqq \pi/2$ the convex curve of maximum area is the circle of radius $\delta/2$; given the minimum breadth β the curve of minimum length is the circle of radius $\beta/2$.*

(d). If the convex curve K has continuous geodesic curvature, from (9) and (2) we deduce

$$\int_L \sin (h/2)(\rho_g^{-1} \sin (h/2) - \cos (h/2)) ds = 0 \tag{14}$$

and we therefore have the theorem: *In any convex curve on the sphere of unit radius with continuous radius of geodesic curvature there are at least two points for which*

$$\rho_g = \tan (h/2),$$

h being the chord of K orthogonal to K at the point considered.

Analogously from (8) and (2) we deduce

$$\int_L \cos (\alpha/2)(\rho_g^{-1} \sin (\alpha/2) - \cos (\alpha/2)) ds = 0 \tag{15}$$

and therefore: *in any convex curve on the sphere of unit radius with continuous radius of geodesic curvature there are at least two points for which*

$$\rho_g = \tan(\alpha/2),$$

α being the breadth of K corresponding to the point considered.

5. **Passage to the case of the plane.** Considering a sphere of radius R and making $R \to \infty$, the anterior formulas transform themselves into valid formulas for plane convex curves.

The principal formula (8) if K belongs to a sphere of radius R is written

$$\frac{L}{R} = \int_K \sin\frac{\alpha}{R}\, d\tau - \int_L \cos\frac{\alpha}{R}\,\frac{ds}{R};$$

multiplying both sides by R and making $R \to \infty$ we obtain the well known formula (1).

The formula (9) taking into account (2) may be written

$$\int_K \sin^2\frac{h}{2}\, d\tau = \int_L \sin\frac{h}{2}\cos\frac{h}{2}\, ds.$$

Writing this formula for a sphere of radius R, multiplying afterwards both sides by R^2 and making $R \to \infty$ we obtain

$$J = \int_L h ds = 2^{-1}\int_0^{2\pi} h^2 d\tau. \tag{16}$$

In this formula h is the chord of the plane convex curve K normal to this at the point s (or τ).

The element hds which appears in the first expression of J in (16) is the area of a strip of K with the height h and base ds. If a point $P(x, y)$ interior to K belongs to ν of these strips, from P there will be ν normals to K and the element of area $dxdy$ will be counted ν times in (16). Hence, if ν has a finite value for any P interior to K, the integral J is also equal to

$$J = \iint_F \nu\, dxdy, \tag{17}$$

ν being the number of normals to K from the point $P(x, y)$ interior to K.

If there are points with $\nu = \infty$ (for instance the case of the circle) the integral (17) must be considered as the limit of the similar ones referring to convex curves K_n (with ν always finite) which tend to K.

It is interesting to study the limits between which the integral J

is bounded. From any point $P(x, y)$ interior to K may be traced always at least two normals to K, joining in straight lines P with the points of K of minimum and maximum distance to P. Consequently

$$J \geqq 2F. \tag{18}$$

From another aspect, the second expression for J in (16) shows that J is equal to the area that is obtained by extending from a fixed point of the plane segments of length h parallel to the direction τ; as h is less than or equal to the length of the maximum chord of K which also has direction perpendicular to the line of support of direction τ, a known theorem of Rademacher about vector regions of convex curves [**3**; **2**, p. 105] tells us that

$$J \leqq 6F. \tag{19}$$

(18) cannot be bettered, it being sufficient to consider a triangle with an angle approximating to π in order that J comes near $2F$ in such degree as we should wish.

The inequality (19), on the contrary, is probably excessive. For the convex curves of constant breadth α we have

$$J = L\alpha = \pi\alpha^2. \tag{20}$$

If we recall that the triangle of Reuleaux [**2**, p. 132] is the figure which, given the constant breadth α, has minimum area of value $(\pi - 3^{1/2})\alpha^2/2$, we see that $F \geqq (\pi - 3^{1/2})\alpha^2/2$ and consequently from (20) we deduce

$$J \leqq 2\pi F/(\pi - 3^{1/2}) = 4{,}4576 \cdots F. \tag{21}$$

It is probable that this superior bound is valid not only for convex curves of constant breadth but also for any convex curve of the plane. Nevertheless I have not been able to prove this last assertion.

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THE DETERMINATION OF SOME PROPERTIES OF A FUNCTION SATISFYING A PARTIAL DIFFERENTIAL EQUATION FROM ITS SERIES DEVELOPMENT

STEFAN BERGMAN

1. The method of integral operators. The solution of an equation and its associate. The method of integral operators in the theory of linear partial differential equations of the type[1]

$$(1.1)\qquad \begin{aligned} L(U) &\equiv (U_{xx}+U_{yy})/4 + A(x,y)U_x/2 + B(x,y)U_y/2 + C(x,y)U \\ &\equiv U_{z\bar z} + 2\,\mathrm{Re}\,[a(z,\bar z)U_z] + c(z,\bar z)U = 0, \end{aligned}$$

where

$$z = x + iy, \qquad \bar z = x - iy, \qquad U_z = [(\partial U/\partial x) - i(\partial U/\partial y)]/2,$$
$$U_{\bar z} = [(\partial U/\partial x) + i(\partial U/\partial y)]/2,$$

consists in associating with an arbitrary analytic function $f(\zeta)$ of a complex variable ζ, by means of an operator of the form

$$(1.2)\qquad U(z,\bar z) = \boldsymbol{M}(f) \equiv \mathrm{Re}\,[P(f)],$$

$$(1.3)\qquad u(z,\bar z) = P(f) = \int_{-1}^{1} \mathrm{E}(z,\bar z,t) f(z(1-t^2)/2)\,dt/(1-t^2)^{1/2},$$

a solution $U(z,\bar z)$ of the equation (1.1).

$\mathrm{E} = \mathrm{E}(z,\bar z,t)$, $|t| \leqq 1$, is any analytic function of z and $\bar z$ which satisfies the equation

$$(1.4)\qquad G(\mathrm{E}) \equiv (1-t^2)(\mathrm{E}_{zt} + a\mathrm{E}_t) - t^{-1}(\mathrm{E}_z + a\mathrm{E}) + 2ztL(\mathrm{E}) = 0,$$

is regular in a sufficiently large domain and has the property that $(\mathrm{E}_{\bar z} + A\mathrm{E})/zt$ is continuous at $\bar z = 0$, $t = 0$.

REMARK. An operator (1.2) is determined by choosing a particular function E (the *generating function* of the operator) which satisfies the above requirements.

Let $U(z,\bar z)$ be a function which satisfies the equation $L(U) = 0$ and which is an entire function of two complex variables x and y, that is, a solution of (1.1) whose series development

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[1] Since we consider the functions $u(z,\bar z)$ for real values of x and y, that is for z and $\bar z$ conjugate, it would be, of course, sufficient to write simple $u(z)$. We shall, however, use the first notation in order to stress the fact that $u(z,\bar z)$ is a (complex) analytic function of two *real* variables x, y, reserving $u(z)$ for analytic functions of *one* complex variable.

$$(1.5)\qquad U(z, \bar{z}) = \sum_{m,n=0}^{\infty} B_{mn} x^m y^n \equiv \sum_{m,n=0}^{\infty} D_{mn} z^m \bar{z}^n, \qquad D_{mn} = \overline{D}_{nm},$$

converges for $|x| < \infty$, $|y| < \infty$.

The problem with which we shall be concerned in §§2 and 3 of this note is to determine an upper bound for the growth of $U(x, y)$ for *real* values of x, y from certain subsequences of the coefficients, D_{mn}, of its series development (1.5). Further we shall give in §4 a condition in terms of A_{m0} for $V = \sum_{m,n=0}^{\infty} A_{m,n} z^m \bar{z}^n$, $L(V) = 0$, to have only one branch point of certain type on its circle of regularity.

If $f(z/2)$ is an analytic function which is regular in the circle $|z| \leqq \rho$, then clearly $u(z, \bar{z})$ will be an analytic function of two real variables in the circle $x^2 + y^2 < \rho^2$. The converse is not necessarily true.

If $\mathrm{E}(z, \bar{z}, t)$ possesses the property that to every $u(z, \bar{z})$ which is regular in a circle $x^2 + y^2 < \rho^2$ there corresponds a unique function $f(z/2)$ which is regular in the circle $|z| < \rho_1, \rho_1 > 0$ and independent of f, then we shall say that E is a *generating function of the type* $\mathcal{A}$, or that E belongs to $\mathcal{A}$. If in addition to this, for every f, $\rho_1 = \rho$ then E will be said to be a *generating function belonging to* $\mathcal{B}$. In a previous paper, [1a],[2] it was shown that to every equation $L(U) = 0$ where a and c are entire functions of z and $\bar{z}$, there exists an $\mathrm{E}(z, \bar{z}, t)$ (a generating function of the first kind) of the form

$$(1.6)\qquad \mathrm{E}(z, \bar{z}, t) = \left[\exp\left(-\int_0^{\bar{z}} a(z, \bar{z})d\bar{z}\right)\right]\left[1 + z\bar{z}t^2\mathrm{E}^*(z, \bar{z}, t)\right]$$

where E^* is a regular function.

The operator (1.3) where E is of the form (1.6) possesses the property that if $u(z, \bar{z})$ is regular in a simply connected domain which includes the origin, then $f(z/2)$ is also regular in this domain, and hence E in this case belongs to $\mathcal{B}$.

The function f which upon application of the operator P yields u will be denoted as the associate of u with respect to the generating function E. (Often, however, we omit the last phrase.)

REMARK. If we change the generating function E, we change the law connecting the real and imaginary parts, U and V, of the obtained (complex) solution, u. That is to say, if for a given L we have two generating functions E_1 and E_2 and two functions f_1, f_2, such that $U_1+iV_1=P_1(f_1)$ and $U_1+iV_2=P_2(f_2)$, then V_1 in general differs from V_2.

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[2] The numbers in brackets refer to the following papers: **1.** S. Bergman, a. Rec. Math. (Mat. Sbornik) N.S. vol. 2 (1937) pp. 1169–1198; b. Trans. Amer. Math. Soc. vol. 53 (1943) pp. 139–155; c. *The hodograph method in the theory of compressible fluids*, Brown University, 1942; d. Proc. Nat. Acad. Sci. U.S.A. vol. 29 (1943) pp. 276–281; **2.** L. Bers and A. Gelbart, Quarterly of Applied Mathematics vol. 1 (1943) pp. 168–188. **3.** L. Bieberbach, *Funktionentheorie*, vol. 2, 1927; **4.** G. Darboux, *La théorie génerale de surfaces*, vol. 2, 1915; **5.** K. L. Nielsen and B. P. Ramsay, Bull. Amer. Math. Soc. vol. 49 (1943) pp. 156–162; **6.** A. Zygmund, *Trigonometrical series*, 1935.

In the case of the generating function of the first kind the corresponding complex solutions

$$u(z, \bar{z}) = \sum_{m,n=0}^{\infty} A_{mn} z^m \bar{z}^n, \quad A_{00} \text{ real}, \tag{1.7}$$

have the property that

$$c_m A_{m0} = a_m, \qquad c_m = 2^m \Gamma(m+1)/\pi^{1/2}\Gamma(m+1/2), \tag{1.8}$$
$$\sum_{m=0}^{\infty} A_{0m} \bar{z}^m = A_{00} \exp\left[-\int_0^{\bar{z}} a(0, \bar{z}) d\bar{z}\right],$$

where the a_m are the coefficients of the series development $f(\zeta) = \sum_{n=0}^{\infty} a_n \zeta^n$ of the associate function f.

In this case we may also express the a_m in terms of the coefficients D_{mn} of the real part of $u(z, \bar{z})$, $U(z, \bar{z})$. Substituting $\bar{z}=0$ in

$$\begin{aligned}\sum_{m,n=0}^{\infty} D_{mn} z^m \bar{z}^n = \frac{1}{2}\int_{-1}^{1} &[E(z, \bar{z}, t) f(z(1-t^2)/2) \\ &+ \bar{E}(\bar{z}, z, t)\bar{f}(\bar{z}(1-t^2)/2)] dt/(1-t^2)^{1/2}\end{aligned} \tag{1.9}$$

we have by (1.5)

$$\begin{aligned}\sum_{m=0}^{\infty} D_{m0} z^m = \frac{1}{2}\Bigg\{ &\sum_{m=0}^{\infty} 2^{-m} a_m \Gamma(m+1/2)\pi^{1/2}) [\Gamma(m+1)]^{-1} z^m \\ &+ \bar{f}(0)\pi \exp\left(-\int_0^{z} \bar{a}(0, z) dz\right)\Bigg\}.\end{aligned} \tag{1.10}$$

Thus from given D_{m0} and $\bar{a}(0, z)$ we may determine the a_m and, therefore, by the classical results of the theory of functions the order of the growth of the associate function f in (1.3). Consequently if in addition we know the order of the growth of E we can obtain by (1.3) an upper bound for the order of the complex solution, and hence for its real part, U.

2. Relations between the series developments of U and its associate in the general case. However in applying this method it is extremely important to obtain the lowest possible order of growth for the function E, and it is for this reason that in some instances we shall consider functions E belonging to $\mathcal{A}$ but of different structure than those of the first kind. The price we have to pay for this is that the laws connecting the coefficients D_{mn} and the a_m will be considerably more complicated than those which follow from (1.8).[3] If now we put $\bar{z}=0$ in (1.9) and set

$$\frac{1}{2}\int_{-1}^{1}\bar{\mathrm{E}}(0, z, t)dt/(1-t^2)^{1/2} = \sum_{m=0}^{\infty} L_m z^m \tag{2.1}$$

then we obtain

$$\sum_{\nu=0}^{n} a_\nu P_{n,\nu} = 2D_{n0} - 2\bar{f}(0)L_n \tag{2.2}$$

where

$$\begin{aligned} P_{n,\nu} &= 2^\nu \int_{-1}^{1} P_n(t)(1-t^2)^{\nu-1/2}dt, \quad \nu \geqq n; \quad P_{0,\nu} \neq 0, \\ \mathrm{E}(z, 0, t) &= \sum_{\nu=0}^{\infty} P_\nu(t) z^\nu. \end{aligned} \tag{2.3}$$

Conversely, we have

$$a_n = \begin{vmatrix} P_{0,0} & 0 & 0 & \cdots & 2(D_{00}-\bar{f}(0)L_0) \\ P_{1,0} & P_{0,1} & 0 & \cdots & 2(D_{10}-\bar{f}(0)L_1) \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ P_{n,0} & P_{n-1,1} & P_{n-2,2} & \cdots & 2(D_{n0}-\bar{f}(0)L_n) \end{vmatrix} \Bigg/ \prod_{\nu=0}^{n} P_{0,\nu} \tag{2.4}$$

instead of (1.8).

3. Upper bounds for the growth of generating functions E of equations L, whose coefficients are connected by certain differential relations.

In §3 of [1], a procedure was indicated to determine the differential equations L, a generating function of which is of the form

$$\mathrm{E}(z, \bar{z}, t) = \exp\left[\mathrm{P}(z, \bar{z}, t)\right] \tag{3.1}$$

where P is a polynomial in t.

Nielsen and Ramsay in [5] applied this method and found additional equations L with generating functions (3.1). In the latter paper, however, the conditions are formulated in a form which is not very convenient for immediate application. The equation (1.1) is assumed already to be reduced to the form $u_{z\bar{z}}+Du_z+Fu=0$, and the conditions are given in the form of assumptions concerning the structure of D and F. It seems useful for applications to formulate the same conditions in the form of differential relations between the coëfficients as was done in Theorem 4 of [1a].[4]

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[3] When we use rough estimates for the growth of $\mathrm{E}(z, \bar{z}, t)$, for example like those given on p. 1176 of [1a], then the results are valid for a very large class of equations L. The majorant for E derived in Theorem 2 of [1a] depends only upon the maxima of $|a|$ and $|c|$ in certain domains. The estimates which we obtain in the following use the property of the coefficients a and c in a much more specific fashion. The estimates are more special and more precise. (Note that the $P_{n,\nu}$ depend upon E, and therefore upon a and c.)

As we shall see, in certain cases several types of the generating functions (3.1) are admissible.

Case 1.

$$a_z = \bar{a}_{\bar z}, \qquad F_z = 0, \qquad F \equiv -a_z - |a|^2 + c. \tag{3.2}$$

Then

$$\mathrm{P} = -\int_0^z a\,d\bar{z} + z^{1/2}\left[C - 4\int_0^z F\,d\bar{z}\right]^{1/2} t \tag{3.3}$$

or

$$\mathrm{P} = -\int_0^z a\,d\bar{z} - k_1 z + z^{1/2}\left[-4\int_0^z F\,d\bar{z}\right]^{1/2} t + k_1 z t^2. \tag{3.4}$$

Case 2.

$$a_z = \bar{a}_{\bar z}, \qquad \phi = 2(zF_z - F)/F_zF_{zz} = \text{constant}. \tag{3.5}$$

Then

$$\mathrm{P} = -\int_0^z a\,d\bar{z} - z^{1/2}\phi^{-1/2}(\acute{z} + \phi F_z)t + 2\phi^{-1/2}t^3/3. \tag{3.6}$$

Case 3.

$$F_z = 0, \qquad F = (a_z - \bar{a}_{\bar z})/2. \tag{3.7}$$

Then

$$\mathrm{P} = -\int_0^z a\,d\bar{z} + z\left[C + 2\int_0^z F\,d\bar{z}\right]t^2 \tag{3.8}$$

where C is an arbitrary constant.

Case 4.

$$\bar{a}_{\bar z} - a_z = F, \qquad F_z = 0. \tag{3.9}$$

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[4] We note that line 4 from bottom, p. 1197 of [1a], should be "$z^{3/2}gM_z=0$" instead of "$zg^{3/2}M_z=0$" and line 2, p. 1197, "$h_z=-A$" instead of "$h_{\bar z}=-A$."

Then

$$
\begin{aligned}
\mathrm{P} = & -\int_0^z \bar{a}\,dz + \int_0^{\bar z}\int_0^z F\,dz\,d\bar z + Cz \\
& - \left[2Cz^2 + z\int_0^{\bar z} F\,d\bar z\right] t^2 + Cz^2 t^4
\end{aligned}
\tag{3.10}
$$

where C is an arbitrary constant.

The procedure indicated in [1a] and [5] yields further types of equations L which possess a generating function of the form (3.1).

The method described in §3 of [1a] can be immediately generalized for a much larger class of equations L.

Consider the equation $u_{xy}+a(x, y)u_x+b(x, y)u_y+c(x, y)u=0$ whose coefficients are supposed to be (complex) analytic functions of two real variables, and where $b(x, y)=\bar{a}(y, x)$. If we replace x by z and y by $\bar z$ then the latter equation becomes $L(u)=0$ (see (1.1)). Generalizing the "method of cascade" to the case of the equations $L(u)=0$, we form the invariants[5]

$$
\begin{aligned}
& F^{(0)}, F^{(1)}, \cdots, \\
& F^{(0)} = -F \equiv a_z + |a|^2 - c, \\
& F^{(n+1)} = 2F^{(n)} - \overline{F}^{(n)} - F^{(n)}\Delta F^{(n)}/4 + |F_z^{(n)}|^2,
\end{aligned}
\tag{3.11}
$$

and the corresponding equations

$$
\begin{aligned}
& L_0, L_1, L_2, \cdots, \\
& L_0 = L,\ L_n(v) = v_{z\bar z} + a^{(n)}v_z + b^{(n)}v_{\bar z} + c_v^{(n)}
\end{aligned}
\tag{3.12}
$$

where

$$
\begin{aligned}
& a^{(n+1)} = a^{(n)}F_z^{(n)}/F^{(n)}, \qquad b^{(n+1)} = b^{(n)}, \\
& c^{(n+1)} = c^{(n)} - a_z^{(n)} - b_{\bar z}^{(n)} - 2b^{(n)}F_{\bar z}^{(n)}/F^{(n)}, \qquad b^{(0)} = \bar{a}.
\end{aligned}
\tag{3.13}
$$

See [4, p. 28].

If the coefficients of one of the equations L_n satisfy any one of the conditions (3.4), (3.5), (3.7), (3.9) or similar relations then the generating function of L_n has the form (3.1). Hence the expression

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[5] If an invariant, say $F^{(n)}$, vanishes identically and, therefore, the sequence (3.11) consists of only a finite number of nonvanishing terms, then the solutions of $L(u)=0$ may be represented in the form $u(z, \bar z)=\sum_{k=0}^{n} B_k(z, \bar z)d^k f(z)/dz^k$, where the B_k are certain fixed functions, and $f(z)$ ranges over the field of analytic functions of a complex variable z. See, for example, [4, p. 42] or [1a, Theorem 4].

$$\begin{aligned} u(z, \bar{z}) &= N\left[\int_{-1}^{1} e^{P(z,\bar{z},t)} f(\bar{z}(1-t^2)/2)dt/(1-t^2)\right] \\ &= \int_{-1}^{1} N[e^{P(z,\bar{z},t)}] f(\bar{z}(1-t^2)/2)dt/(1-t^2)^{1/2}, \end{aligned} \tag{3.14}$$

where N denotes the differential operator

$$\begin{aligned} N[m] \equiv \left[\exp\left(-\int_0^z \tilde{a}dz\right)\right]\frac{1}{F^{(0)}}\frac{\partial}{\partial z}\left\{\frac{1}{F^{(1)}}\frac{\partial}{\partial z}\left[\frac{1}{F^{(2)}}\right.\right. \\ \left.\left.\cdots\left(\frac{1}{F^{(n-1)}}\frac{\partial}{\partial z}\left(m \exp\left(\int \tilde{a}dz\right)\right)\right)\right]\right\} \end{aligned} \tag{3.15}$$

and $f(\zeta)$ is an arbitrary analytic function, will be a solution of $L(u)=0$. See [4, p. 30]. Therefore

$$E(z, \bar{z}, t) = N[e^{P(z,\bar{z},t)}] \tag{3.16}$$

is a generating function of $L(u)=0$.

Example. If L_1 satisfies the conditions (3.2), then

$$N(m) = \left[\exp\left(-\int_0^z \tilde{a}dz\right)\right]\frac{\partial}{\partial z}\left[m \exp\left(\int_0^z \tilde{a}dz\right)\right] \Big/ F^{(0)}.$$

Substituting $m=[z(C-4\int_0^z F^{(1)}d\bar{z})]^{1/2}t$ yields

$$\begin{aligned} E = \left\{\left(b - a^{(1)}\right)F^{-1} + \frac{t}{2}\left[\left(C - 4\int_0^z F^{(1)}d\bar{z} \Big/ z\right]^{1/2} F^{-1}\right\} \\ \cdot\exp\left\{-\int_0^z a^{(1)}d\bar{z} + \left[z\left(C - 4\int_0^z F^{(1)}d\bar{z}\right)\right]^{1/2} t\right\}. \end{aligned}$$

If we now assume that the growth of the coefficients a and c of L is known, then using the above results we can easily determine an upper bound for the growth of E. For instance, if a and c are polynomials, then we obtain an upper bound for E of the form

$$\exp(A|z|^n). \tag{3.17}$$

Now, two cases can occur. Either the order of growth of f is more than n, then the growth of f is decisive for the growth of $u(z, \bar{z})$,[6] or the growth of f is less than n, in which case a rough estimate of the growth of the right-hand side of (1.3) yields the order n. In many instances, however, we may obtain much sharper bounds. It may happen that the order of growth of exp $[\mathrm{P}(z, \bar{z}, t)]$ in the real xy plane or at least for some values of $\phi = \arg z$ is smaller than n. Then it is sometimes possible to find estimates for the order of growth of $u(z, \bar{z})$ from the A_{m0} or D_{m0}. In such cases we may obtain from A_{m0} an upper bound for the growth of u in the whole plane or at least in some directions, despite the fact that the order of growth of f is less than n.

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[6] The order of growth of f can be determined from the coefficients A_{m0} (or D_{m0}) of series developments of (1.7) or (1.5).

Finally we may determine solutions of (1.4) which are bounded at infinity if we approach along a certain direction. We can then obtain upper bounds for the growth of u from the A_{m0} in this direction.

Example. Let $L(u) \equiv \Delta u + u = 0$. Then $\mathrm{E}(z, \bar{z}, t) = \cos\, [t(z\bar{z})^{1/2}]$ is bounded in the whole plane, and we obtain immediately an upper bound in terms of the a_m (and therefore in terms of the A_{m0} or of the D_{m0}) for

$$u(z, \bar{z}) = \int_{-1}^{1} \cos\, [t(z\bar{z})^{1/2}] \left\{ \sum_{n=0}^{\infty} 2^{-n} a_n z^n (1 - t^2)^{n-1/2} \right\} dt.$$

In this connection it would also be possible to apply the well known results on the growth of the function f in a specific direction from the coefficients of its series development.

Remark. In previous papers [1c, p. 23], [1d, p. 277] (see also [2]) an operator, T, was introduced. T is of a different form than (1.3). It transforms the power series, $s = \sum_{n=0}^{\infty}(\alpha_n + i\beta_n)(x + iY)^n$, into functions $\sum_{n=0}^{\infty}(\alpha_n + i\beta_n)(x + iY)^{[n]}$ whose imaginary parts satisfy the equation

$$l(Y) U_{xx} + U_{YY} = 0. \tag{3.18}$$

Concerning the definition of $(x + iY)^{[n]}$, see (2.3) of [1d] and p. 23 of [1c]. After the introduction of a suitable variable, $y = y(Y)$, the equation (3.17) assumes the form

$$L(U) = U_{xx} + U_{yy} + N(2y) U_y = 0. \tag{3.19}$$

Let T^* be the operator which we obtain from T by replacing Y by $Y(y)$. If the equation (1.1), or one of the equations L_n of the sequence (3.12) corresponding to L, has the form (3.19) then by applying the operator NT^* to a power series, s, we obtain a solution of (1.1).

The same reasoning can be applied to all types of equations considered by Bers and Gelbart in [2].

4. Conditions for a branch point on the boundary of the largest regularity circle.

As was proved previously the radius, ρ, of the largest circle with the center at the origin in which $u(z, \bar{z})=\sum_{m,n=0}^{\infty}A_{mn}z^m\bar{z}^n$ is regular is given by[7]

$$\rho = 1/\limsup_{n\to\infty} |A_{n0}|^{1/n}. \tag{4.1}$$

A problem which now arises is to determine how many singular points lie on the boundary of the circle, and the character of these singularities. There exists a very simple test (the test of Hadamard) in the case of analytic functions to decide that on the largest regularity circle one and only one pole lies.

The theory of operators enables us to obtain a similar procedure to decide that one and only one singular point of a certain type lies on the largest circle of regularity. In order to make the essential idea of the procedure clearer we shall at first apply it in the case where $u(z, \bar{z})=u(z)$ is an analytic function. Let $Q^{(n)}(z)$, $n=0, 1, 2, \cdots$, be a set of analytic functions of a complex variable z, such that

$$\sum_{n=0}^{\infty} | z^n Q^{(n)}(|z|) | \tag{4.2}$$

converges uniformly in a sufficiently large circle, say for $|z| \leqq \rho^*$.

The point $2a$ is said to be a branch point of the type $\{Q^{(n)}\}$ of u if the function u can be represented in the neighborhood of $2a$ in the form

$$\begin{aligned} u(z) = \pi^{1/2}\sum_{n=0}^{\infty} 2^{-2n}\Gamma(2n+1)[\Gamma(n+1)]^{-1}Q^{(n)}(z) & \\ \cdot I^{1/2+n}[z^{-1/2}(a - z/2)^{-1}] + g(z) & \end{aligned} \tag{4.3}$$

where $g(z)$ is a regular function at the point $z=2a$, and the I^k denote the integral of kth order. (For the definition of the integrals of non-integral order see [6, p. 222].)

Remark. We have

$$\begin{aligned} & I^{1/2}[z^{-1/2}(a - z/2)^{-1}] \\ & \qquad = \pi^{-1/2}\int_{-1}^{1} [a - z(1 - t^2)/2]^{-1}(1 - t^2)^{-1/2}dt \\ & \qquad = \pi^{1/2}a^{-1/2}(a - z/2)^{-1/2}, \\ & I^{3/2}[z^{-1/2}(a - z/2)^{-1}] \\ & \qquad = 2\pi^{-1/2}\int_{-1}^{1} t^2[a - z(1 - t^2)/2]^{-1}(1 - t^2)^{-1/2}dt \\ & \qquad = 2\pi^{1/2}z^{-1}(a - z/2)^{-1/2}[a^{1/2} - (a - z/2)^{1/2}]. \end{aligned} \tag{4.4}$$

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[7] See [1b, p. 142, formula (7.4)]. We note that this formula should be "lim $\sup_{n\to\infty} (A_{n0})^{1/n}$" instead of "$\lim_{n\to\infty} (A_{n0}c_n)^{1/n}$."

THEOREM. A. *A sufficient condition that*

$$u(z) = \sum_{m=0}^{\infty} A_m z^m \tag{4.5}$$

possess (as a unique singularity on the circle of convergence, $|z| = 2|a|$*) a branch point of the type* $\{Q^{(n)}(z)\}$ *at* $z = 2a$ *is that there exists a constant* k, $k < 1$, *such that*

$$\left| \frac{\begin{vmatrix} H_{0,0} & 0 & 0 & \cdots A_0 \\ H_{1,0} & H_{0,1} & 0 & \cdots A_1 \\ \cdot & \cdot & \cdot & \cdots\cdot \\ H_{n+1,0} & H_{n,1} & H_{n-1,2} & \cdots A_{n+1} \end{vmatrix}}{H_{0,n+1} \begin{vmatrix} H_{0,0} & 0 & 0 & \cdots A_0 \\ H_{1,0} & H_{0,1} & 0 & \cdots A_1 \\ \cdot & \cdot & \cdot & \cdots\cdot \\ H_{n,0} & H_{n-1,1} & H_{n-2,2} & \cdots A_n \end{vmatrix}} - \frac{1}{2a} \right| < k^n \tag{4.6}$$

where

$$H_{n,k} = \int_{-1}^{1} H_n(t)(1 - t^2)^{k-1/2} dt/2^k \tag{4.7}$$

and the $H_n(t)$ *are given by*

$$\mathrm{E}(z, t) = \sum_{n=0}^{\infty} t^{2n} z^n Q^{(n)}(z) = \sum_{\nu=0}^{\infty} H_\nu(t) z^\nu. \tag{4.8}$$

B. *If* $\mathrm{E}(z, t) \in \mathfrak{B}$, see p. 559, *then the above condition is also necessary.*

REMARK. We assume that the radius of convergence, ρ^*, of (4.2) is larger than ρ given by (4.1).

PROOF. A. We note that

$$\begin{aligned} &\int_{-1}^{1} \mathrm{E}(z, t)[a - z(1 - t^2)/2]^{-1} dt/(1 - t^2)^{1/2} \\ &= \sum_{n=0}^{\infty} 2^{-2n}\Gamma(2n + 1)[\Gamma(n + 1)]^{-1} Q^{(n)}(z) I^{n+1/2}[z^{-1/2}(a - z/2)^{-1}]. \end{aligned} \tag{4.9}$$

We consider now the function

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \tag{4.10}$$

where

$$(4.11)\qquad a_n = \begin{vmatrix} H_{0,0} & 0 & 0 & \cdots A_0 \\ H_{0,0} & H_{0,1} & 0 & \cdots A_1 \\ \cdot & \cdot & \cdot & \cdots\cdot \\ H_{n,0} & H_{n-1,1} & H_{n-2,2} & \cdots A_n \end{vmatrix} \Bigg/ \prod_{\nu=0}^{n} H_{0,\nu}.$$

By (4.6) the radius of convergence of this series is $2|a|$ and by the Hadamard test (see, for example, [3, p. 219]) the series is regular in the circle $|z| \leqq 2|a|$ except at the point $z=2a$, where it possesses a simple pole. Therefore

$$(4.12)\qquad f(z/2) = C_0(a - z/2)^{-1} + f_1(z/2)$$

where $f_1(z)$ is regular for $|z| \leqq |a|$.

$$(4.13)\qquad \begin{aligned} &\int_{-1}^{1} \mathrm{E}(z, t) f(z(1 - t^2)/2)dt/(1 - t^2)^{1/2} \\ &= \int_{-1}^{1} \mathrm{E}(z, t) f_1(z(1 - t^2)/2)dt/(1 - t^2)^{1/2} \\ &\quad + C_0 \int_{-1}^{1} \mathrm{E}(z, t)\left[a - z(1 - t^2)/2\right]^{-1} dt/(1 - t^2)^{1/2} \end{aligned}$$

will be a function which is regular for $|z| \leqq 2|a|$ except at $z=2a$, where it has a branch point of the type $\{Q^{(n)}\}$. See (4.9). On the other hand a formal computation yields

$$(4.14)\qquad u(z) = \sum_{n=0}^{\infty} A_n z^n = \sum_{n=0}^{\infty} z^n \left(\sum_{\nu=0}^{n} 2^{-\nu} a_\nu H_{n-\nu,\nu} \right)$$

and therefore $f(z) = \sum_{n=0}^{\infty} a_n z^n$, with a_n given by (4.11), is the associate of $u(z)$ with respect to $\mathrm{E}(z, t)$. Hence (4.6) is a sufficient condition.

B. Suppose now that $\mathrm{E}(z, t) \in \mathfrak{B}$ and that $u(z)$ is regular in $|z| \leqq 2|a|$ except at the point $z=2a$, where $u(z)$ possesses a branch-point of the type $\{Q^{(n)}\}$. Hence

$$(4.15)\qquad \begin{aligned} u_1(z) = u(z) - C_0 \Big[&\sum_{n=0}^{\infty} 2^{-2n} \Gamma(2n + 1)\left[\Gamma(n + 1)\right]^{-1} Q^{(n)}(z) \\ &\cdot I^{n+1/2}\left[z^{-1/2}(a - z/2)^{-1}\right] \Big] \end{aligned}$$

is regular in $|z| \leqq 2|a|$. Since $\mathrm{E} \in \mathfrak{B}$, it follows that the associate $f_1(z)$

of $u_1(z)$ is regular for $|z| \leqq |a|$, and therefore denoting by f the associate of u we obtain

$$(4.16) \qquad f(z/2) = f_1(z/2) + C_0(a - z/2)^{-1}.$$

As we have previously shown, the coefficients a_n of $f(z)$ are given by (4.11). Since the Hadamard relation is a necessary condition in order that $f(z)$ is regular in $|z| \leqq 2|a|$ except for a simple pole at $z=2a$, it follows that in the case $\mathrm{E}(z, t) \in \mathfrak{B}$, (4.6) is a necessary condition.

Clearly, using the results of Hadamard, the same procedure may be applied to obtain conditions that $u(z, \bar{z})$ possesses several branch-points of the type $\{Q^{(n)}\}$.

The generalization of the theorem to the case where $u(z, \bar{z})$ is a solution of the equation (1.1) is evident. Now the functions $Q^{(n)}(z, \bar{z})$ are no longer arbitrary; for in addition to the fact that $\sum_{n=0}^{\infty} |z^n Q^{(n)}(|z|, |\bar{z}|)|$ converges uniformly,

$$(4.17) \qquad \sum_{n=0}^{\infty} t^{2n} z^n Q^{(n)}(z, \bar{z})$$

must be a solution of the equation (1.4).

If we denote by A_{mn} the coefficients of the series development (1.7) of $u(z, \bar{z})$ and by the D_{mn} the coefficients of the development (1.5) of the real part U of u, then the a_n can be determined from either the subsequence A_{m0} or from the D_{m0}'s. In the first case we must replace in (4.11) the A_m by the A_{m0} and obtain the $H_{n,k}$ by substituting in (4.7) the coefficients $H_\nu(t)$ of the series development

$$E(z, 0, t) = \sum_{k=0}^{\infty} t^{2k} z^k Q^{(k)}(z, 0) = \sum_{\nu=0}^{\infty} H_\nu(t) z^\nu.$$

Similar results may be obtained if we express a_n in terms of the D_{m0}.

BROWN UNIVERSITY

ON CERTAIN ARITHMETICAL FUNCTIONS DUE TO G. HUMBERT

M. A. BASOCO

1. **Introduction.** G. Humbert has discussed, in a series of brief notes,[1] a certain class of entire functions with interesting arithmetical properties. These functions are defined, in an essentially unique manner, by certain functional equations. The Fourier series representations of the solutions of these equations are similar in form to those for the elliptic functions *snu*, *cnu*, *dnu*, and so on. They differ from these, however, in that their domain of validity extends throughout the entire complex plane (Z_∞ excluded) and moreover, in that their arithmetized forms involve *incomplete* numerical functions of the divisors of an integer.

In the present paper we extend somewhat the results of Humbert and obtain a relation between his functions and certain pseudo-periodic functions discussed elsewhere by the writer.[2] This relation is, in effect, embodied in a series of twelve identities; these are of some interest in that their arithmetical equivalents (paraphrases) are relatively simple and involve partitions related to the representations of an integer as the sum of five squares.

It is also pointed out that as an immediate consequence of the analytical form of Humbert's functions, it is possible to deduce a series of relations between the greatest integer function $E(x)$ and certain incomplete numerical functions of the divisors of an integer.

2. **The functional equations.** In what follows the notation is that ordinarily used in the theory of the elliptic theta functions.[3] The period $\pi\tau$ is such that $0<\arg\tau<\pi$.

The set of functional equations considered has the form

$$\text{(A)}\qquad \begin{aligned} h(z+\pi) &= (-1)^a h(z),\\ h(z+\pi\tau) &= (-1)^b h(z) + F_{ab}^{(\alpha)}(z), \end{aligned}$$

where a, b take the values zero or unity, and $F_{ab}^{(\alpha)}(z)$, to be defined presently, is an expression which involves the theta function $\vartheta_\alpha(z)$.

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[1] G. Humbert, *Sur quelques fonctions numeriques remarquables*, C. R. Acad. Sci. Paris vol. 158 (1914) pp. 220, 294, and 1841; vol. 163 (1916) p. 412.

[2] M. A. Basoco, Amer. J. Math. vol. 54 (1932) pp. 242–252.

[3] Whittaker and Watson, *Modern analysis*, Cambridge.

We shall denote the integral functions satisfying these equations by the symbol $H_{ab}^{(\alpha)}(z)$. These are readily found on assuming series solutions of the form.

$$H_{ab}^{(\alpha)}(z) = \sum_{k=-\infty}^{\infty} A_k e^{ikz}.$$

These solutions, for $(a, b) = (1, 0)$, $(0, 1)$ and $(1, 1)$, are unique; for $(a, b) = (0, 0)$, the solution is completely determined to within an additive constant. For suppose that for a given (a, b) there exist two distinct solutions. Denote by $D(z)$ their difference; this would likewise be an integral function and would satisfy periodicity relations of the form

$$D(z + \pi) = (-1)^a D(z),$$
$$D(z + \pi\tau) = (-1)^b D(z).$$

The function $D(z)$ is therefore an elliptic function, reducing by Liouville's theorem to a constant. From (A) it follows easily that this constant vanishes except for the case $(a, b) = (0, 0)$, when it remains undetermined. In this case, we have selected the solution which vanishes for $z = 0$.

3. The functions $F_{ab}^{(\alpha)}(z)$. Let λ and μ be the multipliers associated with the reduction of the theta functions of argument $z + \pi\tau/2$ and $z + \pi\tau$ respectively, so that

$$\lambda = q^{-1/4} e^{-iz}, \qquad \mu = q^{-1} e^{-2iz}$$

where

$$q = \exp \pi i \tau, \qquad 0 < \arg \tau < \pi.$$

The functions $F_{ab}^{(\alpha)}(z)$ are defined as follows:

$$\begin{aligned}
F_{00}^{(0)}(z) &= i(1 - \mu)\vartheta_0(z) - 2i; & F_{11}^{(0)}(z) &= 2i\lambda\vartheta_0(z),\\
F_{00}^{(1)}(z) &= 2i(-1 + i\lambda\vartheta_1(z)); & F_{11}^{(1)}(z) &= i(1 + \mu)\vartheta_1(z),\\
F_{00}^{(2)}(z) &= 2i(-1 + \lambda\vartheta_2(z)); & F_{11}^{(2)}(z) &= i(1 - \mu)\vartheta_2(z),\\
F_{00}^{(3)}(z) &= i(1 + \mu)\vartheta_3(z) - 2i; & F_{11}^{(3)}(z) &= 2\lambda\vartheta_3(z),
\end{aligned}$$

. . . .

$$\begin{aligned}
F_{01}^{(0)}(z) &= i(1 + \mu)\vartheta_0(z); & F_{10}^{(0)}(z) &= 2\lambda\vartheta_0(z),\\
F_{01}^{(1)}(z) &= 2i\lambda\vartheta_1(z); & F_{10}^{(1)}(z) &= -i(1 - \mu)\vartheta_1(z),\\
F_{01}^{(2)}(z) &= 2\lambda\vartheta_2(z); & F_{10}^{(2)}(z) &= i(1 + \mu)\vartheta_2(z),\\
F_{01}^{(3)}(z) &= i(1 - \mu)\vartheta_3(z); & F_{10}^{(3)}(z) &= 2i\vartheta_3(z).
\end{aligned}$$

These functions satisfy the condition

$$F_{ab}^{(\alpha)}(z+\pi) = (-1)^{a} F_{ab}^{(\alpha)}(z),$$

which is implied by equations (A).

4. **The solutions $H_{ab}^{(\alpha)}(z)$.** The procedure indicated in §2 yields the solutions $H_{ab}^{(\alpha)}(z)$ of equations (A) corresponding to the choices of $F_{ab}^{(\alpha)}(z)$ listed in §3. These solutions are valid for all values of z. We thus obtain:

$$H_{00}^{(3)}(z) = 2\sum_{(n)} \frac{q^{n^2+2n} + q^{n^2}}{1 - q^{2n}} \sin 2nz, \tag{4.1}$$

$$H_{00}^{(2)}(z) = 4\sum_{(n)} \frac{q^{n^2+n}}{1 - q^{2n}} \sin 2nz, \tag{4.2}$$

$$H_{01}^{(3)}(z) = 2\sum_{(n)} \frac{q^{n^2+2n} - q^{n^2}}{1 + q^{2n}} \sin 2nz, \tag{4.3}$$

$$H_{01}^{(2)}(z) = 1 + 4\sum_{(n)} \frac{q^{n^2+n}}{1 + q^{2n}} \cos 2nz, \tag{4.4}$$

$$H_{10}^{(3)}(z) = 4\sum_{(m)} \frac{q^{(m^2+2m)/4}}{1 - q^{m}} \sin mz, \tag{4.5}$$

$$H_{10}^{(2)}(z) = 2\sum_{(m)} \frac{q^{(m^2+4m)/4} + q^{m^2/4}}{1 - q^{m}} \sin mz, \tag{4.6}$$

$$H_{11}^{(3)}(z) = 4\sum_{(m)} \frac{q^{(m^2+2m)/4}}{1 + q^{m}} \cos mz, \tag{4.7}$$

$$H_{11}^{(2)}(z) = 2\sum_{(m)} \frac{q^{(m^2+4m)/4} - q^{m^2/4}}{1 + q^{m}} \sin mz. \tag{4.8}$$

In the preceding the index of summation n ranges over all the positive integers while the index m ranges over the odd positive integers.

The remaining functions $H_{ab}^{(c)}(z)$ $(c=0, 1)$ may be obtained from the above results upon replacing z by $z+\pi/2$.

5. **Arithmetized form.** In this section we list the arithmetical forms of the trigonometric series (4.1) to (4.8). The following notation is used: n denotes an arbitrary positive integer; m is an arbitrary positive *odd* integer; α is a positive integer of the form $4k+1$ and β is one of the form $4k+3$; $\sum'$ refers to the conjugate divisors (d, δ) of n and (t, τ) of α or β such that $\delta < d$, $\tau < t$; $\epsilon(n)$ is one or zero according as n is or is not the square of an integer.

$$(5.1)\quad H_{00}^{(3)}(z) = 2\sum_{(n)} q^{n}\epsilon(n)\sin 2n^{1/2}z + 4\sum_{(n)} q^{n}\Big\{\sum{}'\sin 2\delta z\Big\}, \qquad (\delta - d \equiv 0, \bmod 2).$$

$$(5.2)\quad H_{00}^{(2)}(z) = 4\sum_{(n)} q^{n}\Big\{\sum{}'\sin 2\delta z\Big\}, \qquad (\delta - d \equiv 1, \bmod 2).$$

$$(5.3)\quad \begin{aligned} H_{01}^{(3)}(z) = &- 2\sum_{(n)} q^{n}\epsilon(n)\sin 2n^{1/2}z \\ &- 4\sum_{(n)} q^{n}\Big\{\sum{}'(-1)^{(d-\delta)/2}\sin 2\delta z\Big\}, \end{aligned} \qquad (\delta - d \equiv 0, \bmod 2).$$

$$(5.4)\quad H_{01}^{(2)}(z) = 1 + 4\sum_{(n)} q^{n}\Big\{\sum{}'(-1)^{(\delta-d-1)/2}\cos 2\delta z\Big\}, \qquad (\delta - d \equiv 1, \bmod 2).$$

$$(5.5)\quad H_{10}^{(3)}(z) = 4\sum_{(\beta)} q^{\beta/4}\Big\{\sum{}'\sin \tau z\Big\},$$

$$(5.6)\quad H_{10}^{(2)}(z) = 2\sum_{(\alpha)} q^{\alpha/4}\epsilon(\alpha)\sin \alpha^{1/2}z + 4\sum q^{\alpha/4}\Big\{\sum{}'\sin \tau z\Big\},$$

$$(5.7)\quad H_{11}^{(3)}(z) = 4\sum_{(\beta)} q^{\beta/4}\Big\{\sum{}'(-1)^{(t-\tau-2)/4}\cos \tau z\Big\},$$

$$(5.8)\quad \begin{aligned} H_{11}^{(2)}(z) = &- 2\sum_{(\alpha)} q^{\alpha/4}\epsilon(\alpha)\sin \alpha^{1/2}z \\ &- 4\sum_{(\alpha)} q^{\alpha/4}\Big\{\sum{}'(-1)^{(t-\tau)/4}\sin \tau z\Big\}. \end{aligned}$$

The remaining eight functions will not be listed explicitly; they can be obtained quite readily from what precedes.

6. **Identities.** In a former paper[4] the writer obtained the trigonometric developments for the sixteen theta quotients of the form

$$\Theta_{\alpha\beta\gamma}(x, y) \equiv \vartheta_1'^2\,\frac{\vartheta_\alpha(x + y)}{\vartheta_\beta^2(x)\vartheta_\gamma(y)}.$$

A comparison of these expansions (with $(x, y) = (0, 2)$ and $\beta \neq 1$) with those given in the preceding sections leads to the following identities:

$$(6.1)\quad \begin{aligned} \vartheta_1'(z)H_{01}^{(1)}(z) = &\;\vartheta_0^2\vartheta_3^2\vartheta_2(z) \\ &- 4\vartheta_1(z)\sum_{(n)}{}' q^{n}\Big\{\sum{}'(-1)^{(d+\delta-1)/2}(d+\delta)\sin 2\delta z\Big\} \end{aligned} \qquad (\delta - d \equiv 1, \bmod 2),$$

$$(6.2)\quad \begin{aligned} \vartheta_2'(z)H_{01}^{(2)}(z) = &- \vartheta_0^2\vartheta_3^2\vartheta_1(z) \\ &- 4\vartheta_2(z)\sum_{(n)} q^{n}\Big\{\sum{}'(-1)^{(d-\delta-1)/2}(d+\delta)\sin 2\delta z\Big\} \end{aligned} \qquad (\delta - d \equiv 1, \bmod 2),$$

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[4] Loc. cit.

$$\vartheta_3'(z)H_{10}^{(3)}(z) = -\vartheta_2^2\vartheta_3^2\vartheta_3(z) + 2\vartheta_3(z)\sum_{(\beta)} q^{\beta/4}\Big\{\sum{}'(t+\tau)\cos\tau z\Big\}, \tag{6.3}$$

$$\begin{aligned}\vartheta_3'(z)H_{11}^{(3)}(z) &= \vartheta_0^2\vartheta_2^2\vartheta_1(z)\\ &\quad - 2\vartheta_3(z)\sum_{(\beta)} q^{\beta/4}\Big\{\sum{}'(-1)^{(t-\tau-2)/4}(t+\tau)\sin\tau z\Big\},\end{aligned} \tag{6.4}$$

$$\begin{aligned}\vartheta_0'(z)H_{11}^{(0)}(z) &= -\vartheta_0^2\vartheta_2^2\vartheta_2(z)\\ &\quad - 2\vartheta_3(z)\sum_{(\beta)} q^{\beta/4}\Big\{\sum{}'(-1)^{(t+\tau)/4}(t+\tau)\cos\tau z\Big\},\end{aligned} \tag{6.5}$$

$$\begin{aligned}\vartheta_0'(z)H_{10}^{(0)}(z) &= \vartheta_2^2\vartheta_3^2\vartheta_1(z)\\ &\quad - 2\vartheta_0(z)\sum_{(\beta)} q^{\beta/4}\Big\{\sum{}'(-1)^{(\tau-1)/2}(t+\tau)\sin\tau z\Big\},\end{aligned} \tag{6.6}$$

$$\begin{aligned}\vartheta_1'(z)H_{10}^{(1)}(z) &= \vartheta_2^2\vartheta_3^2\vartheta_0(z)\\ &\quad - 2\vartheta_1(z)\sum_{(\alpha)} q^{\alpha/4}\Big\{(-1)^{(\alpha^{1/2}-1)/2}\epsilon(\alpha)\alpha^{1/2}\sin\alpha^{1/2}z\\ &\qquad + \sum{}'(-1)^{(\tau-1)/2}(t+\tau)\sin\tau z\Big\},\end{aligned} \tag{6.7}$$

$$\begin{aligned}\vartheta_1'(z)H_{11}^{(1)}(z) &= \vartheta_0^2\vartheta_2^2\vartheta_3(z)\\ &\quad - 2\vartheta_1(z)\sum_{(\alpha)} q^{\alpha/4}\Big\{(-1)^{(\alpha^{1/2}-1)/2}\epsilon(\alpha)\alpha^{1/2}\sin\alpha^{1/2}z\\ &\qquad + \sum{}'(-1)^{(t+\tau-2)/4}(t+\tau)\sin\tau z\Big\},\end{aligned} \tag{6.8}$$

$$\begin{aligned}\vartheta_2'(z)H_{10}^{(2)}(z) &= -\vartheta_2^2\vartheta_3^2\vartheta_3(z) + 2\vartheta_2(z)\sum_{(\alpha)} q^{\alpha/4}\Big\{\epsilon(\alpha)\alpha^{1/2}\cos\alpha^{1/2}z\\ &\qquad + \sum{}'(t+\tau)\cos\tau z\Big\},\end{aligned} \tag{6.9}$$

$$\begin{aligned}\vartheta_2'(z)H_{11}^{(2)}(z) &= -\vartheta_0^2\vartheta_2^2\vartheta_0(z) + 2\vartheta_2(z)\sum_{(\alpha)} q^{\alpha/4}\Big\{\epsilon(\alpha)\alpha^{1/2}\cos\alpha^{1/2}z\\ &\qquad + \sum{}'(-1)^{(t-\tau)/4}(t+\tau)\cos\tau z\Big\},\end{aligned} \tag{6.10}$$

$$\begin{aligned}\vartheta_0'(z)H_{01}^{(0)}(z) &= \vartheta_0^2\vartheta_3^2\vartheta_3(z) - \vartheta_0(z)\Big\{\psi(z)\\ &\qquad - 4\sum_{(n)} q^{n}\Big\{\sum{}'(-1)^{(d+\delta)/2}(d+\delta)\cos 2\delta z\Big\}\Big\}\\ &\qquad\qquad (\delta - d \equiv 0, \bmod 2),\end{aligned} \tag{6.11}$$

$$\begin{aligned}\vartheta_3'(z)H_{01}^{(3)}(z) &= -\vartheta_0^2\vartheta_3^2\vartheta_0(z) + \vartheta_3(z)\Big\{\phi(z)\\ &\qquad - 4\sum_{(n)} q^{n}\Big\{\sum{}'(-1)^{(\delta-d)/2}(d+\delta)\cos 2\delta z\Big\}\Big\}\\ &\qquad\qquad (\delta - d \equiv 0, \bmod 2),\end{aligned} \tag{6.12}$$

where

$$\psi(z) = 1 - 4\sum_{n=1}^{\infty} (-1)^n n q^{n^2} \cos 2nz$$

and

$$\phi(z) = \psi(z + \pi/2).$$

The above yield, therefore, identities for the product $\vartheta_\alpha'(z)H_{ab}^{(\alpha)}(z)$, the case $(a, b) = (0, 0)$ being, however, excluded. This is because in the expansions for the functions $\Theta_{\alpha\beta\gamma}(x, y)$ with $\beta = 1$ it is not possible to set $x = 0$, this point being a pole of these functions.

7. **Paraphrases.** The preceding set of identities may be paraphrased[5] into rather simple arithmetical equivalents. It is of interest to note that the partitions involved in these paraphrases refer to the representations of numbers in certain linear forms as the sum of five squares.

The notation used is as follows: α, β, m, m_1, n, n_1, t, τ, d, δ are positive integers; $\alpha \equiv 1$ and $\beta \equiv 3 \pmod 4$ while m, m_1, τ are odd; n and n_1 are unrestricted. The sets of conjugate divisors (d, δ) and (t, τ) are subject to the condition $\delta < d$ and $\tau < t$; further restrictions on these divisors will be indicated as needed. h, z_i are unrestricted integers, positive, negative, or zero, while μ, $n_i \gtrless 0$ are odd; the $w_i \leqq 0$, w_i are even integers, positive, negative, or zero. Moreover, $\epsilon(n) = 1$ or 0 according as n is or is not the square of an integer and $\tau(n) = 1$ or 0 accordingly as n is or is not the sum of two integral squares. Finally, the functions $f(x)$ and $g(x)$ are quite arbitrary except that they must be respectively even and odd and be well defined for integral values of the argument.

The partitions of the integers α, β, $2m$, n which appear in our results are as follows:

$$(i) \qquad \alpha = x_1^2 + w_1^2 + w_2^2 + w_3^2 + w_4^2 = \mu^2 + 4d\delta \qquad (\delta - d \equiv 1, \bmod 2),$$

$$(j) \qquad \beta = x_1^2 + x_2^2 + x_3^2 + w_1^2 + w_2^2 = 4h^2 + t\tau,$$

$$(k) \qquad 2m = x_1^2 + x_2^2 + w_1^2 + w_2^2 + w_3^2 = \mu^2 + t\tau = \mu^2 + m_1^2,$$

$$(l) \qquad n = z_1^2 + z_2^2 + z_3^2 + z_4^2 + z_5^2 = h^2 + d\delta = h^2 + n_1^2 \qquad (\delta - d \equiv 0, \bmod 2).$$

The arithmetical equivalents of (6.1) to (6.12) follow in the same order; the necessary partition is denoted by placing the proper letter i, j, k or l under the first $\sum$. In the relations (7.1) and (7.2) the divisors (d, δ) are of opposite parity while in (7.11) and (7.12) they have like parity.

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[5] E. T. Bell, Trans. Amer. Math. Soc. vol. 22 (1921) pp. 1–39 and 198–219.

$$\begin{aligned}\sum_{(i)} (-1)^{(w_1+w_2)/2} f(x_1) = 2\epsilon(\alpha)\alpha^{1/2}(-1)^{(\alpha^{1/2}-1)/2} f(\alpha^{1/2})&\\ +4\sum{}'(-1)^{(d+\delta+\mu)/2}(d+\delta-\mu)f(2\delta+\mu),&\end{aligned}\tag{7.1}$$

$$\begin{aligned}\sum_{(i)} (-1)^{(w_1+w_2+x_1-1)/2} g(x_1) = 2\epsilon(\alpha)\alpha^{1/2} g(\alpha^{1/2})&\\ +4\sum{}'(-1)^{(\delta-d+1)/2}(d+\delta-\mu)g(2\delta+\mu),&\end{aligned}\tag{7.2}$$

$$\sum_{(j)} f(x_1) = 2\sum{}'(t+\tau-4h)f(\tau+2h),\tag{7.3}$$

$$\sum_{(j)} (-1)^{(w_1+w_2+x_1-1)/2} g(x_1) = 2\sum{}'(-1)^{(t-\tau-2)/4}(t+\tau-4h)g(\tau+2h),\tag{7.4}$$

$$\sum_{(j)} (-1)^{(w_1+w_2)/2} f(x_1) = -2\sum{}'(-1)^{(t+\tau+4h)/2}(t+\tau-4h)f(\tau+2h),\tag{7.5}$$

$$\sum_{(j)} (-1)^{(x_1-1)/2} g(x_1) = 2\sum{}'(-1)^{(\tau+2h-1)/2}(t+\tau-4h)g(\tau+2h),\tag{7.6}$$

$$\begin{aligned}\sum_{(k)} (-1)^{w_1/2} f(w_1) = 2\sum{}'(-1)^{(\tau+\mu)/2}(t+\tau-2\mu)f(\tau+\mu)&\\ +2\lambda(2m)\sum (-1)^{(m_1+\mu)/2}(m_1-\mu)f(m_1+\mu),&\end{aligned}\tag{7.7}$$

$$\begin{aligned}\sum_{(k)} (-1)^{(w_1+w_2)/2} f(w_3) = 2\sum{}'(-1)^{(t+\tau+2\mu)/4}(t+\tau-2\mu)f(\tau+\mu)&\\ +2\lambda(2m)\sum (-1)^{(m_1+\mu)/2}(m_1-\mu)f(m_1+\mu),&\end{aligned}\tag{7.8}$$

$$\sum_{(k)} f(w_1) = 2\sum{}'(t+\tau-2\mu)f(\tau+\mu)+2\lambda(2m)\sum (m_1-\mu)f(m_1+\mu),\tag{7.9}$$

$$\begin{aligned}\sum_{(k)} (-1)^{(w_1+w_2+w_3)/2} f(w_1) = 2\sum{}'(-1)^{(t-\tau)/4}(t+\tau-2\mu)f(\tau+\mu)&\\ +2\lambda(2m)\sum (m_1-\mu)f(m_1+\mu),&\end{aligned}\tag{7.10}$$

$$\begin{aligned}\sum_{(l)} (-1)^{z_1+z_2} f(z_1) = f(0)+2\epsilon(n)(-1)^{n^{1/2}} f(n^{1/2})&\\ +4\lambda(n)\sum (-1)^{h+n_1}(h-n_1)f(h+n_1)&\\ -4\sum{}'(-1)^{(d+\delta+2h)/2}(d+\delta-2h)f(\delta+h),&\end{aligned}\tag{7.11}$$

$$\begin{aligned}\sum_{(l)} (-1)^{z_1+z_2+z_3} f(z_1) = f(0)+2\epsilon(n)f(n^{1/2})&\\ +4\lambda(n)\sum (h-n_1)f(h+n_1)&\\ -4\sum{}'(-1)^{(d-\delta)/2}(d+\delta-2h)f(\delta+h).&\end{aligned}\tag{7.12}$$

The preceding formulae with $f(x)=1$ yield enumerations relative to the number of representations of a number as the sum of five squares. The most interesting results are those deduced from (7.3) and (7.9). We thus obtain the following theorems.

THEOREM A. *If* $\beta \equiv 3$, mod 4, *and* $x, y, z \gtrless 0$ *are odd while* u, v *are even integers, positive, negative, or zero, then*

$$N[\beta = x^2 + y^2 + z^2 + u^2 + v^2] = 2\phi(\beta) + 4\sum_{(r)} \phi(\beta - 4r^2),$$

where $r = 1, 2, 3, \cdots, [\beta^{1/2}/2]$, *and* $\phi(n)$ *is the sum of the positive integral divisors of* n.

THEOREM B. *If* $m \equiv 1$, mod 2, *and* $x, y \gtrless 0$ *are odd while* u, v, w *are even and positive, negative, or zero, then*

$$N[2m = x^2 + y^2 + u^2 + v^2 + w^2] = 4\sum_{(s)} \phi(2m - s^2),$$

where $s = 1, 3, 5, 7, \cdots, [(2m)^{1/2}]$, *provided* $2m$ *is not representable as the sum of two squares. If, however,* $2m$ *is so representable, the quantity* $G(2m)$ *must be added to the preceding sum, where* $G(2m) = 4\sum x$, *the sum being extended over all solutions of* $2m = x^2 + y^2$, $x, y > 0$ *and odd.* $\phi(n)$ *is as in the preceding theorem.*

8. **Application to the function** $E(x)$. The analytical form of the functions defined by (4.1) to (4.8) suggests the application of a device due to Hermite,[6] which yields identities involving the greatest integer function $E(x)$. Hermite's method depends on the following generating functions:

$$\frac{u^b}{(1-u)(1-u^a)} = \sum_{(n)} E\left(\frac{n+a-b}{a}\right)u^n;$$

$$\frac{u^b}{(1-u)(1+u^a)} = \sum_{(n)} E_1\left(\frac{n+a-b}{2a}\right)u^n,$$

where a, b are positive integers and

$$E_1(x) = E(2x) - 2E(x) = E(x + 1/2) - E(x).$$

The following two relations are typical of the set of sixteen which may be deduced from our results. Let $F(z)$ be an arbitrary function, r a positive integer and (d, δ) conjugate integral divisors of r. Define $P_1(x, r)$ and $P_2(x, r)$ by the following:

$$P_1(x, r) = \epsilon(r)F(2r^{1/2}x) + 2\sum{}' F(2\delta n) \quad (r = d\delta,\ d - \delta \equiv 0, \text{mod } 2),$$

$$P_2(x, r) = \sum{}' F(2\delta x) \quad (r = d\delta,\ d - \delta \equiv 1, \text{mod } 2).$$

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[6] Hermite, Acta Math. vol. 5 (1884–1885) pp. 297–330; J. Reine Angew. Math. vol. 100 (1887) pp. 51–65; *Oeuvres*, vol. 5, pp. 151–159. See also a paper by the present writer in Bull. Amer. Math. Soc. vol. 42 (1936) pp. 720–726.

Let n be an arbitrary fixed positive integer; then

$$(8.1)\qquad \sum_{r=1}^{n} P_1(x, r) = 2\sum_{(s)} E\left(\frac{n - s^2}{2s}\right)F(2sx) + \sum_{s=1}^{[n^{1/2}]} F(2sx),$$

$$(8.2)\qquad \sum_{r=1}^{n} P_2(x, r) = \sum_{(s)} E\left(\frac{n + s - s^2}{2s}\right)F(2sx),$$

where s ranges over the values $s=1, 2, 3, \cdots$ so long as the argument in $E(x)$ is greater than or equal to 1.

In particular, let $F(z)=z^k$, where k is a positive integer. Then the preceding reduce to the following:

$$(8.3)\qquad \sum_{r=1}^{n} X_1(r) = 2\sum_{(s)} E\left(\frac{n - s^2}{s}\right)s^k + \sum_{s=1}^{[n^{1/2}]} s^k,$$

$$(8.4)\qquad \sum_{r=1}^{n} X_2(r) = \sum_{(s)} E\left(\frac{n + s - s^2}{2s}\right)s^k,$$

where,

$$X_1(r) = \epsilon(r)r^{1/2} + 2\sum{}'\delta^k \qquad (r = d\delta,\ \delta < d,\ \delta - d \equiv 0, \bmod 2),$$
$$X_2(r) = \sum{}'\delta^k \qquad (r = d\delta,\ \delta < d,\ \delta - d \equiv 1, \bmod 2).$$

These results follow from (4.1), (4.2) and their equivalents (5.1) and (5.2) respectively.

THE UNIVERSITY OF NEBRASKA

SIMPLIFIED TECHNIQUE FOR CONSTRUCTING ORTHONORMAL FUNCTIONS

M. O. PEACH

1. Introduction. An orthonormalization process starts with a set of linearly independent functions

$$f_1, f_2, \cdots,$$

and the set of complex conjugate functions

$$\bar{f}_1, \bar{f}_2, \cdots,$$

all defined over a given region R. From these are constructed a set of functions

$$g_1, g_2, \cdots,$$

and the set of complex conjugate functions

$$\bar{g}_1, \bar{g}_2, \cdots,$$

defined over R and such that

$$\int_R g_m \bar{g}_n dR = \begin{cases} 0, & \text{if } m = n, \\ 1, & \text{if } m \neq n. \end{cases}$$

The standard method[1] of constructing orthonormal functions, while completely satisfying logically, has certain practical disadvantages. For example, if the integrations must be done numerically (as would be necessary if either the f_i or the boundary of R were complicated functions, or if the f_i were tabular functions) then the mere tabulation of the intermediate functions which appear becomes burdensome. One would prefer to perform the necessary integrations on the original functions f_i and then proceed by a purely algebraic or numerical process to obtain the g_i. This can be done. If we let N_i be the numerator and D_i the denominator of the orthonormal function g_i, and if we put $F_{ij} = \int_R f_i \bar{f}_j dR$, then the standard orthonormalization process can be shown, by simple algebra, to result in the following:

$$N_1 = f_1, \qquad D_1^2 = F_{11},$$

$$N_2 = \begin{vmatrix} F_{11} & f_1 \\ F_{21} & f_2 \end{vmatrix}, \qquad D_2^2 = F_{11} \cdot \begin{vmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{vmatrix},$$

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[1] Courant and Hilbert, *Methoden der matematischen Physik*, vol. 1, p. 41.

$$N_3 = \begin{Vmatrix} \begin{vmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{vmatrix} & \begin{vmatrix} F_{11} & f_1 \\ F_{21} & f_2 \end{vmatrix} \\ \begin{vmatrix} F_{11} & F_{13} \\ F_{21} & F_{23} \end{vmatrix} & \begin{vmatrix} F_{11} & f_1 \\ F_{31} & f_3 \end{vmatrix} \end{Vmatrix},$$

$$D_3^2 = F_{11} \cdot \begin{vmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{vmatrix} \cdot \begin{Vmatrix} \begin{vmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{vmatrix} & \begin{vmatrix} F_{11} & F_{13} \\ F_{21} & F_{23} \end{vmatrix} \\ \begin{vmatrix} F_{11} & F_{13} \\ F_{21} & F_{23} \end{vmatrix} & \begin{vmatrix} F_{11} & F_{13} \\ F_{31} & F_{33} \end{vmatrix} \end{Vmatrix}.$$

The next quantities N_4 and D_4 would each occupy eight lines; N_5 and D_5 sixteen lines, and so on. The quantity N_4 would be a second order determinant whose elements would also be second order determinants, which in turn would have as elements second order determinants whose elements would be the F_{ij} and f_i. We observe that the construction of N_i and D_i reduces to the successive evaluation of second order determinants. All determinants are obtained from a matrix of the original quantities F_{ij} and f_i by a uniform and simple procedure which is described later as the "process P."

The present paper generalizes the above observation into a technique which (a) provides a definite arrangement of work which minimizes repetition of symbols and lessens the chance of computational errors, (b) can be used by a person whose mathematical training extends no further than a knowledge of the algebraic rules of sign, (c) is adapted to the use of modern computing machines.

To carry through the proof of the method, it is necessary to introduce a more compact notation than that used above. Therefore the proof is somewhat abstract, so we illustrate the extreme simplicity of the final result by an example, namely, the evaluation of the first four Legendre polynomials. The advantages of the method become more pronounced as the number of orthonormal functions desired increases; but space limitations forbid consideration of more than four.

2. **Example.** We shall construct the first four orthonormal functions corresponding to the linearly independent set of functions 1, $x, x^2, \cdots$, and the region $-1 \leqq x \leqq +1$.

Using the standard notation $(f, g) = \int_R f(x)g(x)dR$ we obtain by integration $(1, 1) = 2.000$, $(1, x) = 0$, $(1, x^2) = 0.667$, $(1, x^3) = 0$, $(x, x) = 0.667$, $(x, x^2) = 0$, $(x, x^3) = 0.400$, $(x^2, x^2) = 0.400$, $(x^2, x^3) = 0$, $(x^3, x^3) = 0.286$. With these values as elements we form a symmetric determinant of order four, and adjoin to it the unit matrix of order four, thus obtaining the following 4×8 Matrix No. 1:

$$\left\|\begin{array}{cccc|cccc} 2.000 & 0 & 0.667 & 0 & 1.000 & 0 & 0 & 0 \\ 0 & 0.667 & 0 & 0.400 & 0 & 1.000 & 0 & 0 \\ 0.667 & 0 & 0.400 & 0 & 0 & 0 & 1.000 & 0 \\ 0 & 0.400 & 0 & 0.286 & 0 & 0 & 0 & 1.000 \end{array}\right\|.$$

Now for every element not in either the first row or first column we can define a determinant of order two by blotting out all rows and all columns except those containing either the element under consideration or the leading element of the matrix. For example, associated with the element 0.400 appearing in the third row and third column is the determinant

$$\left|\begin{array}{cc} 2.000 & 0.667 \\ 0.667 & 0.400 \end{array}\right| = +\,0.356.$$

All such determinants are evaluated and the quantities so obtained are placed in the same position as their associated elements of Matrix No. 1, thus forming a 3×7 Matrix No. 2:

$$\left\|\begin{array}{ccc|cccc} +1.334 & 0 & +0.800 & 0 & +2.000 & 0 & 0 \\ 0 & +0.356 & 0 & -0.667 & 0 & +2.000 & 0 \\ +0.800 & 0 & +0.571 & 0 & 0 & 0 & +2.000 \end{array}\right\|.$$

This is treated in the same way as Matrix No. 1, thus forming the 2×6 Matrix No. 3:

$$\left\|\begin{array}{cc|cccc} +0.474 & 0 & -0.889 & 0 & +2.667 & 0 \\ 0 & +0.122 & 0 & -1.600 & 0 & +2.667 \end{array}\right\|.$$

In similar fashion we form the 1×5 Matrix No. 4:

$$\left\|\begin{array}{c|cccc} +0.058 & 0 & -0.759 & 0 & +1.264 \end{array}\right\|.$$

The rth orthonormal function can now be written down as a fraction whose denominator is the square root of the product of the leading elements of all the matrices up to and including the rth matrix, and whose numerator is a polynomial in the original functions $1, x, x^2, \cdots$, with coefficients the last n elements in the top row of the rth matrix. The first four are:

$$g_1(x) = \frac{1\cdot 1 + 0\cdot x + 0\cdot x^2 + 0\cdot x^3}{(2.000)^{1/2}} = 0.707,$$

$$g_2(x) = \frac{0\cdot 1 + 2.000x + 0\cdot x^2 + 0\cdot x^3}{(2.000 \times 1.334)^{1/2}} = 1.222x,$$

$$g_3(x) = \frac{(-0.889)\cdot 1 + 0\cdot x + 2.667x^2 + 0\cdot x^3}{(2.000 \times 1.334 \times 0.474)^{1/2}} = -0.790 + 2.37x^2,$$

$$g_4(x) = \frac{0\cdot 1 + (-0.759)x + 0\cdot x^2 + 1.264x^3}{(2.000 \times 1.334 \times 0.474 \times 0.058)^{1/2}} = -0.280x + 4.67x^3.$$

In actual computation the original matrix is laid out on a large sheet with sufficient space to enter the new quantities as they are determined, so that when the computation is finished all n matrices appear superimposed upon the same sheet. Two L-shaped guides prevent errors in picking out the elements of each determinant. It is seen that the unit operation is the evaluation of a second order determinant, which operation requires but slight mathematical knowledge. This operation can be performed as a single step on most modern computing machines without writing down the intermediate products.

3. **The process** P. Let us be given an nth order determinant

$$D^1 = | d^1_{p,q} |, \qquad p, q = 1, \cdots, n,$$

where the element $d^1_{p,q}$ lies in the pth row and qth column. We define the process P by the functional equation

$$P(d^1_{p,q}) = d^2_{p,q}, \qquad p, q = 2, \cdots, n,$$

where

$$d^2_{p,q} = \begin{vmatrix} d^1_{1,1} & d^1_{1,q} \\ d^1_{p,1} & d^1_{p,q} \end{vmatrix}.$$

Notice that $P(d^1_{p,1})$ and $P(d^1_{1,q})$ are not defined. Suppressing these undefined quantities we form a determinant of order $n-1$

$$D^2 = | d^2_{p,q} |, \qquad p, q = 2, \cdots, n.$$

Ordinarily, we would regard this determinant as possessing rows and columns numbered $1, 2, \cdots, (n-1)$. It is more convenient for our present purpose to regard the first row and first column as being absent, so we agree instead that D^2 contains rows and columns numbered $2, 3, \cdots, n$. With this convention, we complete the definition of D^2 by saying that the element $d^2_{p,q}$ lies in the pth row (row numbered p) and qth column (column numbered q) of the determinant D^2.

In a similar way we define

$$P^2(d^1_{p,q}) = P(d^2_{p,q}) = \begin{vmatrix} d^2_{2,2} & d^2_{2,q} \\ d^2_{p,2} & d^2_{p,q} \end{vmatrix} = d^3_{p,q}, \qquad p, q = 3, \cdots, n,$$

and, in general:

$$P^{r-1}(d^{1}_{p,q}) = P(d^{r-1}_{p,q}) = \begin{vmatrix} d^{r-1}_{r-1,r-1} & d^{r-1}_{r-1,q} \\ d^{r-1}_{p,r-1} & d^{r-1}_{p,q} \end{vmatrix} = d^{r}_{p,q}, \qquad p, q = r, \cdots, n.$$

Also:

$$P^{r-1}(D^{1}) = D^{r} = | d^{r}_{p,q} |, \qquad p, q = r, \cdots, n.$$

Notice that $P(d^{r}_{p,s})$ and $P(d^{r}_{s,q})$ are not defined, when $s<r$. We agree that D^r is of order $n-r+1$ and contains rows and columns numbered $r, r+1, \cdots, n$, the element $d^{r}_{p,q}$ lying in the pth row (row numbered p) and the qth column (column numbered q). This convention as to the suppression of undefined quantities and the numbering of elements will be extended to all determinants used in this paper.

Suppose now that the elements in the nth column of D^1 are integrable functions defined over a given region R, while all other elements are constants. Denote these functions by

$$d^{1}_{p,n} = f_p, \qquad 1 \leqq p \leqq n.$$

Let g be an arbitrary integrable function defined over R. Then, by the simple properties of determinants, it follows easily that

(a) The elements in the nth column of D^r are linear combinations of the functions f_i,

(b) $\int_R g \cdot D^r dR$ differs from D^r only in that every f_i appearing in D^r is replaced by the quantity $\int_R g \cdot f_i dR$,

(c) $\int_R g \cdot P^r(D^1) dR = P^r(\int_R g \cdot D^1 dR)$.

4. Proof of the method. Let us be given a linearly independent set of complex functions of real variables defined in a given region R, and integrable over R:

$$f = f_1, f_2, \cdots, f_n, \cdots.$$

Form the set of conjugate complex functions:

$$\bar{f} = \bar{f}_1, \bar{f}_2, \cdots, \bar{f}_n, \cdots.$$

Define the three determinants of order n:

$$D^{1} = | d^{1}_{p,q} |, \quad d^{1}_{p,q} = \int_R f_p \bar{f}_q dR, \qquad p, q = 1, \cdots, n,$$

$$F^{1} = | f^{1}_{p,q} |, \quad f^{1}_{p,q} = d^{1}_{p,q}, \quad q \neq n, \qquad p, q = 1, \cdots, n,$$

$$f^{1}_{p,n} = f_p,$$

$$\bar{F}^1 = |\bar{f}^1_{p,q}|, \quad \bar{f}^1_{p,q} = d^1_{p,q}, \quad p \neq n, \qquad p, q = 1, \cdots, n,$$
$$\bar{f}^1_{n,q} = \bar{f}_q.$$

Define $F^r = P^{r-1}(F^1)$ and $\bar{F}^r = P^{r-1}(\bar{F}^1)$. Define $f^r_{p,q}$ as the element lying in the pth row and qth column of F^r, and $\bar{f}^r_{p,q}$ as the element lying in the pth row and qth column of $\bar{F}^r$.

We now prove in succession a set of results to culminate in our main theorem, Theorem J, and its corollary.

THEOREM A. *$\int_R f^r_{p,n} \cdot \bar{f}_r dR = d^r_{p,r}$, if $r \leqq p \leqq n$.*

PROOF. $\int_R f^r_{p,n} \cdot \bar{f}_r dR = \int_R P^{r-1}(f^1_{p,n}) \cdot \bar{f}_r dR = P^{r-1}(\int_R F_p \cdot \bar{f}_r dR)$ $= P^{r-1} D^1_{p,r} = d^r_{p,r}$, by (c) of §3 and definitions.

THEOREM B. *$\int_R f^{r+1}_{p,n} \cdot \bar{f}_r dR = 0$, if $r+1 \leqq p \leqq n$.*

PROOF:

$$\int_R f^{r+1}_{p,n} \cdot \bar{f}_r dR = \int_R \begin{vmatrix} d^r_{r,r} & f^r_{r,n} \\ d^r_{p,r} & f^r_{p,n} \end{vmatrix} \cdot \bar{f}_r dR,$$

$$= \begin{vmatrix} d^r_{r,r} & \int_R f^r_{r,n} \cdot \bar{f}_r dR \\ d^r_{p,r} & \int_R f^r_{p,n} \cdot \bar{f}_r dR \end{vmatrix} = 0,$$

since, by Theorem A, the last column is identical to the first.

THEOREM C. *$\int_R f^r_{p,n} \cdot \bar{f}_s dR = 0$, if $s < r \leqq p \leqq n$.*

PROOF. Assume the theorem true for $r = k$. Then

$$\int_R f^{k+1}_{p,n} \cdot \bar{f}_s dR = \begin{vmatrix} f^k_{k,k} & \int_R f^k_{k,n} \cdot \bar{f}_s dR \\ f^k_{p,k} & \int_R f^k_{p,n} \cdot \bar{f}_s dR \end{vmatrix}.$$

By hypothesis the elements in the last column are zeros, and thus the theorem holds for $k+1$. By Theorem B it holds for $r = s+1$, hence it holds for all r satisfying the conditions of the theorem.

THEOREM D. *$\int_R f^n_{n,n} \cdot \bar{f}_s dR = 0$ if $s < n$ and equals $d^n_{n,n}$ if $s = n$.*

This follows from Theorems A and C by placing $r = p = n$.

THEOREM E. *An analogue to each of the previous theorems is obtained by taking conjugates of each side of the respective equations.*

THEOREM F. $\int_R f^n_{n,n}\cdot\bar{f}^n_{n,n}dR=\prod_{s=1}^n d^s_{s,s}$.

PROOF. Write

$$\bar{f}^n_{n,n}=\sum_{s=1}^n A_s\bar{f}_s, \qquad A_s = \text{constants}.$$

Then

$$\int_R f^n_{n,n}\sum_{s=1}^n A_s\bar{f}_s dR=\sum_{s=1}^n A_s\int_R f^n_{n,n}\cdot\bar{f}_s dR=\sum_{s=1}^n A_sQ_s,$$

where

$$Q_s=\begin{cases}0, & \text{if } s\neq n,\\ d^n_{n,n}, & \text{if } s=n.\end{cases}$$

We see that if we replace $\bar{f}_s$ by Q_s we change $\bar{f}^n_{n,n}$ into $\int_R f^n_{n,n}\cdot\bar{f}^n_{n,n}dR$. Suppose we make this replacement in the determinant $\bar{F}^1$ forming a new determinant $\bar{Q}^1$. Then since $P^{n-1}(\bar{F}^1)=\bar{F}^n$ it follows that $P^{n-1}(\bar{Q}^1)=\bar{Q}^n$ where the element $\bar{q}^n_{n,n}=\int_R f^n_{n,n}\cdot\bar{f}^n_{n,n}dR$ of $\bar{Q}^n$ corresponds to the element $\bar{f}^n_{n,n}$ of $\bar{F}^n$. Consider $\bar{Q}^1$. It is identical to $\bar{D}^1$ except that all the elements in the bottom row, except the one in the last column, are replaced by zeros. Therefore the process P applied to $\bar{q}^1_{n,n}$ of this determinant reduces to multiplication. We have

$$P(\bar{q}^1_{n,n})=(d^n_{n,n})(d^1_{1,1}),$$
$$P^2(\bar{q}^1_{n,n})=(d^n_{n,n})(d^1_{1,1})(d^2_{2,2}),$$
$$\cdots\cdots\cdots\cdots\cdots,$$
$$P^{n-1}(\bar{q}^1_{n,n})=\prod_{s=1}^n d^s_{s,s}.$$

THEOREM G. *Suppose that in addition to the determinants already mentioned we have a determinant $\bar{F}^{o1}$ defined exactly the same as $\bar{F}^1$ except that it is of order m where $m\neq n$. Then*

$$\int_R f^n_{n,n}\cdot\bar{f}^{om}_{m,m}dR=0, \qquad m\neq n.$$

PROOF. Suppose $m<n$. Then

$$\bar{f}^{om}_{m,m}=\sum_{s=1}^m A^o_s\bar{f}_s, \qquad A^o_s = \text{constants},$$

$$\int_R f^n_{n,n}\sum_{s=1}^m A^o_s\bar{f}_s dR=\sum_{s=1}^m A^o_s\int_R f^n_{n,n}\cdot f_s dR=0,$$

by Theorem D.

If $m>n$ we expand $f^n_{n,n}$ as a linear combination of the functions f_s, $s=1, 2, \cdots, n$, and use the conjugate theorem of Theorem D.

THEOREM H. *The function*

$$g_n = f^n_{n,n} \Big/ \left(\prod_{s=1}^{n} d^s_{s,s} \right)^{1/2}$$

is the nth orthonormal function for the given set f and the given domain R.

This follows immediately from Theorems F and G.

THEOREM J. *Let us adjoin the unit matrix of order n to the determinant D^1 and apply the process P^{n-1} to the combined matrix. Form a matrix whose first row is the top row of the unit matrix, whose second row is the top row of the first transform of the unit matrix, and, in general, whose rth row is the top row of the $(r-1)$st transform of the unit matrix. Let this matrix be denoted by*

$$\begin{Vmatrix} B_{1,1} & B_{1,2} \cdots & B_{1,n} \\ B_{2,1} & B_{2,2} \cdots & B_{2,n} \\ \cdot & \cdot \quad \cdots & \cdot \\ B_{n,1} & B_{n,2} \cdots & B_{n,n} \end{Vmatrix}.$$

Then the nth orthonormal function is given by:

$$g_n = \frac{B_{n,1}f_1 + B_{n,2}f_2 + \cdots + B_{n,n}f_n}{(d^1_{1,1})(d^2_{2,2}) \cdots (d^n_{n,n})}.$$

PROOF. We know $f^n_{n,n} = k_{n,1}f_1 + k_{n,2}f_2 + \cdots + k_{n,n}f_n$, where the $k_{n,s}$ are constants. To evaluate $k_{n,s}$ we put $f_s=1$ and all other $f=0$ and see what happens to $f^n_{n,n}$. Suppose we make these replacements in F^1 forming a new determinant K^1, that is, we replace the nth column of F^1 by the sth column of the unit matrix. Then $P^n(F^1)$ is identical to $P^n(K^1)$ except that its nth column is replaced by the sth column of the transformed unit matrix. In particular the element $f^n_{n,n}$ is replaced by the quantity $B_{n,s}$. Hence $k_{n,s}=B_{n,s}$ and the theorem is proved.

COROLLARY. *In Theorem J it is also true that*

$$g_r = \frac{B_{r,1}f_1 + B_{r,2}f_2 + \cdots + B_{r,n}f_n}{(d^1_{1,1})(d^2_{2,2}) \cdots (d^r_{r,r})},$$

where $r<n$.

PROOF. Apply Theorem J for the case $n=r$. We obtain for g_r an expression which differs from the one just written only in the fact that the terms $B_{r,r+1}f_{r+1}+B_{r,r+2}f_{r+2}+\cdots+B_{r,n}f_n$ are missing from its numerator. But the coefficients $B_{r,s}=0$ when $r<s$. Hence the two expressions are equal.

REMARK. The generalization of the method to orthonormalization with respect to a general norming or weight function p is obvious. One applies the process described to the functions $(p)^{1/2}f_i$ and obtains functions g_i (=linear combinations of the $(p)^{1/2}f_i$) which are orthonormal with respect to the weight function unity. Dividing through by the common factor $(p)^{1/2}$ one forms functions $g_i(p)^{-1/2}$ which are orthonormal with respect to p.

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VERTICES OF PLANE CURVES

S. B. JACKSON

1. Introduction. The Four-Vertex Theorem, proved first by Mukhopadhyaya [1],[1] states that on any oval, not a circle, there are at least four vertices, that is, extrema of the curvature. This result was extended by Fog [2] and Graustein [3] to any simple closed curve with continuous curvature. The discussion by Graustein makes it clear that the Four-Vertex Theorem is valid also for a very large number of non-simple curves. Indeed the class of curves having only two vertices is relatively quite small. The main object of the present paper is to characterize geometrically, as far as possible, the curves with just two vertices. It is thus a proof of the Four-Vertex Theorem by exclusion.

Since a curve with just two vertices consists of two arcs of monotone curvature, a study is made of such arcs (§2). The most useful fact is that this monotone character of an arc is invariant under direct circular transformations. The property that a point be a vertex of a curve is similarly invariant. This makes it possible to simplify many of the discussions by suitably chosen transformations. Monotone arcs are found to be essentially simple and possess a spiral character.

The existence of vertices on certain types of arcs is established (§4)

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[1] Numbers in brackets refer to the Bibliography at the end of the paper.

and these results applied to give two new proofs of the theorem of Fog and Graustein mentioned above. It is then possible to obtain five results describing the structural character of curves with just two vertices (Theorem 5.1).

It is well known that an oval which meets any circle at most four times has exactly four vertices [4, 5]. This result is generalized to simple closed curves (Theorem 6.1), and two other results are obtained concerning the relation of such a curve to its extreme circles of curvature (§6). Another familiar result on ovals, namely that an oval which meets a circle $2n$ times has at least $2n$ vertices [6], is also extended to simple closed curves (§7). In this case an additional condition on the cyclic character of the points of intersection is necessary since the curve need not be convex.

At several points in the discussion it is convenient to make use of the following result (Lemma 3.1), which is of some interest in its own right. If a Jordan curve bounding a simply connected region is divided in any manner into three arcs, there exists a circle interior to the region and having points in common with all three arcs.

2. Vertices, and arcs of monotone curvature. By a vertex of a curve of class C'' is meant a point or a circular arc of the curve for which the curvature is a relative extremum with respect to the neighboring arcs. The term *relative extremum* is to be understood in the following sense. If $1/A$ denotes the curvature at the given point (or arc) and $1/R$ the curvature at an arbitrary point, then in a sufficiently small neighborhood of the point (arc) $1/R-1/A\geqq 0$ or $\leqq 0$. The equality sign may be valid at any number of points on the adjacent arcs, but is not identically true for either of them.

An arc on which the curvature is monotone nondecreasing or monotone nonincreasing has no vertices, and conversely, an arc with no vertices has this property. For brevity such arcs will be referred to as monotone arcs. The term arc in this paper is used to denote a map of a line segment which is locally topological but which is not necessarily simple in the large. Thus any segment of a curve will be referred to as an arc of the curve. An arc without double points is a simple arc. In the following discussion all arcs and curves will be understood to be of class C'' unless the contrary is expressly stated.

LEMMA 2.1. *Let two curves, C_1 and C_2, be tangent in the same direction at a point or coincide along an arc, but not coincide along the neighboring arcs. If $1/R_i$ denotes the curvature of C_i and if $1/R_1\geqq 1/R_2$ in a neighborhood of the given point or arc, then in a sufficiently small neighborhood C_1 lies entirely to the left of C_2 except for the given point or arc of contact.*

Let P_0 be the given point or the last point of the given arc of coincidence, and let s be the arc length on C_i measured from P_0. It will be sufficient to prove the lemma for positive values of s, for if the directions on the C_i are reversed the same proof will then apply to the arcs in that direction. If P_0 is taken as the origin of coordinates and the common directed tangent as the x-axis, the curves C_i are given locally by the equations

$$(2.1) \qquad x_i = \int_0^s \cos \phi_i ds, \qquad y_i = \int_0^s \sin \phi_i ds,$$

where $\phi_i = \int_0^s ds/R_i$.

Since $1/R_1 \geqq 1/R_2$ and the equality sign is not identically true, it follows that for all values of s greater than zero

$$\phi_1(s) = \int_0^s ds/R_1 > \int_0^s ds/R_2 = \phi_2(s).$$

By (2.1) this implies at once the following relation:

$$(2.2) \qquad y_1 = \int_0^s \sin \phi_1 ds > \int_0^s \sin \phi_2 ds = y_2.$$

Thus if P_1 and P_2 denote the points on C_1 and C_2 corresponding to the same arc length s, then $P_1 \neq P_2$ for all sufficiently small positive s. The slope of the line P_1P_2 may be computed from (2.1) and the limit of this slope as s approaches zero found by l'Hospital's Rule,

$$\begin{aligned} \lim_{s\to 0} m_{P_1P_2} &= \lim_{s\to 0} \frac{\int_0^s \sin \phi_1 ds - \int_0^s \sin \phi_2 ds}{\int_0^s \cos \phi_1 ds - \int_0^s \cos \phi_2 ds} = \lim_{s\to 0} \frac{\sin \phi_1 - \sin \phi_2}{\cos \phi_1 - \cos \phi_2} \\ &= \lim_{s\to 0} \frac{2 \cos 2^{-1}(\phi_1 + \phi_2) \sin 2^{-1}(\phi_1 - \phi_2)}{-2 \sin 2^{-1}(\phi_1 + \phi_2) \sin 2^{-1}(\phi_1 - \phi_2)} = \infty. \end{aligned}$$

By taking s sufficiently small the lines P_1P_2 can be made as nearly parallel to the y-axis as we desire. But since C_1 and C_2 are both perpendicular to the y-axis at P_0, it is clear that for sufficiently small s the lines P_1P_2 meet each C_i only once in this neighborhood. Inasmuch as $P_1 \neq P_2$, C_1 and C_2 cannot meet in this neighborhood. Since by (2.2), $y_1 > y_2$, P_1 lies always above, that is, to the left of C_2, and the lemma is proved.

COROLLARY 2.1.1. *In a neighborhood of a vertex of a curve, the circle of curvature lies entirely to the left or entirely to the right of the curve according as the curvature at the vertex is a maximum or a minimum.*

This is an immediate consequence of the lemma with the curve and its circle of curvature taken as the curves C_i.

COROLLARY 2.1.2. *At every interior point (or circular arc) of a monotone arc, $\mathcal{A}$, the arc crosses its circle of curvature. The crossing is from right to left or from left to right according as the curvature is monotone nondecreasing or monotone nonincreasing.*

This also is a direct consequence of the lemma if the curves C_i are taken as follows. Let one of them be the path traced by a moving point which follows the circle of curvature as it approaches the contact point (or arc) and follows $\mathcal{A}$ as it leaves this contact, while the other is the path traced if the point follows $\mathcal{A}$ as it approaches the contact, and the circle as it leaves it.

LEMMA 2.2. *If, at every interior point (or circular arc), an arc, $\mathcal{A}$, crosses its circle of curvature, then $\mathcal{A}$ is a monotone arc.*

For if $\mathcal{A}$ were not monotone it would have vertices, which is impossible by Corollary 2.1.1. Since Lemma 2.2 and Corollary 2.1.2 completely characterize the monotone arcs and since a point not interior to a monotone arc is necessarily either a vertex or a limit point of vertices, the following result is established by exclusion.

LEMMA 2.3. *If for a point (or circular arc) interior to an arc, $\mathcal{A}$, the circle of curvature lies locally entirely to one side of $\mathcal{A}$ then this point (or arc) is either a vertex or a limit point of vertices.*

LEMMA 2.4. *A monotone arc of nondecreasing (nonincreasing) curvature is carried into a monotone arc of nondecreasing (nonincreasing) curvature by any direct circular transformation. A vertex of an arc is carried by a direct circular transformation either into a vertex of the same kind on the transformed arc or into a limit point of such vertices.*

This result follows immediately from the fact that Corollary 2.1.1, Corollary 2.1.2, Lemma 2.2, and Lemma 2.3 characterize vertices and monotone arcs in terms that are invariant under direct circular transformations. This fact affords a simple proof of the following well known result.

LEMMA 2.5. *If P_0 is any point of a monotone arc $\mathcal{A}$ and P is any point further along $\mathcal{A}$ in the positive direction, the circle of curvature at P lies to the left or to the right of the circle of curvature at P_0 according as the curvature on $\mathcal{A}$ is nondecreasing or nonincreasing. The circles have no point in common unless they are identical and $\mathcal{A}$ contains the circular arc from P_0 to P.*

It is clearly sufficient to prove the lemma for points P in an arbitrarily small neighborhood of P_0. Suppose the circles of curvature at P and P_0 have a point Q in common. By a direct circular transformation let Q be carried into the point at infinity. The transformed arc, $\mathcal{A}'$, then has zero curvature at the transforms of P_0 and P, which, by the monotone character of $\mathcal{A}'$, can occur only if the curvature is zero on the entire arc between the two points. The lemma follows by transforming back to $\mathcal{A}$. This argument is valid even if Q coincides with P_0. The fact that the circle of curvature at P lies to the left or right of that at P_0 according as the curvature is nondecreasing or nonincreasing is a consequence of Corollary 2.1.2.

COROLLARY 2.5.1. *A monotone arc $\mathcal{A}$ is simple unless it contains a complete circle traced one or more times. In this latter case it is tangent to itself without crossing at a single point or along a single arc of this circle.*

COROLLARY 2.5.2. *If a monotone arc, $\mathcal{A}$, has positive curvature, it is an inwinding or an outwinding spiral according as the curvature is nondecreasing or nonincreasing. It is simple except for any complete circles it may contain.*

Both corollaries are immediate consequences of the lemma. For Corollary 2.5.2 the spiral character follows from the fact that for a point of positive curvature the inside of the circle of curvature lies to the left of the arc.

3. **A lemma on circles.** At several points in the remainder of the paper it will be convenient to make use of the following lemma.

LEMMA 3.1. *Let $\mathfrak{R}$ be a closed simply connected region of the plane bounded by a Jordan curve, and let the Jordan curve be divided into three arcs, $\mathcal{A}_1$, $\mathcal{A}_2$, $\mathcal{A}_3$. Then there exists a circle contained in $\mathfrak{R}$ and having points in common with all three arcs.*

The following simple proof is due to Paul Erdös and is given in preference to the more complicated proof originally given by the author.

Let A_i be the set of points of $\mathfrak{R}$ whose distances from arc $\mathcal{A}_i$ do not exceed their distances from the other $\mathcal{A}$'s. The sets A_i are clearly closed. Moreover they are connected, for if X, $Y \in A_i$ and if P, Q are the corresponding nearest points of $\mathcal{A}_i$, then the line segments XP and YQ and the arc PQ of $\mathcal{A}_i$ all belong to A_i. Now $A_1 \cup A_2 \cup A_3 = \mathfrak{R}$ and since each of the arcs has an end point in common with the other two, $A_i \cap A_j \neq 0$. Thus $A_1 \cup A_2$ is a closed connected set.

Suppose $A_1 \cap A_2 \cap A_3 = 0$. Then the intersection of A_3 with $A_1 \cup A_2$ consists of the two disjoint nonvacuous closed sets $A_1 \cap A_3$ and $A_2 \cap A_3$, and is therefore not connected. This is impossible since it contradicts the unicoherence of $\mathcal{R}$.[2] Thus there is some point P_0 in $A_1 \cap A_2 \cap A_3$. By definition, the distances from P_0 to the three $\mathcal{A}_i$ have the same value, say r. The circle with center at P_0 and radius r is the required circle.

Corollary 3.1.1. *If a simply connected region is bounded by n differentiable arcs* ($n \geq 3$), *and if the angles formed by the arcs interior to the region are all less than or equal to* π, *then there exists a circle contained in the region and tangent to at least three of the arcs.*

If two consecutive arcs are taken as the arcs $\mathcal{A}_1$ and $\mathcal{A}_2$ of Lemma 3.1 and all the other arcs together taken as $\mathcal{A}_3$, then by the lemma there is a circle interior to the region and having points in common with at least three of the arcs. If, at their common point, two consecutive arcs make an angle less than π interior to the region, there is no circle through this point which lies interior to the region. Such a point thus cannot be on the given circle. But if a circle meets a differentiable arc at an interior point without crossing it, they are tangent, whence all the points common to the circle and the arcs are tangencies. Two of the contact points may coincide, for if two consecutive arcs make an angle of π at their common point, the circle may be tangent to them both at this point.

4. **Location of vertices on a curve.** The following lemma establishes the existence of vertices on certain types of arcs, and is of fundamental importance in the succeeding work.

Lemma 4.1. *If a simple, noncircular arc, AB, is tangent to a circle (or line) in the same direction at A and B, and never crosses this circle (line), there is a maximum or a minimum of curvature interior to AB according as AB lies to the right or left of the given directed circle (line).*

It is sufficient to prove the lemma for the case when AB meets the circle only at A and B, since otherwise AB may be replaced by a subarc which does have this property. Suppose AB lies to the left of the directed circle. Let a point P_0 of the directed circular arc BA be carried into the point at infinity by a direct circular transformation. The circle goes into a straight line and the arc AB into an arc $A'B'$ tangent to this directed line at A' and B' and lying to its left. The points A', B' appear in that order on the directed line. It is clear, from the Gauss-Bonnet formula, that the angular measure of the arc $A'B'$ is zero. Since at A' and B' the arc lies to the left of the line there are points of positive curvature arbitrarily near A' and B', but since the angular measure is zero there must be points of negative curvature on $A'B'$. Thus there is at least one minimum of curvature interior to $A'B'$. By Lemma 2.4 this implies that there is a minimum interior to AB. This completes the proof of the lemma when AB lies to the left of the circle. The other case reduces to this by reversing the direction on the arc.

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[2] A continuum, that is, a closed connected set, is said to be unicoherent if, when it is written in any manner as the sum of two continua, C_1 and C_2, the set $C_1 \cap C_2$ is also a continuum. This is a well known property of closed simply connected regions of the plane.

COROLLARY 4.1.1. *If a simple, noncircular arc, AB, is tangent to a circle (or line) in the same direction at A and B and lies to the left of the circle (line) near A and B there is at least one minimum of curvature interior to AB. If AB lies to the right of the circle near A and B there is at least one maximum of curvature interior to AB.*

Let AB lie locally to the left of the given circle at A and B. By adjoining the directed circle arc, BA, to AB a closed curve $\overline{C}$ of class C' is obtained. Since, by a direct circular transformation, this can be transformed into a curve where BA is a line segment, the proof will be confined to this case.

If there are points of negative curvature the result is trivial, since there are points of positive curvature arbitrarily close to A and B. Suppose then that the curvature is positive and consider the smallest circle containing $\overline{C}$. There is a smallest such circle since $\overline{C}$ is a closed set, and there must be at least two distinct contact points of $\overline{C}$ with this circle, P and Q. Neither of the contact points can be on the line segment BA for there would then be points of $\overline{C}$ outside the circle. Arc AB is simple so the tangencies are in the same direction at P and Q. Since AB has positive curvature and is interior to the circle, it lies to the left of the directed circle. The arc PQ therefore has a minimum of curvature by Lemma 4.1, or if PQ is a circular arc it is a minimum of curvature itself. This proves the first part of the corollary. The second part may be reduced to this by reversing the direction on AB.

An arc AB for which $A=B$ will be called a closed arc. If it contains no other double points it is a simple closed arc.

LEMMA 4.2. *A closed arc, not a circle, contains at least one vertex interior to the arc.*

For if an arc has no vertices it is monotone, and by Lemma 2.5 the first and last points can coincide only if the entire arc is a circle traced one or more times.

LEMMA 4.3. *Let AB be a simple closed arc, not a circle, bounding a region $\mathfrak{R}$, and let θ be the angle interior to $\mathfrak{R}$ between the two tangents at the double point. If $\theta \leqq \pi$, there is a maximum or a minimum of curvature interior to AB according as $\mathfrak{R}$ lies to the left or the right of the arc. If $\theta \geqq \pi$, there is a maximum or a minimum of the curvature interior to AB according as $\mathfrak{R}$ lies to the right or left of the arc.*

First let θ be less than or equal to π. Let AB be divided into three arcs, AP, PQ, QB, so that PQ is not an arc of a circle. This can be done since AB is not a circle. By Corollary 3.1.1 there is a circle entirely in $\mathfrak{R}$ and tangent to all three of these arcs. The subarc of AB from the contact point on AP to the contact point on QB satisfies the conditions of Lemma 4.1, whence the lemma follows at once since the arc is to the right or left of the circle according as $\mathfrak{R}$ is to the left or right of the arc. The case when $\theta \geqq \pi$ may be reduced at once to the above by a direct circular transformation taking a point of $\mathfrak{R}$ into the point at infinity. It should be noted that under this transformation $\mathfrak{R}$ corresponds to the exterior, not the interior of the transformed arc.

When $\theta = \pi$, that is, when the directed tangents at A and B coincide, both parts of Lemma 4.3 apply, and we obtain the following special case.

COROLLARY 4.3.1. *If AB is a simple closed arc, not a circle, and the directed tangents at A and B coincide, there is both a maximum and a minimum of curvature interior to the arc.*

Lemma 4.1 is a direct generalization of the results obtained by Graustein [3] on arcs of type Ω. The material of this section also affords an easy proof of the following result of Graustein [3] and Fog [2].

THEOREM 4.1. *Every simple closed curve, C, not a circle, has at least four vertices.*

Let C be directed so that its interior is to the left of the curve, and let P_0 be the point of absolute minimum curvature. Then C, considered as a simple closed arc from P_0 to P_0, satisfies the conditions of Lemma 4.3, and so has a minimum interior to the arc. Thus there are two minima of the curvature on C, whence there are two maxima and so four vertices.

An alternative proof can be given as follows. Let C be divided into three arcs, none of them merely circular arcs. By Corollary 3.1.1 there is a circle entirely inside C and tangent to each of the three arcs. At least two of the contact points are distinct and all tangencies are in the same direction. The arcs into which these contact points divide C both satisfy Lemma 4.1, whence C has two minima (or two maxima) and thus four vertices.

5. **Curves with two vertices.** If a curve, C, is to have exactly two vertices it must consist of two monotone arcs, whose characteristics were discussed in §2. Corollary 2.5.1 leads at once to the following result.

LEMMA 5.1. *A curve, C, with exactly two vertices may be divided into two arcs, each of which is simple unless it contains a full circle. In this latter case the arc is tangent to itself, without crossing, either at a single point or along a single arc of the circle.*

If, from a curve C with just two vertices, all arcs which are complete circles are deleted, the resulting curve, $\overline{C}$, will still be a closed curve of class C''. Moreover the operation neither adds nor takes away vertices, whence $\overline{C}$ consists of two simple monotone arcs. The curve $\overline{C}$ is called a normalized curve.

A point where a curve meets itself is called a double point. A double point is called simple if the curve passes through it exactly twice.

LEMMA 5.2. *A normalized curve, $\overline{C}$, having exactly two vertices has double points, and all the double points are simple.*

If there were no double points $\overline{C}$ would be a simple closed curve, which is impossible by Theorem 4.1. If any double point were not simple it would divide $\overline{C}$ into at least three arcs, none of which are circles since $\overline{C}$ is normalized. But by Lemma 4.2 each arc would then contain a vertex, which is impossible.

Let us consider the circle of curvature at the point of minimum curvature on any curve C with two vertices. Since the curve consists of two monotone arcs, it follows from Lemma 2.5 that the entire curve C lies on or to the left of this circle. By a direct circular transformation, a point to the right of this circle may be taken into the point at infinity. C is then carried into a curve C' whose curvature is always positive and which is inside the transformed circle. If M and m are the points of maximum and minimum curvature on C', all the circles of curvature lie inside that at m and outside that at M, by Lemma 2.5. The center of curvature, O, at the point M is interior to every circle of curvature. If P is a point moving in the positive direction on C' the radius vector, OP, always turns in the same direction, namely counterclockwise. Thus if C' is tangent to itself at any point the directed tangents at this point must coincide.

LEMMA 5.3. *If a curve, C, having exactly two vertices is tangent to itself at any point, the directed tangents coincide.*

For otherwise C' would also have oppositely directed tangents at a point, and this has just been shown impossible.

A simple closed arc of a curve is called a loop. If the remainder of the curve never crosses this arc the loop is called a simple loop.

LEMMA 5.4. *A curve, C, having exactly two vertices, has exactly two simple loops, one containing each vertex.*

Let C be transformed into curve C' as in the last proof. It is sufficient to prove the lemma for C'. We shall proceed to establish the existence of a simple loop containing the maximum of curvature. If the maximum of curvature occurs on a circular arc which contains a full circle, this arc coincides with the circle of maximum curvature. In this case the circle itself constitutes the desired simple loop since it is completely within all other circles of curvature.

Suppose then that the maximum curvature occurs at a point or on an arc less than a full circle. As before let M and m be points of maximum and minimum curvature respectively. By Corollary 2.5.2 the arcs mM and Mm are inwinding and outwinding spirals respectively, each arc being simple except for tangencies without crossing which occur whenever a complete circle is contained in the arc. Since by Lemma 5.2 even the normalized curve $\overline{C}$ is not simple it follows that arcs mM and Mm intersect. Let A be the first point where Mm, as it is traced from M, meets mM. We shall show that arc AMA is the required simple loop. If A is a double point of mM, AM will mean the arc from A to M which does not pass again through A.

If AM were not simple it would contain a full circle which would have M in its interior and A on its exterior, whence A could not be the first point where Mm meets mM. Similarly MA is simple. Thus AMA is a loop, not a circle, containing the maximum of curvature. Since the entire curve C' spirals in the counterclockwise direction about the circle of curvature at M, this circle lies interior to and to the left of AMA. The angle formed at A interior to the loop is less than π, for otherwise by Lemma 4.3 the arc contains a minimum of curvature, which is false.

No points of mM lie interior to AMA since by its spiral character it never crosses itself and by construction it never crosses MA. Let a moving point P trace Mm. At A it definitely passes to the exterior of the loop. Let B be the first point where P meets the loop again. If B does not exist the loop is simple. If B lies on MA the arc from B to B is a full circle containing the loop in its interior. The outwinding spiral character of Mm assures that P can never get to the interior of the loop, so again it is simple. Suppose finally B is on AM. The arc $BMAB$ is then a simple closed arc bounding a region to its left. Moreover the angle at B interior to this region is clearly greater than or equal to π. By Lemma 4.3 this arc would then contain a minimum, which is impossible. Thus the loop AMA is simple. This establishes the existence of a simple loop containing the vertex of maximum curvature. But since a mere reversal of direction on a curve interchanges maximum and minimum curvatures, the same proof applies to the other case.

By Lemma 4.2 every loop, not a circle, contains at least one vertex, whence there can be no other loops on the curve which are not circles. Moreover since every circle in the curve distinct from the above loops is a circle of curvature on a monotone arc, it is crossed by the arc and is not a simple loop. This completes the proof of the lemma.

A closed curve in the plane divides the plane into a certain number of regions bounded by arcs of the curve. The definition of a simple loop implies that the loop constitutes the entire boundary of one of these regions. For the loop containing the maximum point this region was shown above to lie to the left of the loop, while for the loop containing the minimum point the region bounded is to the right. It should be noted that one of the regions bounded by the curve is always an infinite one.

LEMMA 5.5. *The region bounded by a simple loop of a curve C with exactly two vertices lies to the right or left of the loop according as this loop contains the minimum or the maximum of curvature.*

LEMMA 5.6. *If C is a curve with exactly two vertices, the only regions determined by C which are bounded in the same sense by all their bounding arcs are those bounded by the two simple loops.*

To prove this, let C be transformed into a curve C' of positive curvature as before. The regions bounded by the two loops in this case are the infinite region and the region containing the center of curvature, O, at M. Let P be a point of any other region and consider the ray OP. By the spiral character of C', ray OP always crosses it in the same direction, namely from left to right. Point P is thus to the right of one of the arcs bounding its region and to the left of another since there is at least one crossing between O and P and another between P

and the point at infinity. This establishes the lemma for C' and thus for C.

The results of this section may be summarized in the following theorem.

THEOREM 5.1. *A curve, C, with exactly two vertices, has the following properties*:

(a) *The corresponding normalized curve,* $\bar{C}$, *may be divided into two simple arcs*;

(b) *The corresponding normalized curve,* $\bar{C}$, *must contain double points, and all the double points are simple*;

(c) *The curve, C, has exactly two simple loops, one containing the maximum of curvature and bounding a region to its left, the other containing the minimum of curvature and bounding a region to its right*;

(d) *No region determined by C, other than those mentioned in* (c), *is bounded in the same sense by all its bounding arcs*;

(e) *At any point where C is tangent to itself the directed tangents coincide.*

6. **Curves with four vertices.** It is a well known theorem that an oval which has at most four intersections with any circle has exactly four vertices [4, 5]. This result may be generalized as follows.

THEOREM 6.1. *A simple closed curve, C, which meets any circle or straight line at most four times has exactly four vertices.*

Let the curve be directed so the interior lies to its left. First let us show that the circle of curvature at any point (or arc) of maximum curvature on C can have no further points in common with the curve. In a neighborhood of the given vertex the circle is interior to C by Corollary 2.1.1. In this neighborhood let points P and Q of the circle be chosen on opposite sides of the vertex. Suppose the circle contains a point R outside C. Now let the curvature of the circle decrease, keeping it tangent to C at the maximum point, M, so that the new circle lies to the right of the original one. This deformation may be taken so small that the points P, Q, R do not cross C. But by Lemma 2.1 the new circle lies to the right of C in a neighborhood of M. Thus, in addition to M itself, there are crossings on MP, PR, RQ, and QM. This contradicts the assumption of at most four intersections. If the circle of curvature at M contains no exterior points but does contain a point of tangency with C, the curve lies to the right of the directed circle at this point, R. The deformation above then takes R to a point outside C and the contradiction is obtained as before.

Suppose Theorem 6.1 is false, that is, suppose C has at least six vertices. If M_1, M_2, M_3 are three maximum points, let C be divided into arcs M_1M_2, M_2M_3, M_3M_1. By Corollary 3.1.1 there is a circle interior to C and tangent to all three arcs. None of the points M_i can be points of contact since we have shown that the largest circles at the M_i which are locally interior to C can never meet C again. Thus there are three distinct points of contact, C_1, C_2, C_3. A slightly larger concentric circle will then have the M_i outside and the C_i inside, contradicting the hypothesis of only four intersections. Thus the assumption is false and the theorem is proved.

THEOREM 6.2. *If a simple closed curve, C, has exactly four vertices, the circles of curvature at the vertices have no further points in common with C.*

Let M be a point of maximum curvature and suppose the circle of curvature at M meets C at a point other than this vertex. Consider the family of circles tangent to C at M and lying on or to the left of the circle of curvature at M. Since the circles of this family which have sufficiently small radius meet C only at M, while the circle of curvature does meet C at some other point, there is a circle of the family lying entirely inside C but tangent to it at some point P distinct from M. Since C is simple, the circle is tangent to C in the same direction at P and M, lying to the left of C at both points. The arcs MP and PM then each contain a maximum of curvature by Lemma 4.1. This gives three maximum points and thus at least six vertices. This contradicts the hypothesis of only four vertices so the circle of curvature cannot meet C again. Since a reversal of direction on C interchanges maximum and minimum points, the same proof shows that the circles of curvature at the minimum points also do not meet the curve again.

THEOREM 6.3. *If a simple closed curve, C, has exactly four vertices, a necessary and sufficient condition that it can be carried into an oval by a direct circular transformation is that either the two maximum or the two minimum circles of curvature intersect.*

The term oval here means a simple closed curve with nonvanishing curvature. If an oval with exactly four vertices is directed so the curvature is positive, the entire curve lies inside the circles of curvature at the points of minimum curvature by Theorem 6.2. Thus each such circle contains an arc of the other near the opposite minimum point, and they must intersect. The condition of the theorem is thus necessary, since the properties in question are preserved by direct circular transformations.

Consider now a simple closed curve, C, with just four vertices, having two of its extreme circles of curvature intersecting. The curve may be so directed that these are the circles of curvature at the minimum points. The entire curve then lies to the left of both circles. Since the circles intersect, there is a point to the right of both circles. Let this point be taken into the point at infinity by a direct circular transformation. The two circles of minimum curvature for the new curve now have positive curvature since their interiors lie to their left. But since there are only four vertices, and monotone arcs go into monotone arcs, the curvature is positive at all points. The transformed curve is therefore an oval, and the proof is complete.

7. Curves with $2n$ vertices. It is a well known fact that an oval which crosses a circle $2n$ times has $2n$ vertices [6]. This fact may be generalized to simple closed curves as follows.

THEOREM 7.1. *Let a simple closed curve, C, be met by a circle $\overline{C}$. If, among the arcs into which $\overline{C}$ divides C, there are $2n-1$ arcs $P_{2i-1}P_{2i}$ $(i=1, 2, \cdots, 2n-1)$ such that the points P_k are in the same cyclic order on C and $\overline{C}$, then C has at least $2n$ vertices.*

The point P_{2i} may coincide with P_{2i+1}. If C is an oval the condition on the cyclic order of the points is automatically fulfilled. It should be noted that the theorem does not require that the chosen arcs be all the arcs of C.

Any one of the chosen arcs lies either wholly inside or wholly outside of $\overline{C}$. There are therefore at least n of these arcs in one place or the other. We may suppose the n arcs interior to $\overline{C}$ since the other case can be reduced to this by a direct circular transformation. Let these n arcs, in order, be denoted by $\mathcal{A}_i$, their end points by Q_i, R_i, and the arc of $\overline{C}$ from R_i to Q_{i+1} by $\overline{C}_i$. It may be assumed without loss of generality that none of the points Q_i, R_i, are points of tangency of C and $\overline{C}$, for if they are $\overline{C}$ may be replaced by a slightly smaller concentric circle for which this is not true. This implies that no $\overline{C}_i$ reduces to a point. By hypothesis the region interior to $\overline{C}$ bounded by the $\mathcal{A}_i$ and the $\overline{C}_i$ is bounded in the same sense by all these arcs. By suitably directing C the region may be made to lie to the left of these arcs.

Let $\mathcal{A}_j$ and $\mathcal{A}_{j+1}$ be any two consecutive $\mathcal{A}_i$, and consider the region bounded by $\mathcal{A}_j$, $\mathcal{A}_{j+1}$, $\overline{C}_j$, and the directed arc of $\overline{C}$ from R_{j+1} to Q_j. This is a simply connected region bounded by differentiable arcs. Let the boundary be divided into three arcs as follows: $\mathcal{A}_j$, $\overline{C}_j$, and the remainder of the boundary. By Lemma 3.1 there is a circle in the region having points in common with all three arcs, and these are all points of tangency as in Corollary 3.1.1. Since there is a contact point on $\bar{C}_j$, there cannot be one on the rest of $\bar{C}$, whence there are contact points on $\mathcal{A}_j$ and $\mathcal{A}_{j+1}$ which we shall call T_j and S_{j+1} respectively. These points are distinct since $\mathcal{A}_j$ and $\mathcal{A}_{j+1}$ have no point in common and are distinct from R_j and Q_{j+1} since C is not tangent to $\bar{C}$ at these latter points. The tangencies at T_j and S_{j+1} are clearly in the same direction on the circle.

If the same operation is carried out for arcs $\mathcal{A}_{j-1}$ and $\mathcal{A}_j$ we obtain points T_{j-1} and S_j on $\mathcal{A}_{j-1}$ and $\mathcal{A}_j$ respectively. We shall show that the points S_j, T_j occur in that order as $\mathcal{A}_j$ is traced from Q_j to R_j. Let the circles constructed with contact points on $\bar{C}_{j-1}$ and $\bar{C}_j$ be denoted by O_{j-1} and O_j respectively. Suppose that arc Q_jT_j does not contain S_j, and consider the region bounded by Q_jT_j, the arc of O_j from T_j to its contact point on $\bar{C}_j$, and the arc of $\bar{C}$ from Q_j to this same contact point. This region is exterior to O_j and contains the arc $T_jS_jR_j$ by hypothesis. The point of contact of O_{j-1} and $\bar{C}_{j-1}$ is exterior to this region and also exterior to O_j. Thus each of the arcs of O_{j-1} from its contact point with $\bar{C}_{j-1}$ to S_j must cross O_j twice. This is impossible, since two distinct circles can meet in only two points. Thus S_j and T_j occur in that order on $\mathcal{A}_j$. The various arcs $T_{j-1}S_j$ of the given curve C are therefore distinct arcs with no points in common. But each of these n arcs of C satisfies the conditions of Corollary 4.1.1, and thus contains a maximum point since it lies locally to the right of O_{j-1} at T_{j-1} and S_j. This proves the existence of n maximum points and thus $2n$ vertices.

It should be noted that if a simple closed curve intersects a circle an infinite number of times any number of arcs may be found satisfying the conditions of the theorem. The following result then follows at once.

Corollary 7.1.1. *If a simple closed curve intersects a circle infinitely often, it has an infinite number of vertices.*

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ON A THEOREM OF BOHR AND PÁL

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Let D be the domain bounded by a simple closed plane Jordan curve of equations $x=f(t)$, $y=g(t)$, where f and g are continuous and of period 2π. Fejér [1][1] has proved that the power series representing the function mapping conformally the interior of the unit circle $|z|<1$ into D converges uniformly on the circle $|z|=1$; hence that there exists a continuous strictly increasing function $t(\theta)$ $(t(0)=0,\ t(2\pi)=2\pi)$ such that the Fourier series of $F(\theta)=f(t(\theta))$ and of $G(\theta)=g(t(\theta))$ converge uniformly for $0\leqq\theta\leqq 2\pi$. Using this theorem, J. Pál [2] has proved that given any continuous function $\phi(t)$ of period 2π there exists a function $t(\theta)$ of the above described type such that the Fourier series of $\phi(t(\theta))$ converges everywhere, and uniformly in the interval $\delta\leqq\theta\leqq 2\pi-\delta$, for any positive δ. H. Bohr [3] has removed the restriction on the uniform convergence in Pál's theorem by proving that the function $t(\theta)$ can be chosen such that the Fourier series of $\phi(t(\theta))$ converges uniformly for $0\leqq\theta\leqq 2\pi$. Bohr's argument involves some delicate considerations. The purpose of this paper is to give a short and simple proof of Bohr's result.

Let $\phi(t)$ be continuous, and of period 2π. Without loss of generality we can, by adding to ϕ a suitable constant, assume that $\int_0^{2\pi}\phi(t)dt=0$. Then there are values of t for which $\phi(t)$ vanishes and we can assume that $t\equiv 0 \pmod{2\pi}$ is one of these values. Thus $\phi(0)=\phi(2\pi)=0$. The mean value of the function being zero, there exists at least another point a $(0<a<2\pi)$ such that $\phi(a)=0$.

Suppose first that, in the open interval $(0, 2\pi)$, a is the only point at which $\phi(t)$ vanishes. Then $\phi(t)$ is strictly positive in one of the open intervals $(0, a)$, $(a, 2\pi)$, and strictly negative in the other one. Let $\alpha(t)$ be any function, continuous, of period 2π, such that $\alpha(0)=\alpha(2\pi)=0$ and such that $\alpha(t)$ is strictly increasing in $(0, a)$ and strictly decreasing in $(a, 2\pi)$. Then the equations $x=\alpha(t)$, $y=\phi(t)$ represent a simple closed Jordan curve and we have only to apply the theorem of Fejér quoted above to get our result for the function $\phi(t)$.

Suppose now that a is not the only point in the open interval $(0, 2\pi)$ at which $\phi(t)$ vanishes. Let M_1 be the maximum of $|\phi(t)|$ for $0\leqq t\leqq a$ and let t_1 be a point $(0<t_1<a)$ such that $|\phi(t_1)|=M_1$. In the same way let M_2 be the maximum of $|\phi(t)|$ in $a\leqq t\leqq 2\pi$ and let t_2 be

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[1] Numbers in brackets refer to the references cited at the end of the paper.

a point ($a < t_2 < 2\pi$) such that $|\phi(t_2)| = M_2$. Consider the function $\omega(t)$, continuous, of period 2π, and defined for every t ($0 \leqq t \leqq 2\pi$) as follows:

$$\omega(t) = \max_{0 \leqq t' \leqq t} |\phi(t')| + \sin(\pi t/a) \qquad \text{for } 0 \leqq t \leqq t_1,$$

$$\omega(t) = \max_{t \leqq t' \leqq a} |\phi(t')| + \sin(\pi t/a) \qquad \text{for } t_1 \leqq t \leqq a,$$

and in the same way

$$\omega(t) = -\max_{a \leqq t' \leqq t} |\phi(t')| - \sin[\pi(t-a)/(2\pi - a)] \qquad \text{for } a \leqq t \leqq t_2,$$

$$\omega(t) = -\max_{t \leqq t' \leqq 2\pi} |\phi(t')| - \sin[\pi(t-a)/(2\pi - a)] \qquad \text{for } t_2 \leqq t \leqq 2\pi,$$

where the maxima are taken with respect to t'.

It is immediately seen that $\omega(t)$ is of bounded variation, that $\phi_1(t) = \phi(t) + \omega(t)$ vanishes only at $t = 0$, $t = a$, $t = 2\pi$, for $0 \leqq t \leqq 2\pi$, and that $\phi_1(t)$ is strictly positive in the open interval $(0, a)$ and strictly negative in the open interval $(a, 2\pi)$. Hence applying our first result we can find a function $t(\theta)$ of the above described type such that the Fourier series of $\phi_1(t(\theta))$ converges uniformly for $0 \leqq \theta \leqq 2\pi$. But since $\omega(t)$ is continuous and of bounded variation the same result holds for $\phi(t)$, which proves the theorem.

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FOURIER SERIES WITH COEFFICIENTS IN A BANACH SPACE

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Let $f(t)$ be a function on $(0, 1)$ to the complex Banach space B. Bochner has shown that the older theory of Fourier series carries over to functions of this character, but breaks down in the fundamental L^2 theory [1, pp. 273–276].[1]

Suppose $f(t)$ belongs to L^2 in the sense of Bochner [1]. Define

$$c_n = \int_0^1 f(t)e^{2\pi int}dt. \tag{1}$$

We should expect that the Parseval relation carries over, or at least that the Bessel inequality

$$\sum \|c_n\|^2 \leqq \int_0^1 \|f(t)\|^2dt \tag{2}$$

is valid. This, however, is not the case; for suitable B we may have $\sum\|c_n\|^2 = \infty$ [1, pp. 275–276].

In this note we detect the root of the trouble by proving that for the validity of (2), B must possess a special character.

Theorem. *If* (2) *is valid for all* $f(t)$ *in* L^2 *then* B *is unitary, and conversely.*

B is *unitary* if it admits a *scalar* product with the usual properties [3] (cf. the "normed ring" of Gelfand [2]).

The latter part of the theorem is trivial; we need only apply the classical proof with notational modifications [4, p. 58].

To establish the sufficiency suppose a and b are elements of B. Define

$$f(t) = \begin{matrix} 2a, & (0, 1/2), \\ 2b, & (1/2, 1). \end{matrix}$$

Then

$$\int_0^1 \|f(t)\|^2dt = 2[\|a\|^2 + \|b\|^2]. \tag{3}$$

By (i) we have

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$$c_0 = a + b,$$
$$c_{2n} = 0, \quad n > 0$$
$$c_{2n+1} = (b - a)2/\pi(2n + 1)i.$$

Then

$$\sum \|c_n\|^2 = \|a + b\|^2 + \|a - b\|^2, \tag{4}$$

since

$$\sum_{-\infty}^{\infty} \frac{1}{(2n + 1)^2} = \frac{\pi^2}{4}.$$

By (2), (3), (4)

$$\|a + b\|^2 + \|a - b\|^2 \leq 2[\|a\|^2 + \|b\|^2]. \tag{5}$$

Replace a by $a+b$, b by $a-b$; this simply reverses the inequality of (5). Then for all a, b

$$\|a + b\|^2 + \|a - b\|^2 = 2[\|a\|^2 + \|b\|^2].$$

Hence B is unitary [3].

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KENYON COLLEGE

THE MEAN CONVERGENCE OF NON-HARMONIC SERIES

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The theory of non-harmonic series deals with the completeness and expansion properties of sets of functions $\{e^{i\lambda_n x}\}$, $-\infty < n < \infty$. Extending results of Paley and Wiener, Levinson has obtained a theory of these functions in the spaces L^p, $1 < p \leqq 2$ [5, p. 113; 4, p. 48].[1] His principal result states that if

$$(1) \qquad |\lambda_n - n| \leqq \delta \qquad (n = 0, \pm 1, \cdots)$$

with $\delta < (p-1)/2p$, λ_n real, then any function $f(x) \in L^p$ has a Fourier expansion in the above functions equiconvergent with its ordinary Fourier series in any closed interval $(-\pi+\epsilon, \pi-\epsilon)$.

This, however, leaves open the question whether the expansion converges to $f(x)$ in the topology of $L^p(-\pi, \pi)$. This is known to be true for all $p > 1$ if $\lambda_n = n$ [6, p. 153]. For the non-harmonic case the only available results concern $p = 2$: by refining an earlier result of Paley and Wiener, Duffin and Eachus [2, p. 855] obtain mean convergence for $\delta < \pi^{-1} \log 2$. (This is less than the $\delta < 1/4$ to be hoped for from Levinson's result.)

In this paper we shall generalize the Duffin-Eachus result to L^p, $p > 1$. Our hypothesis is, however, more stringent than (1). We require that

$$(2) \qquad \left(\sum_{-\infty}^{\infty} |\lambda_n - n|^a \right)^{1/a} = \delta < \pi^{-1} \log 2, \qquad a = \frac{2p}{|2-p|}.$$

If $p \to 2$ this becomes $\sup |\lambda_n - n| < \pi^{-1} \log 2$, which is the result we are extending.

It is interesting to observe that (2) remains the same if we replace p by its conjugate p', where $1/p + 1/p' = 1$. For then $|2p/(p-2)| = |2p'/(2-p')|$. We note also that the λ_n need not be real.

THEOREM. *Let* $\{\lambda_n\}$ *satisfy* (2). *Then any function* $f(x) \in L^p$, $p > 1$, *admits a unique representation*

$$(3) \qquad f(x) = \underset{N\to\infty}{\overset{(p)}{\text{l.i.m.}}} \sum_{-N}^{N} c_n e^{i\lambda_n x}.$$

Moreover

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$$(4)\qquad \begin{aligned} \left(\sum_{-\infty}^{\infty}|c_n|^{p'}\right)^{1/p'} &\leqq (2-e^{\pi\delta})^{-1}\|f\|_p \qquad (1<p\leqq 2),\\ e^{\pi\delta}\left(\sum_{-\infty}^{\infty}|c_n|^{p'}\right)^{1/p'} &\geqq \|f\|_p \qquad (2\leqq p<\infty), \end{aligned}$$

where $1/p+1/p'=1$ *and*

$$(5)\qquad \|f\|_p=\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}|f(x)|^p dx\right)^{1/p}.$$

Levinson's methods are a combination of Fourier analysis and contour integration. We shall revert to the original real variable methods of Paley and Wiener, using the fact, observed by Boas, that their basic completeness theorem generalizes in a simple manner to general Banach spaces.

Boas' generalization, slightly amended, reads as follows [1, p. 469].

LEMMA 1. *Let* $\{x_n\}$, $\{y_n\}$ *be sequences in a Banach space B such that for some* λ, $0<\lambda<1$, *and all finite sequences* $\{a_n\}$

$$(6)\qquad \left\|\sum a_n(x_n-y_n)\right\|\leqq\lambda\left\|\sum a_nx_n\right\|.$$

Then if $\{x_n\}$ *is a base, so is* $\{y_n\}$. *If* $x\in B$ *has the expansion* $\sum_1^\infty c_ny_n$, *then*

$$(7)\qquad \frac{1}{1+\lambda}\|x\|\leqq\left\|\sum_1^\infty c_nx_n\right\|\leqq\frac{1}{1-\lambda}\|x\|.$$

LEMMA 2 [3, p. 202]. *If* $f(x)\in L^p$ $(-\pi,\pi)$ *and* $\{c_n\}$ *are its Fourier coefficients, then*

$$\begin{aligned} \left(\sum_{-\infty}^{\infty}|c_n|^{p'}\right)^{1/p'} &\leqq \|f\|_p \qquad (1<p\leqq 2),\\ \left(\sum_{-\infty}^{\infty}|c_n|^{p'}\right)^{1/p'} &\geqq \|f\|_p \qquad (2\leqq p<\infty). \end{aligned}$$

We remind the reader of our definition (5) of $\|f\|_p$.

LEMMA 3 [3, p. 144]. *If* $f(x)\in L^s$ *then*

$$\|f\|_r\leqq\|f\|_s \qquad (0<r\leqq s).$$

To prove our theorem we shall apply Lemma 1 with $x_n=e^{inx}$, $y_n=e^{i\lambda_nx}$, $B=L^p$ (with modified norm (5)). (6) then becomes

$$(8)\qquad \left\|\sum a_n(e^{inx}-e^{i\lambda_nx})\right\|_p\leqq\lambda\left\|\sum a_ne^{inx}\right\|_p.$$

We denote the right-hand side by R, the left-hand by L. Then we wish to show $L \leqq R$ for a value of λ, $0<\lambda<1$.

$$L = \left\| \sum a_n e^{inx} \sum_{k=1}^{\infty} \frac{i^k(\lambda_n - n)^k x^k}{k!} \right\|_p$$

$$= \left\| \sum_{k=1}^{\infty} \frac{i^k x^k}{k!} \sum a_n(\lambda_n - n)^k e^{inx} \right\|_p .$$

Hence

$$L \leqq \sum_{k=1}^{\infty} \frac{\pi^k}{k!} \left\| \sum a_n(\lambda_n - n)^k e^{inx} \right\|_p . \tag{9}$$

Case (i). $1<p\leqq 2$. Then by Lemma 3

$$\begin{aligned} L &\leqq \sum_{k=1}^{\infty} \frac{\pi^k}{k!} \left\| \sum a_n(\lambda_n - n)^k e^{inx} \right\|_2 \\ &= \sum_{k=1}^{\infty} \frac{\pi^k}{k!} \left(\sum \left| a_n(\lambda_n - n)^k \right|^2\right)^{1/2} \\ &\leqq \sum_{k=1}^{\infty} \frac{\pi^k}{k!} \left(\sum |a_n|^{p'}\right)^{1/p'} \left(\sum |\lambda_n - n|^{k\cdot 2p'/(p'-2)}\right)^{(p'-2)/2p'} \\ &\leqq \left(\sum |a_n|^{p'}\right)^{1/p'} \sum_{k=1}^{\infty} \frac{\pi^k}{k!} \left(\sum_{-\infty}^{\infty} |\lambda_n - n|^{2p'/(p'-2)} \right)^{k(p'-2)/2p'} . \end{aligned} \tag{10}$$

Also by Lemma 2

$$R \geqq \lambda\left(\sum |a_n|^{p'}\right)^{1/p'} . \tag{11}$$

Now write $\delta = \left(\sum_{-\infty}^{\infty} |\lambda_n - n|^{2p'/(p'-2)}\right)^{(p'-2)/2p'}$. By (10) and (11), (8) is satisfied if

$$e^{\pi\delta} - 1 < 1, \qquad \delta < \pi^{-1} \log 2.$$

But this is precisely our hypothesis (2).

Then the hypotheses of Lemma 1 are satisfied, (3) follows immediately, (4) is a consequence of (7) and the first part of Lemma 2.

Case (ii). $2 \leqq p < \infty$. We have by Lemma 2 and (9)

$$\begin{aligned} L &\leqq \sum_{k=1}^{\infty} \frac{\pi^k}{k!} \left(\sum \left| a_n(\lambda_n - n)^k \right|^{p'}\right)^{1/p'} \\ &\leqq \sum_{k=1}^{\infty} \frac{\pi^k}{k!} \left(\sum |a_n|^2\right)^{1/2} \left(\sum |\lambda_n - n|^{k\cdot 2p'/(2-p')}\right)^{(2-p')/2p'} . \end{aligned}$$

Also by Lemma 3

$$R \geqq \lambda \left\| \sum a_n e^{inx} \right\|_2 = \lambda \left(\sum | a_n |^2 \right)^{1/2}.$$

From here on the argument is analogous to that of Case (i).

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ON UNIFORM CONVERGENCE OF FOURIER SERIES

OTTO SZÁSZ

1. Introduction. In this section we collect some known concepts and simple facts, pertinent to our subject.

Given a sequence of real numbers s_n, $n \geqq 0$, consider for any $\lambda > 1$

$$\limsup_{n\to\infty} \max_{n<m\leqq\lambda n} (s_m - s_n) = u(\lambda) \leqq +\infty;$$

clearly $u(\lambda)$ decreases as $\lambda \downarrow 1$; if

$$\lim_{\lambda\to 1} u(\lambda) \leqq 0, \tag{1.1}$$

then the sequence $\{s_n\}$ is called slowly oscillating from above; similarly slow oscillation from below is defined by

$$\lim_{\lambda\to 1} \liminf_{n\to\infty} \min_{n<m\leqq\lambda n} (s_m - s_n) \geqq 0. \tag{1.2}$$

If both (1.1) and (1.2) hold, that is if

$$\lim_{\lambda\to 1} \limsup_{n\to\infty} \max_{n<m\leqq\lambda n} | s_m - s_n | = 0, \tag{1.3}$$

then the sequence is called simply slowly oscillating. If $s_n = \sum_0^n a_\nu$ is the nth partial sum of a series $\sum_0^\infty a_\nu$, then (1.3) can be written as

$$\lim_{\lambda\to 1} \limsup_{n\to\infty} \max_{n<m\leqq\lambda n} \left| \sum_{n+1}^{m} a_\nu \right| = 0. \tag{1.4}$$

A more restricted class of series is defined by

$$\lim_{\lambda\to 1} \limsup_{n\to\infty} \sum_{n<\nu\leqq\lambda n} | a_\nu | = 0. \tag{1.5}$$

Special cases: If for some $p > 0$, $n|a_n| < p$ for all n, then

$$\sum_{n<\nu\leqq\lambda n} | a_\nu | < p \sum_{n}^{\lambda n} \frac{1}{\nu} = O(\log \lambda).$$

Hence (1.5) holds.

If only

$$na_n > -p \qquad \text{for all } n,$$

then

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$$n(|a_n| - a_n) < 2p,$$

hence

$$\lim_{\lambda\to 1}\limsup_{n\to\infty}\sum_{n}^{\lambda n}(|a_\nu| - a_\nu) = 0. \tag{1.6}$$

This relation implies (1.2), but not necessarily (1.4). The following lemma is immediate:

LEMMA 1. *Every convergent series satisfies* (1.3); *furthermore* (1.3) *and* (1.6) *imply* (1.5).

A sequence of functions $s_n(t)$, defined at a point set $\mathcal{E}$ having $t=\tau$ for a limit point, is said to be uniformly convergent at $t=\tau$ if $\lim s_n(t_n)$ exists for any sequence $t_n\to\tau$. It is an immediate consequence of the definition that the limit of $s_n(t_n)$ is then unique.

If for each n, $s_n(t)$ is defined and continuous at $t=\tau$, then clearly a necessary condition for uniform convergence at $t=\tau$ is that $\lim_{n\to\infty}s_n(\tau) = s$ exists.

We restrict ourselves to such sequences; then the following lemma holds:

LEMMA 2. *The following two properties are equivalent*:

(a) $s_n(t_n)\to s$ *as* $t_n\to\tau$;

(b) $s_n(\tau)\to s$, *and* $|s_n(\tau)-s_n(t)| < \epsilon$ *for any* $\epsilon>0$, *and for* $|\tau-t| < \delta(\epsilon)$, $n>n_0(\delta, \epsilon)=n_0(\epsilon)$.

Thus either (a) or (b) defines uniform convergence at $t=\tau$.

For the proof assume that (a) holds; if (b) would not hold, there would exist an $\epsilon=\epsilon_0$, so that $\limsup_{t_n\to\tau}|s_n(\tau)-s_n(t_n)| > \epsilon_0$. But this contradicts (a). Similarly if (b) holds, then (a) follows.

2. **The cosine series.** We now prove the following theorem.

THEOREM 1. *Suppose that the coefficients of the Fourier cosine series*

$$\phi(t) \sim a_0/2 + \sum_1^\infty a_n \cos nt \tag{2.1}$$

satisfy the condition (1.6), *and that* $\phi(t)$ *is continuous at* $t=0$; *then the series* (2.1) *is uniformly convergent at* $t=0$.

Let

$$s_0 = \frac{a_0}{2}, \quad s_n(t) = \frac{a_0}{2} + \sum_1^n a_\nu \cos \nu t, \quad \sigma_n(t) = \frac{1}{n+1}\sum_0^n s_\nu(t).$$

By a theorem of Fejér [1][1]

$$\sigma_n(t_n) \to \phi(0) \qquad \text{as } t_n \to 0; \tag{2.2}$$

in particular

$$\sigma_n(0) \to \phi(0) \qquad \text{as } n \to \infty. \tag{2.3}$$

By a well known theorem of Tauberian type, (2.3) and (1.6) (or only (1.2)) imply that

$$s_n(0) \to \phi(0). \tag{2.4}$$

By Lemma 1, (1.6) and (2.4) imply (1.5).

We next employ the often used identity

$$s_n - \sigma_{n+\nu} = \frac{n}{\nu+1}(\sigma_{n+\nu} - \sigma_{n-1}) - \frac{1}{\nu+1}\sum_{k=1}^{\nu}(\nu - k + 1)c_{n+k}, \qquad n \geqq 1,\ \nu \geqq 1, \tag{2.5}$$

where s_n, σ_n are the partial sums and arithmetical means respectively of the series $\sum c_n$. Thus

$$\begin{aligned} s_n(0) - s_n(t) &- \{\sigma_{n+\nu}(0) - \sigma_{n+\nu}(t)\} \\ &= \frac{n}{\nu+1}\{\sigma_{n+\nu}(0) - \sigma_{n+\nu}(t) - [\sigma_{n-1}(0) - \sigma_{n-1}(t)]\} \\ &\quad - \frac{1}{\nu+1}\sum_{1}^{\nu}(\nu - k + 1)[1 - \cos(n+k)t]a_{n+k}. \end{aligned} \tag{2.6}$$

By (2.2) and Lemma 2

$$|\sigma_n(0) - \sigma_n(t)| < \epsilon \quad \text{for} \quad |t| < \delta(\epsilon) \quad \text{and} \quad n \geqq n_0(\epsilon);$$

hence, from (2.6)

$$\begin{aligned} |s_n(0) - s_n(t)| &< \epsilon + \frac{2n\epsilon}{\nu+1} + \frac{2}{\nu+1}\sum_{1}^{\nu}(\nu - k + 1)|a_{n+k}| \\ &< \epsilon + \frac{2n\epsilon}{\nu+1} + 2\sum_{1}^{\nu}|a_{n+k}|, \\ &\qquad |t| < \delta(\epsilon),\ n > n_0(\epsilon). \end{aligned} \tag{2.7}$$

Write

$$\limsup_{n\to\infty} \sum_{n}^{\lambda n} |a_\nu| = \omega(\lambda),$$

then

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[1] Numbers in brackets refer to the literature listed at the end of the paper.

$$\sum_{n}^{\lambda n} | a_\nu | < \epsilon + \omega(\lambda) \qquad \text{for } n > n_1(\epsilon, \lambda). \tag{2.8}$$

Given $\epsilon>0$, choose $\nu=[n\epsilon^{1/2}]$, and $\lambda=1+\epsilon^{1/2}$, then, from (2.7) and (2.8),

$$| s_n(0) - s_n(t) | < \epsilon + 2\epsilon^{1/2} + 2(\epsilon + \omega(1 + \epsilon^{1/2})) \qquad \text{for } n > n_2(\epsilon),$$

when n_2 is the larger of the two numbers n_0, n_1. The theorem now follows from (1.5) and Lemma 2.

The identity

$$s_n(t) - \sigma_n(t) = \frac{1}{n+1} \sum_{1}^{n} \nu a_\nu \cos \nu t$$

yields the corollary:

COROLLARY TO THEOREM 1. *Under the assumptions of Theorem* 1 $n^{-1}\sum_1^n \nu a_\nu \cos \nu t \to 0$ *uniformly at* $t=0$.

3. **The sine series.** In this case convergence at $t=0$ is trivial; we introduce two lemmas.

LEMMA 3. *Suppose that the coefficients of the Fourier sine series*

$$\psi(t) \sim \sum_{1}^{\infty} b_n \sin nt \tag{3.1}$$

satisfy the condition (1.2) *with* $s_n=\sum_1^n b_\nu$, *and that*

$$2h^{-1} \int_0^h \psi(t)dt \to d \qquad \text{as } h \downarrow 0,$$

then

$$n^{-1} \sum_{1}^{n} \nu b_\nu \to \pi^{-1} d.$$

This is Lemma 6 of our paper [6].

LEMMA 4. *If for a sequence* $\{b_n\}$

$$\lim n^{-1} \sum_{1}^{n} \nu b_\nu = l$$

exists, and if

$$\lim_{\lambda \downarrow 1} \limsup_{n \to \infty} \sum_{n}^{\lambda n} (| b_\nu | - b_\nu) = 0, \tag{3.2}$$

then

$$\lim_{\lambda\downarrow 1}\limsup_{n\to\infty}\sum_{n}^{\lambda n}|b_\nu| = 0. \tag{3.3}$$

Write

$$\limsup_{n\to\infty}\sum_{n}^{\lambda n}(|b_\nu| - b_\nu) = \xi(\lambda), \qquad \lambda > 1,$$

then by (3.2), $\xi(\lambda)\to 0$ as $\lambda\downarrow 1$. We have

$$\sum_{n}^{\lambda n}\nu(|b_\nu| - b_\nu) \leqq \lambda n\sum_{n}^{\lambda n}(|b_\nu| - b_\nu),$$

hence

$$\limsup n^{-1}\sum_{n}^{\lambda n}\nu(|b_\nu| - b_\nu) \leqq \lambda\xi(\lambda).$$

Furthermore

$$n^{-1}\sum_{n}^{\lambda n}\nu b_\nu = \lambda(\lambda n)^{-1}\sum_{1}^{\lambda n}\nu b_\nu - n^{-1}\sum_{1}^{n-1}\nu b_\nu \to (\lambda - 1)l,$$

hence

$$\limsup n^{-1}\sum_{n}^{\lambda n}\nu|b_\nu| \leqq (\lambda - 1)l + \lambda\xi(\lambda).$$

But

$$\sum_{n}^{\lambda n}|b_\nu| \leqq n^{-1}\sum_{n}^{\lambda n}\nu|b_\nu|,$$

hence

$$\limsup\sum_{n}^{\lambda n}|b_\nu| \leqq (\lambda - 1)l + \lambda\xi(\lambda).$$

Letting $\lambda\downarrow 1$, we get (3.3).

THEOREM 2. *Suppose that the function $\psi(t)$ is continuous at $t=0$, that is $\psi(0)=0$, and that its Fourier coefficients satisfy* (3.2). *Then $\sum_1^n \nu b_\nu = o(n)$, and the series* (3.1) *is uniformly convergent at $t=0$.*

We now write

$$s_n(t) = \sum_{1}^{n} b_\nu \sin \nu t, \qquad \sigma_n(t) = \frac{1}{n+1}\sum_{1}^{n} s_\nu(t);$$

then by the theorem of Fejér

$$\sigma_n(t_n) \to 0 \qquad \text{as } t_n \to 0. \tag{3.4}$$

Also by Lemma 3

$$\sum_{1}^{n}\nu b_\nu = o(n),$$

and Lemma 4 now yields (3.3). Finally from (2.5) with $c_n = b_n \sin nt$, applying (3.4) and Lemma 2,

$$| s_n(t) | < \epsilon + \frac{2n\epsilon}{\nu + 1} + \sum_{n+1}^{n+\nu} | b_k |, \qquad \text{for } |t| < \delta(\epsilon) \text{ and } n > n_0(\epsilon).$$

Write

$$\limsup \sum_{n}^{\lambda n} | b_\nu | = v(\lambda),$$

then by (3.3)

$$v(\lambda) \to 0 \quad \text{as} \quad \lambda \downarrow 1.$$

We now choose $\nu = [n\epsilon^{1/2}]$, then, as in §2,

$$| s_n(t) | < 3\epsilon^{1/2} + 2v(1 + \epsilon^{1/2}) \qquad \text{for } |t| < \delta(\epsilon) \text{ and } n > n_1(\epsilon),$$

which proves the theorem.

COROLLARY. *Under the assumptions of Theorem* 2,

$$n^{-1}\sum_{1}^{n} \nu b_\nu \sin \nu t \to 0 \textit{ uniformly at } t = 0.$$

This follows from $s_n(t) - \sigma_n(t) = (n+1)^{-1}\sum_1^n \nu b_\nu \sin \nu t$.

4. **A converse theorem.** To prove a converse of Theorem 2, we introduce the lemma.

LEMMA 5. *Suppose that* $B_n \geqq 0$, *that for some* $c > 0$

$$B_{n+1} \leqq (1 + c/n)B_n, \qquad n = 1, 2, 3, \cdots, \tag{4.1}$$

and that the sequence $\{B_n\}$ *is Abel summable to* B; *then* $B_n \to B$.

It is known that $B_n \geqq 0$ and Abel summability imply $(C, 1)B_n \to B$, that is

$$B_n' = \sum_1^n B_\nu \sim nB. \tag{4.2}$$

From (4.1)

$$B_{n+k} \leqq (1 + c/n)^k B_n, \qquad k = 0, 1, 2, \cdots,$$

hence

$$\sum_{k=0}^{\nu} B_{n+k} \leqq B_n \sum_{k=0}^{\nu} (1 + c/n)^k = nB_n c^{-1}\{(1 + c/n)^{\nu+1} - 1\},$$

or $B_n \geqq cn^{-1}\{(1+c/n)^{\nu+1} - 1\}^{-1}(B_{n+\nu}' - B_{n-1}')$. To any given $\delta > 0$ choose $\nu = [\delta n]$, so that $\nu n^{-1} \to \delta$. Then

$$\liminf_{n\to\infty} B_n \geqq c(e^{c\delta}-1)^{-1}\{(1+\delta)B - B\} = c\delta B(e^{c\delta}-1)^{-1};$$

letting $\delta \downarrow 0$, we get

$$\liminf B_n \geqq B. \tag{4.3}$$

Similarly from (4.1) by induction

$$B_{n-k} \geqq (1 + c/(n-k))^{-k-1}B_{n+1} \geqq (1 + c/\nu)^{-k-1}B_{n+1}$$
$$\text{for } n - k \geqq \nu > 0,$$

hence

$$\sum_{k=0}^{n-\nu} B_{n-k} \geqq B_{n+1}\sum_{k=0}^{n-\nu}\left(1+\frac{c}{\nu}\right)^{-k-1} = \nu B_{n+1}c^{-1}\left\{1-\left(1+\frac{c}{\nu}\right)^{\nu-n-1}\right\},$$

or

$$B_{n+1} \leqq c\nu^{-1}\{1-(1+c/\nu)^{\nu-n-1}\}^{-1}(B'_n - B'_{\nu-1}).$$

Again to any given positive $\delta<1$ choose $\nu = [n\delta]$; then

$$\limsup B_{n+1} \leqq c\{1-e^{c-c\delta^{-1}}\}^{-1}(B\delta^{-1}-B) = c\delta^{-1}B\frac{1-\delta}{1-\exp(c-c\delta^{-1})}.$$

Letting $\delta \uparrow 1$ we find

$$\limsup B_n \leqq B. \tag{4.4}$$

(4.3) and (4.4) prove the lemma.

It is easily seen that the assumption (4.1) is equivalent to saying $n^{-\gamma}B_n$ is decreasing for some γ; our lemma is in close connection to a lemma due to Hardy [3, p. 442].

THEOREM 3. *Suppose that* $\psi(t) \sim \sum b_n \sin nt$, *that*

$$\psi(t) \to \pi A/2 \qquad \text{as } t \downarrow 0, \tag{4.5}$$

and that for some constants p *and* c

$$0 \leqq (n+1)b_{n+1} + p \leqq (1 + c/n)(nb_n + p), \qquad n \geqq 1. \tag{4.6}$$

Then $nb_n \to A$.

Let

$$g(t) = (\pi - t)/2 = \sum_1^\infty n^{-1}\sin nt, \qquad 0 < t \leqq \pi,$$

and

$$\chi(t) = \psi(t) - Ag(t) \sim \sum (b_n - A_n^{-1})\sin nt \equiv \sum \beta_n \sin nt, \tag{4.7}$$

then, from (4.5),

$$\chi(t) \to 0 \qquad \text{as } t \downarrow 0.$$

Furthermore

$$n\beta_n = nb_n - A \geqq -p - A = -q,$$

say, and

$$(n+1)\beta_{n+1} + q = (n+1)b_{n+1} + p \leqq (1 + c/n)(n\beta_n + q).$$

Thus we need only prove $n\beta_n \to 0$, that is Theorem 3 is reduced to the case $A=0$. Now for this case Theorem 2 yields $\sum_1^n \nu b_\nu = o(n)$; finally Lemma 5 applied to $B_n = nb_n + p$ gives Theorem 3.

A special case. Let $p=0$; $c=1$; then (4.6) reduces to $0 \leqq b_{n+1} \leqq b_n$.

For this case and $A=0$ the theorem is due to Chaundy and Jolliffe, while for $A \neq 0$ it is due to Hardy [3, 4]. As Hardy remarked, here the case $A \neq 0$ is not immediately reducible to the case $A=0$. Our generalization has the advantage of such reduction.

5. **On Gibbs' phenomenon.** We shall apply Theorem 2 to the Gibbs' phenomenon (cf. [7, p. 181]). Consider again the assumption (4.5); that is $\psi(t)$ has the jump πA, while $\chi(t)$ is continuous at $t=0$. We assume in addition (3.2); then evidently the β_n satisfy the same assumption, hence by Theorem 2

$$\sum_1^n \nu\beta_\nu = \sum_1^n \nu b_\nu - nA = o(n),$$

and the series (4.7) is uniformly convergent at $t=0$. On the other hand Fejér proved that

$$\limsup_{t_n \downarrow 0} \sum_1^n \nu^{-1} \sin \nu t_n = \lim_{nt_n \to \pi} \sum_1^n \nu^{-1} \sin \nu t_n = \int_0^\pi t^{-1} \sin t dt,$$

hence assuming, as we may, $A > 0$,

$$(5.1) \quad \limsup_{t_n \downarrow 0} \sum_1^n b_\nu \sin \nu t_n = \lim_{nt_n \to \pi} \sum_1^n b_\nu \sin \nu t_n = A \int_0^\pi t^{-1} \sin t dt.$$

We have thus proved the theorem:

THEOREM 4. *Suppose that* $\psi(t) \sim \sum b_n \sin nt$ *satisfies the conditions* (4.5) *and* (3.2); *then*

$$\sum_1^n \nu b_\nu \sim An,$$

and

$$\sum_1^n b_\nu \sin \nu t_n - A \sum_1^n \nu^{-1} \sin \nu t_n \to 0 \qquad \text{as } t_n \to 0;$$

in particular (5.1) *holds, that is the two series of* $\psi(t)$ *and* $Ag(t)$ *present the same phenomenon of Gibbs.*

For the special case $nb_n = O(1)$ Gibbs' phenomenon was observed by Rogosinski [5, pp. 134–135], however it is difficult to follow his argument.

6. **A contre example.** We cannot replace in Theorems 1 and 2 the conditions (1.6) and (3.2) by (1.3) with $s_n = \sum_1^n a_\nu$ or $s_n = \sum_1^n b_\nu$ respectively. This is seen from an example constructed by Fejér [2] for a similar purpose. It is a power series $\sum_{k=1}^\infty c_k z^k$ with the following properties [2, pp. 38–46]: The coefficients are all real; the power series is convergent for $|z| \leqq 1$; the function $f(z) = \sum c_k z^k$ is continuous for $|z| \leqq 1$; the power series is uniformly convergent for $z = e^{it}$, $\epsilon \leqq t \leqq 2\pi - \epsilon$, $\epsilon > 0$, but neither of the series $\sum a_k \cos kt$, $\sum a_k \sin kt$ is uniformly convergent for $|t| \leqq \epsilon$. It follows easily that neither series is uniformly convergent at $t = 0$, for this would imply uniform convergence on the entire unit circle.

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SUMMABILITY OF SUBSEQUENCES

RALPH PALMER AGNEW

1. **Introduction.** Let a_{nk} $(n, k=1, 2, \cdots)$ be a matrix of real or complex constants for which

$$(1.1)\qquad \lim_{n\to\infty} a_{nk} = 0, \qquad k = 1, 2, 3, \cdots,$$

$$(1.2)\qquad \lim_{n\to\infty} \sum_{k=1}^{\infty} a_{nk} = 1; \qquad \sum_{k=1}^{\infty} | a_{nk} | < M, \qquad n = 1, 2, 3, \cdots,$$

M being a constant. This matrix defines a regular method of summability by means of which a sequence x_n of real or complex numbers is summable to X if $X_n=\sum_{k=1}^{\infty}a_{nk}x_k$, $n=1, 2, 3, \cdots$, exists and $\lim X_n=X$. It has recently been shown by R. C. Buck[1] that if the sequence x_n is real, bounded, and divergent, then the sequence has a subsequence not summable A. This note proves the following more general theorem.

THEOREM. *Let A be regular and let x_n be a bounded complex sequence. Then there exists a subsequence y_n of x_n such that the set L_Y of limit points of the transform Y_n of y_n includes the set L_x of limit points of the sequence x_n.*

If x_n is a bounded divergent sequence, then L_x and hence also L_Y must contain at least two distinct points and accordingly the subsequence y_n is not summable A. Applying the theorem to the divergent sequence 0, 1, 0, 1, $\cdots$, we obtain the result of Steinhaus[2] that there is a sequence of 0's and 1's not summable A.

2. **Proof of the theorem.** Let L_x be the set of limit points of the bounded complex sequence x_n. Since the complex plane is separable and L_x is a closed set, there is a countable (finite or infinite) subset E of L_x such that the closure $\bar{E}$ of E is the set L_x itself. Let $u_1, u_2, u_3, \cdots$ be a sequence containing all of the points of E; in case E is a finite set, the points $u_1, u_2, u_3, \cdots$ are not distinct. Let the elements of the sequence

$$(2.1)\qquad u_1;\ u_1, u_2;\ u_1, u_2, u_3;\ \cdots;\ u_1, u_2, \cdots, u_n;\ \cdots$$

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[1] R. C. Buck, *A note on subsequences*, Bull. Amer. Math. Soc. vol. 49 (1943) pp. 898–899.

[2] H. Steinhaus, *Some remarks on the generalization of limit* (in Polish), Prace Matematyczno-fizyczne vol. 22 (1911) pp. 121–134.

be denoted by $v_1, v_2, v_3, \cdots$. The sequence v_n is a sequence of points in L_x, and the set of limit points of the sequence is the set L_x. For each $p=1, 2, 3, \cdots$, let

$$x(n_{p1}),\ x(n_{p2}),\ x(n_{p3}),\ \cdots \tag{2.2}$$

be a subsequence of x_n having the limit v_p.

To simplify typography, we write $a(n, k)$ for a_{nk}. Since A is regular, (1.1) and (1.2) hold. Hence sequences $n_1<n_2<n_3< \cdots$ and $k_1<k_2<k_3< \cdots$ of indices exist such that for each $p=1, 2, 3, \cdots$

$$\sum_{k=1}^{k_p} | a(n_p, k) | < \frac{1}{p}, \qquad \sum_{k=k_p+1}^{\infty} | a(n_p, k) | < \frac{1}{p}. \tag{2.3}$$

It follows that

$$\begin{aligned} \sum_{k=k_p+1}^{k_{p+1}} a(n_p, k) &= \sum_{k=1}^{\infty} a(n_p, k) - \sum_{k=1}^{k_p} a(n_p, k) \\ &\quad - \sum_{k=k_{p+1}+1}^{\infty} a(n_p, k) = 1 + \epsilon_p \end{aligned} \tag{2.4}$$

where, here and hereafter, ϵ_p denotes generically a sequence for which $\epsilon_p \to 0$ as $p \to \infty$.

The subsequence $y(n)$ of the given sequence x_n is now selected as follows. Assuming that, for a fixed index p, $y(k)$ has been selected for each $k \leqq k_p$, let $y(k_p+1), \cdots, y(k_{p+1})$ be selected from the sequence (2.2) in such a way that $y(j)$ is a predecessor of $y(k)$ in the sequence x_n when $j<k$ and

$$| y(k) - v_p | < 1/p, \qquad k_p < k \leqq k_{p+1}. \tag{2.5}$$

Since x_n is bounded, say $|x_n| \leqq B$, the subsequence $y(n)$ thus defined by induction is bounded and accordingly possesses a transform

$$Y_n = \sum_{k=1}^{\infty} a_{n,k} y_k. \tag{2.6}$$

For each $p=1, 2, 3, \cdots$

$$\begin{aligned} Y(n_p) &= \sum_{k=1}^{k_p} a(n_p, k) y_k + \sum_{k=k_{p+1}+1}^{\infty} a(n_p, k) y_k + \sum_{k=k_p+1}^{k_{p+1}} a(n_p, k) y_k \\ &= \epsilon_p + \sum_{k=k_p+1}^{k_{p+1}} a(n_p, k) y_k, \end{aligned} \tag{2.7}$$

since each of the first two terms of the second member of (2.7) is dominated by B/p. Moreover

$$\begin{aligned} \sum_{k=k_p+1}^{k_{p+1}} a(n_p, k)y_k &= v_p \sum_{k=k_p+1}^{k_{p+1}} a(n_p, k) + \sum_{k=k_p+1}^{k_{p+1}} a(n_p, k)(y_k - v_p) \\ &= v_p(1 + \epsilon_p) + \epsilon_p = v_p + \epsilon_p. \end{aligned} \tag{2.8}$$

Therefore the sequences $Y(n_p)$ and v_p have the same limit points and accordingly the set of limit points of the sequence $Y(n_p)$ is identical with the set L_x of limit points of the sequence x_n. The set L_Y of limit points of the complete sequence Y_n therefore includes the set L_x and the theorem is proved.

3. **Conclusion.** It is apparent from the proof of the theorem that if x_n is a bounded divergent sequence, then it is possible to construct many subsequences y_n not summable A. However, the class of such subsequences y_n thus constructed seems to be a "small" subclass of the class of all subsequences of x_n. This observation is in agreement with the fact, recently proved by Buck and Pollard,[3] that if A is either convergence or the Cesàro method of order 1 and s_n is a real bounded sequence summable A, then there is a sense in which "almost all" of the subsequences of x_n are summable A.

The theorem states that the set L_Y of limit points of the transform Y_n of the subsequence y_n of x_n includes the set L_x of limit points of x_n. In some cases, L_Y is identical with L_x. This is so when A is convergence. In some cases, L_Y is more extensive than L_x. This is so when A is a Cesàro method of positive order and the set L_x is not connected since, as was shown by Barone,[4] the set of limit points of the transform of each bounded sequence must be connected. The same is true for methods of Hölder, Riesz, de la Vallee Poussin, and Euler.

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INTEGRAL THEOREMS IN THREE-DIMENSIONAL POTENTIAL FLOW

R. v. MISES

1. Introduction. A potential flow can be described as a vector distribution $\bar{q}(\bar{r})$ ($\bar{q}$=velocity vector, $\bar{r}$=position vector) subject to the two conditions

$$\operatorname{div} \bar{q} = 0, \qquad \operatorname{curl} \bar{q} = 0. \tag{1}$$

Instead of (1) one can also ask that a scalar function $\phi(\bar{r})$ exist so that

$$\bar{q} = \operatorname{grad} \phi, \qquad \Delta\phi = 0. \tag{2}$$

If both $\bar{q}$ and $\bar{r}$ are restricted to two dimensions, a third form of representation is possible. One can combine the components x, y of $\bar{r}$ and u, v of $\bar{q}$ to two complex numbers

$$x + iy = \zeta, \qquad u - iv = \upsilon \tag{3}$$

and then state that υ is an analytic function of ζ. In this case, the Cauchy formula holds,

$$\int f(\upsilon, \zeta)d\zeta = 0, \tag{4}$$

if f is an analytic function of υ and ζ and the integral is extended over the complete boundary of a region in which f is regular.

In the dynamics of the two-dimensional potential flow several equations of the form (4) play a decisive role. It must be expected that the analogous theorems are valid in three-dimensional potential flow also. But the question has not yet been answered: *For what vector functions $\bar{f}$ of $\bar{q}$ and $\bar{r}$ is the equation*

$$\int \bar{f}(\bar{q}, \bar{r})\cdot d\bar{S} = 0 \tag{5}$$

correct if the integral is extended over the complete boundary of a region in which $\bar{f}$ has continuous derivatives of the first order with respect to the six components x, y, z of $\bar{r}$ and u, v, w of $\bar{q}$? Here, obviously, $d\bar{S}$ is the vectorial area element whose direction is that of the outward normal, and the dot means scalar multiplication. The surface may consist of a finite number of analytic pieces.

The main results of this paper were presented in an address delivered at the New York meeting of the Society on April 4, 1942, by invitation of the Program Committee; received by the editors May 19, 1944.

The present note will answer this and some related questions and illustrate the applications which can be made in the theory of three-dimensional potential flow.

2. **The general form of the function $\bar{f}(\bar{q}, \bar{r})$.** As $\bar{q}$ is a function of $\bar{r}$ according to (1), one may write $\bar{f}(\bar{q}, \bar{r}) = \bar{F}(\bar{r})$. Then the necessary and sufficient condition for (5) being correct for all surfaces is

$$\operatorname{div} \bar{F} = 0. \tag{6}$$

If f_x, f_y, f_z denote the components of $\bar{f}$ we find by differentiation:

$$\begin{aligned} \operatorname{div} F = {} & \frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y} + \frac{\partial f_z}{\partial z} + \frac{\partial u}{\partial x}\frac{\partial f_x}{\partial u} + \frac{\partial v}{\partial x}\frac{\partial f_x}{\partial v} + \frac{\partial w}{\partial x}\frac{\partial f_x}{\partial w} \\ & + \frac{\partial u}{\partial y}\frac{\partial f_y}{\partial u} + \frac{\partial v}{\partial y}\frac{\partial f_y}{\partial v} + \cdots + \cdots . \end{aligned} \tag{7}$$

It can be seen easily that the right-hand expression is zero, by virtue of (1), if the following conditions are fulfilled:

$$\text{(a)} \qquad \partial f_x/\partial x + \partial f_y/\partial y + \partial f_z/\partial z = 0,$$

$$\text{(b)} \qquad \partial f_x/\partial u = \partial f_y/\partial v = \partial f_z/\partial w,$$

$$\text{(c)} \qquad \partial f_z/\partial v + \partial f_y/\partial w = \partial f_x/\partial w + \partial f_z/\partial u = \partial f_y/\partial u + \partial f_x/\partial v = 0.$$

On the other hand, these conditions are also necessary, as the following examples show. First, take $u=1$, $v=1$, $w=1$, then condition (a) follows. Second, take $u=x$, $v=-y$, $w=0$ in accordance with (1), then the first equality (b) follows. Finally, choose $u=y$, $v=x$, $w=0$ which also fulfills (1), then the last equality (c) appears as necessary. Thus, the three components f_x, f_y, f_z have to satisfy the six differential equations (a), (b), (c).

The five equations (b) and (c) express the fact that the infinitesimal transformation of the $\bar{q}$-space, given by

$$d\bar{q}/dt = \bar{f}(\bar{q}, \bar{r}) \tag{8}$$

for any constant $\bar{r}$ (t = scalar parameter), is *conformal* (angle preserving). Without using this fact the general solution of the equations (b) and (c) can be found in the following way.

Assume that f_x, f_y, f_z are developable into a power series with respect to u, v, w within some finite region. Then (b) and (c) furnish relations between the coefficients of terms of the same order n. There are three times $(n+1)(n+2)/2$ such coefficients and five times $n(n+1)/2$ relations. It can be seen that these relations are linearly independent of each other. The excess of the number of coefficients over the number of equations is

$$\begin{gathered} e = 3(n+1)(n+2)/2 - 5n(n+1)/2 = (n+1)(3-n), \\ \text{that is, } e = 3,\ 4,\ 3,\ 0,\ < 0 \\ \text{for } n = 0,\ 1,\ 2,\ 3,\ > 3. \end{gathered} \tag{9}$$

Consequently, there exist ∞^3 solutions of order zero, ∞^4 solutions of order 1, and ∞^3 solutions of order 2 and no other ones. All these solutions can be found by setting up the linear equations for the coefficients, and so on. But, at least for $n=0$ and $n=1$, they are obvious:

For $n=0$ we have the translations

$$\bar{f} = \bar{A}_0 \tag{10a}$$

where the vector $\bar{A}_0$ is independent of $\bar{q}$. For $n=1$ one has the homography and the rotation

$$\bar{f} = K\bar{q} + (\bar{A} \times \bar{q}) \tag{10b}$$

where the scalar K and the vector $\bar{A}$ are functions of $\bar{r}$ alone (cross denoting the vector multiplication). For $n=2$ one may find by computation or by certain geometrical considerations

$$\bar{f} = \bar{L}q^2 - 2(\bar{L}\cdot\bar{q})\bar{q} \tag{10c}$$

where $q^2 = u^2+v^2+w^2$ and $\bar{L}$ is a vector depending on $\bar{r}$ but not on $\bar{q}$. Each infinitesimal transformation of this group consists of an inversion, an infinitesimal translation in direction of $\bar{L}$, and another inversion.

The general, ten-parametric, expression for $\bar{f}$ is thus

$$\bar{f} = \bar{A}_0 + K\bar{q} + (\bar{A} \times \bar{q}) + \bar{L}q^2 - 2(\bar{L}\cdot\bar{q})\bar{q}. \tag{10}$$

This expression has now to satisfy the condition (a) independently of $\bar{q}$. For $\bar{A}_0$, K, $\bar{A}$ one finds immediately

$$\operatorname{div} \bar{A}_0 = 0, \tag{11}$$

$$\operatorname{grad} K + \operatorname{curl} \bar{A} = 0, \cdots \qquad (\Delta K = 0), \tag{12}$$

while $\bar{L}$ has to fulfill the six equations

$$\begin{gathered} \partial L_x/\partial x = \partial L_y/\partial y = \partial L_z/\partial z = 0, \\ \partial L_z/\partial y + \partial L_y/\partial z = \partial L_x/\partial z + \partial L_z/\partial x = \partial L_y/\partial x + \partial L_x/\partial y = 0. \end{gathered}$$

This means that the space transformation $d\bar{r}/dt = \bar{L}$ is angle preserving and length preserving. Thus

$$\bar{L} = \bar{L}_0 + (\bar{L}_1 \times \bar{r}) \tag{13}$$

with $\bar{L}_0$ and $\bar{L}_1$ constants.

The result can be stated: *The most general function $\bar{f}(\bar{q}, \bar{r})$ which satisfies* (5) *for an arbitrary surface S and any $\bar{q}(\bar{r})$ fulfilling* (1) *is given by* (10) *provided the functions $\bar{A}_0$, K, $\bar{A}$, and $\bar{L}$ of $\bar{r}$ satisfy* (11), (12), *and* (13) *with $\bar{L}_0$ and $\bar{L}_1$ constants.*

3. **Simple examples.** The first term, $\bar{A}_0$, in the expression for $\bar{f}$ which is independent of $\bar{q}$ can be disregarded.

The simplest linear function is determined by $K=$const., $\bar{A}=0$. This gives nothing else than the integral form of the divergence condition:

$$\int \bar{q}\cdot d\bar{S} = \int (\bar{q}\cdot\bar{n})dS = \int q_n dS = 0. \tag{14}$$

Here and in the following, $\bar{n}$ is the unit vector in the direction of the outward normal and the subscript n refers to the vector component in this direction.

If $K=0$ and $\bar{A}=$const. one finds, using the formula $(\bar{A}\times\bar{q})\cdot\bar{n} = (\bar{q}\times\bar{n})\cdot\bar{A}$, that each component of

$$\int (\bar{q} \times \bar{n})dS = 0 \tag{15}$$

is justified. This vector integral may be considered as a *surface circulation.* If it is extended over a surface surrounding a rigid body (that is, not over the complete boundary of a region of regular $\bar{q}$), it may have a constant value different from zero. If the region outside the body includes a discontinuity surface S', one has to add the integral of $(\bar{q}_1-\bar{q}_2)\times\bar{n}'$, extended over S' where $\bar{n}'$ is the unit vector of the normal to S' in the direction from 1 to 2, and $\bar{q}_1$ and $\bar{q}_2$ are the velocities on both sides. This vector $(\bar{q}_1-\bar{q}_2)\times\bar{n}'$ has the direction of the vortex line and the magnitude of the vortex density on S' if the discontinuity surface is regarded as a vortex sheet. In a field of vortex motion the left-hand side of (15) equals the volume integral of $-\text{curl}\,\bar{q}$.

As a next step one may take, with $K=0$, for $\bar{A}$ some gradient, for example a function $g(r)\,\bar{r}$. Then the formulae

$$\int g(r)(\bar{r} \times \bar{q})\cdot d\bar{S} = \int g(r)(\bar{q} \times \bar{n})\cdot\bar{r}dS = 0 \tag{16}$$

result. These integrals are zero even when extended over a surface surrounding an obstacle since they vanish for the spheres $r=\text{const.}$ It is understood that singular points of $g(r)$ have to be excluded.

If K is chosen to equal x, the vector $\bar{A}$ must be (except for a gradient which always can be added) $\bar{r}\times\bar{i}_x/2$ where $\bar{i}_x$ is the unit vector in x-direction. Combining this with the analogous formulae for y and z, the following vector equation results:

$$\tag{17} \begin{aligned} &\int [\bar{r}(\bar{q}\cdot\bar{n}) - (\bar{r}/2)\times(\bar{q}\times\bar{n})]dS \\ &\qquad = \int [\bar{r}(\bar{q}\cdot\bar{n}) - \bar{q}(\bar{r}\cdot\bar{n})/2 + \bar{n}(\bar{r}\cdot\bar{q})/2Sd] = 0. \end{aligned}$$

This may be considered as a relation for the *moment of surface circulation.*

If any spherical harmonic is taken for K, it is easy to find a corresponding vector $\bar{A}$. Two examples may suffice:

$$\tag{18} K = xy, \qquad A_x = 0, \qquad A_y = 0, \qquad A_z = (y^2 - x^2)/2$$

and

$$\tag{19} K = 1/r,\ A_x = zy/r(x^2 + y^2),\ A_y = -zx/r(x^2 + y^2),\ A_z = 0.$$

Again, one may combine three expressions of each kind and thus form a vector integral, like (17), which vanishes when taken over the complete boundary of a region of regularity.

4. Cauchy formula. In the two-dimensional case the value of any analytic function of $x+iy$ and $u-iv$ in an arbitrary point can be expressed in terms of its values on a curve surrounding this point. This is based on a formula of the type (4) where the f has such a singularity at $\zeta=0$ that the integral over a circle of radius ρ has a finite limit when ρ tends toward zero. To achieve this kind of singularity one has simply to take $f(v, \zeta)=g(v, \zeta)/\zeta$ where g is regular in $\zeta=0$, since

$$\tag{20} \lim_{\rho=0} \int_{C_\rho} \frac{d\zeta}{\zeta} = 2\pi i$$

(where C_ρ is the circle $|\zeta|=\rho$). Then, if g is continuous at $\zeta=0$,

$$\tag{21} \lim_{\rho=0} \int \frac{g(v, \zeta)d\zeta}{\zeta} = 2\pi i g(0, v[0]).$$

In order to obtain analogous equations in the three-dimensional case we have to look for functions $\bar{f}$ such that fr^2 remains finite when r

tends to zero. It is seen from the general result in §2 that this can never be the case with a term of second order in $\bar{q}$ since its coefficient is linear in $\bar{r}$. We omit the case of order zero terms, that is, of terms which do not involve $\bar{q}$. Thus only the terms $K\bar{q}$ and $\bar{A}\times\bar{q}$ are left where K must be a harmonic function according to (12).

The only harmonic functions which behave like $1/r^2$ at the origin are x/r^3, y/r^3, z/r^3. Taking the first for K one can easily find from equation (12) the components of A (as in the examples (18), (19)):

$$K = x/r^3, \qquad A_x = 0, \qquad A_y = -z/r^3, \qquad A_z = y/r^3. \tag{22}$$

The components of $K\bar{q}+(\bar{A}\times\bar{q})$ are then, except for the factor $1/r^3$:

$$xu - yv - zw, \qquad xv + yu, \qquad xw + zu \tag{23}$$

and, if y/r^3 and z/r^3 are taken for K:

$$\begin{aligned} &yu + xv, \qquad yv - zw - xu, \qquad yw + zv, \\ &zu + xw, \qquad zv + yw, \qquad zw - xu - yv. \end{aligned} \tag{23'}$$

If each of the three vectors represented by (23) and (23′) is multiplied by $\bar{n}$, the three scalar products form the components of one vector which can be considered as a *particular triple product* of $\bar{r}$, $\bar{q}$, and $\bar{n}$. This product may be designated by $(\bar{r}, \bar{q}; \bar{n})$ and can be expressed in three different forms by combinations of the usual products:

$$\begin{aligned} (\bar{r}, \bar{q}; \bar{n}) &= \bar{r}(\bar{q}\cdot\bar{n}) + \bar{q}(\bar{r}\cdot\bar{n}) - \bar{n}(\bar{q}\cdot\bar{r}) \\ &= \bar{r}(\bar{q}\cdot\bar{n}) + \bar{r}\times(\bar{q}\times\bar{n}) \\ &= \bar{q}(\bar{r}\cdot\bar{n}) + \bar{q}\times(\bar{r}\times\bar{n}). \end{aligned} \tag{24}$$

On a sphere $r=$const., where $\bar{n}=\bar{r}/r$, this reduces to $r\bar{q}$, as the last expression shows. Therefore, the integral over an infinitesimal sphere reduces to

$$\lim_{\rho=0}\int_{(S_\rho)} \frac{r\bar{q}}{r^3}\,dS = \bar{q}(0)\int_{(S_\rho)}\frac{dS}{r^2} = 4\pi\bar{q}(0) \tag{25}$$

and the Cauchy formula for the three-dimensional space reads

$$\bar{q}(0) = \frac{1}{4\pi}\int\frac{(\bar{r}, \bar{q}; \bar{n})}{r^3}\,dS, \tag{26}$$

where the integral has to be extended over a surface surrounding the origin and including no singular point of the velocity distribution.[1]

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[1] This formula has already been given by Fulton and Rainich (Amer. J. Math. vol. 54 (1932) pp. 235–241). See also S. Bergman, Bull. Amer. Math. Soc. vol. 49 (1943) p. 174.

The equation (26) *is the only formula of Cauchy type involving the velocity of a potential flow in three dimensions.* It follows from the preceding that for no other function of $\bar{r}$ and $\bar{q}$, for example the products $\bar{r}\cdot\bar{q}$ and $\bar{r}\times\bar{q}$, its value in an arbitrary point can be computed from the values the function assumes on a surrounding surface. This, however, is only seemingly less general than what is the case in the two-dimensional problem. Here, any analytic function of v and ζ can be computed in the form (21). But giving the values of $g(v, \zeta)$ at all points of the contour is the same as giving here v, while in space the knowledge of a function, for example $\bar{r}\times\bar{q}$, is not sufficient to determine $\bar{q}$. If $\bar{q}$ is given in all points of S, equation (26) supplies the values of any function of $\bar{r}$ and $\bar{q}$ in an arbitrary point (within the range of regularity).

The matrix formed by the nine quantities (22) and (22′) can be considered as a symmetric tensor by which the unit vector $\bar{n}$ has to be multiplied. Using Gibbs' notation of dyadics and the sign I for the unit tensor, one can write this tensor as

$$(\bar{r};\bar{q}) + (\bar{q};\bar{r}) - I(\bar{r}\cdot\bar{q}) \tag{27′}$$

or, applying Jaumann's dyadics also, as

$$(\bar{r};\bar{q}) + (\bar{r}\underset{;}{\times}\bar{q}). \tag{27}$$

The linear transformation determined by this tensor consists of the reflection with respect to the bisectrix of $\bar{r}$ and $\bar{q}$ and a contraction at the rate $\cos(\bar{r}, \bar{q})$ in the direction normal to the plane extended by $\bar{r}$ and $\bar{q}$. In the two-dimensional case the reflection only remains, which is easily expressed in terms of complex numbers.

5. **Biot-Savart formula.** The Cauchy formula (26) shall now be applied to the following case. We consider the space included in a sphere of infinite radius. The velocity at infinity has the constant value $\bar{q}_\infty$. We admit a finite number of discontinuity surfaces S_1, S_2, S_3, $\cdots$ which, of course, are tangential to the velocity vector, that is, $\bar{q}\cdot\bar{n}=0$ on both sides of each S_ν. Some of the S_ν may be closed surfaces, in the inner of which $\bar{q}$ must necessarily vanish. Such S_ν represent rigid bodies (obstacles) that are surrounded by the flow.

If any point not on a discontinuity surface is chosen as origin, equation (26) will give its velocity vector $\bar{q}(0)$, if the integral is extended over the infinite sphere and both sides of each discontinuity surface. The integral over the sphere supplies $4\pi\bar{q}_\infty$, according to what was said in connection with (25). On a closed S_ν the integral for the inner side is zero, since here $\bar{q}=0$. Denoting now by $\bar{n}$ the outward normal of the body (=inside normal of the fluid mass), we have to reverse the sign of the integrand in (26) which then reads, according to $\bar{q}\cdot\bar{n}=0$ and the second expression (24), $(\bar{q}\times\bar{n})\times\bar{r}/r^3$. On each open discontinuity surface, $\bar{q}$ may refer to the side toward which $\bar{n}$ points. Then (26) becomes

$$\bar{q}(0) = \bar{q}_\infty + \frac{1}{4\pi}\int \frac{(\bar{q}\times\bar{n})\times\bar{r}}{r^3}\,dS, \tag{28}$$

where the integral has to be extended over both sides of all discontinuity surfaces. If only the outer side of the closed S, and one side of the open S, is taken and $\bar{q}'$ called the velocity on the opposite sides, one can write instead of (28):

$$\bar{q}(0) = \bar{q}_\infty + \frac{1}{4\pi}\int \frac{[(\bar{q}-\bar{q}')\times\bar{n}]\times\bar{r}}{r^3}\,dS, \tag{28'}$$

where $\bar{q}'=0$ for the closed S, which represent obstacles.

The equations (28) and (28′) are in close analogy to the so-called *Biot-Savart formula* which—in hydrodynamic terms—gives the velocity "induced" at the point $r=0$ by a "vortex line" of vorticity $\bar{C}$:

$$\bar{q}(0) = \frac{1}{4\pi}\int \frac{\bar{C}\times\bar{r}}{r^3}\,ds, \tag{29}$$

the integral being taken over the arc length of the (closed) vortex line. Thus, the result (28′) can be stated as follows: *The velocity excess $\bar{q}-\bar{q}_\infty$ can be considered as induced by vortex sheets extended over all discontinuity surfaces (including the obstacle surfaces), with a vortex-line density of magnitude $\bar{q}-\bar{q}'$, directed perpendicular to $\bar{q}-\bar{q}'$ in the tangential plane ($\bar{q}'=0$ on the closed surfaces).*

This statement is the basis of the three-dimensional wing theory of Lanchester and Prandtl. Its equivalent was deduced by Prandtl and his followers from certain assumptions about impulsive forces and fictitious mass forces inside the bodies.

The formula (28) or (28′) is valid only if the origin O is not a point of one of the discontinuity surfaces. If O approaches S, the limit of the expression on the right-hand side includes, in addition to the integral over S, the quantity $[\bar{q}(0)-\bar{q}'(0)]/2$, as can be shown easily in the usual way. Therefore (28′) becomes, for O on S:

$$[\bar{q}(0)+\bar{q}'(0)]/2 = \bar{q}_\infty + \frac{1}{4\pi}\int \frac{[(\bar{q}-\bar{q}')\times\bar{n}]\times\bar{r}}{r^3}\,dS, \tag{30}$$

where again $\bar{q}'(0)$ vanishes if the point under consideration falls on a closed discontinuity surface. This relation, in connection with the fact that on both sides of a free discontinuity surface the pressure (the velocity amount) must have the same value, leads to the integral equations which determine the vortex distribution on S as well as the shape of those parts of S which constitute the free discontinuity surfaces.

6. The second order terms. Momentum theorems. The last two terms in (10) with $\bar{L}$ satisfying the condition (13) lead to two new integral formulae. First, taking $L_1=0$ we have

$$\int [L_0 q^2 - 2(\bar{L}_0\cdot\bar{q})\bar{q}]\cdot d\bar{S} = \int [q^2(\bar{L}_0\cdot\bar{n}) - 2(\bar{L}_0\cdot\bar{q})(\bar{q}\cdot\bar{n})]dS. \tag{31}$$

If $\bar{L}_0$ is identified with the unit vector in x-direction, (31) is the x-component of the equation

$$\int [q^2\bar{n} - 2(\bar{q}\cdot\bar{n})\bar{q}]dS = 0. \tag{32}$$

The vector $\bar{n}'$ defined by

$$\bar{n}' = \bar{n} - 2((\bar{q}/q)\cdot\bar{n})\bar{q}/q \tag{33}$$

is easily seen to be a unit vector, symmetrical to $\bar{n}$ with respect to the plane normal to $\bar{q}$. Equation (32) can then be written as

$$\int q^2\bar{n}'dS = 0. \tag{32'}$$

If, on the other hand, in equation (13), $\bar{L}_0$ is taken as zero the integral formula will read

$$\int [(\bar{L}_1 \times \bar{r})q^2 - \{2(\bar{L}_1 \times \bar{r})\cdot\bar{q}\}\bar{q}]\cdot d\bar{S} = 0. \tag{34}$$

Setting here $d\bar{S}=\bar{n}dS$ and using the identity $(\bar{a}\times\bar{b})\cdot\bar{c}=\bar{a}\cdot(\bar{b}\times\bar{c})$, equation (34) becomes

$$\int \{q^2\bar{L}_1\cdot(\bar{r}\times\bar{n}) - 2[\bar{L}_1\cdot(\bar{r}\times\bar{q})](\bar{q}\cdot\bar{n})\}dS = \int q^2\bar{L}_1\cdot(\bar{r}\times\bar{n}')dS = 0.$$

This is, if $\bar{L}_1$ is considered as the unit vector in x-direction, the x-component of the equation

$$\int \bar{r}\times[q^2\bar{n} - 2(\bar{q}\cdot\bar{n})\bar{q}]dS = 0 \tag{35}$$

or using (33)

$$\int q^2(\bar{r} \times \bar{n}')dS = 0. \tag{35'}$$

The two relations (32′) and (35′) are the three-dimensional generalizations of $\int v^2 d\zeta = 0$ and $\int \zeta v^2 d\zeta = 0$ which are sometimes known as the *Blasius formulae*. In fact, when applied to a two-dimensional velocity distribution and a cylindrical S, our equations (32′) and (35′) coincide with those complex integrals. The argument of the complex number $v^2 d\zeta$ corresponds to the direction of the "reflected normal" $\bar{n}'$.

The physical meaning of the integrals (32) and (35) can easily be understood. If S is the closed surface of a rigid body, $\bar{q}\cdot\bar{n}$ equals zero on S and $(-\rho/2)q^2$ where ρ is the density can be considered as the pressure value. As $\bar{n}$ is the direction of the thrust upon the body, the integrals of $(-\rho/2)\int q^2\bar{n}'dS$ and $(-\rho/2)\int q^2(\bar{r}\times\bar{n}')dS$ give the resultant force and the moment of the fluid reaction upon the body. Thus the equations (32′) and (35′) allow the following statement: *If S is a closed surface the inner of which consists of some rigid bodies with surfaces S_1, S_2, S_3, $\cdots$ and of a fluid mass with finite velocity distribution satisfying* (1), *then the resultant force $\bar{F}$ and the moment $\bar{M}$ of the fluid reaction upon S_1, S_2, S_3, $\cdots$ are given by*

$$\bar{F} = \frac{\rho}{2}\int q^2\bar{n}'dS, \qquad \bar{M} = \frac{\rho}{2}\int q^2(\bar{r} \times \bar{n}')dS \tag{36}$$

where $\bar{n}'$, at each point of S, is the normal vector reflected with respect to the plane perpendicular to $\bar{q}$.

These are evidently the momentum theorems for the reactions upon bodies surrounded by a potential flow. Note that free discontinuity surfaces inside S do not invalidate (36) since q^2 must have the same value on both sides and $\bar{q}\cdot\bar{n}$ must vanish at each point of such a surface. Another useful form of the momentum theorems would be this: *If the complete boundary S of a region of potential flow with finite velocities consists of two parts S' and S'' where S'' is tangential to the flow, the integrals* (36), *extended over S' only, give the fluid reactions exerted along S''*. This would apply, for example, to the flow through a channel where S'' consists of the walls of the channel and S' of two cross sections, one at the entrance and the other at the exit.

7. D'Alembert's paradox and the generalization of Joukowski's formula. If a rigid body or a group of bodies is moving at constant velocity $-\bar{c}$ through a bulk of fluid at rest, the inverse motion, with the bodies at rest and the fluid velocity $\bar{q}_\infty = \bar{c}$ at infinity, is a potential flow. Setting

$$\bar{q} = \bar{q}_\infty + \bar{q}' = \bar{c} + \bar{q}', \tag{37}$$

$\bar{q}'$ as well as $\bar{q}$ fulfill the equations (1) and $\bar{q}'$ vanishes at infinity.

To find the fluid reactions upon the moving bodies we introduce (37) in the expression $q^2\bar{n} - 2(\bar{q}\cdot\bar{n})\bar{q}$ of (32):

$$\begin{aligned} q^2\bar{n} - 2(\bar{q}\cdot\bar{n})\bar{q} = {} & c^2\bar{n} - 2(\bar{c}\cdot\bar{n})\bar{c} \\ & + 2(\bar{c}\ \bar{q}')\bar{n} - 2(\bar{c}\cdot\bar{n})q' - 2(\bar{q}'\cdot\bar{n})\bar{c} \\ & + q'^2\bar{n} - 2(\bar{q}'\cdot\bar{n})\bar{n}. \end{aligned} \tag{38}$$

The first terms, depending on $\bar{c}$ only, give no contribution to either of the integrals (32) or (35). The bilinear terms in the second line can be written, according to (24), as

$$-2(\bar{c}, \bar{q}'; \bar{n}) = -2\bar{c}(\bar{q}'\cdot\bar{n}) - 2\bar{c} \times (\bar{q}' \times \bar{n}). \tag{39}$$

Let us now assume that $\bar{q}$ can be developed into a power series at $r = \infty$. Then, since $\bar{q}$ is the gradient of a harmonic function corresponding to (2), the first terms are

$$\begin{aligned} \bar{q} &= \bar{c} + \operatorname{grad}\left(\frac{a_0}{r} + \frac{\bar{a}\cdot\bar{r}}{r^3} + \cdots\right), \\ \bar{q}' &= \frac{-a_0\bar{r}}{r^3} + \left[\frac{\bar{a}}{r^3} - \frac{3(\bar{a}\cdot\bar{r})\bar{r}}{r^5}\right] + \cdots. \end{aligned} \tag{40}$$

The first term in $\bar{q}'$ implies a source or sink at infinity. If this is excluded, the lowest terms are of order r^{-3} and that means that the contribution to the integral (32), if it is extended over the sphere $r = \infty$, is zero. Thus the conclusion is reached: *If some bodies are moving at constant velocity $-\bar{c}$ through a bulk of fluid at rest and the velocity distribution is supposed to be developable at infinity and no source or sink admitted, then the total air reaction upon the bodies can consist of a couple only, the resultant force being zero.*

This statement includes the so-called *d'Alembert paradox* which states that a sphere moving through a fluid at rest experiences no resultant force under normal conditions. It is seen from our argument that this is still valid if some discontinuity surfaces, not extending to infinity, are present.

If the boundary of one of the rigid bodies or a discontinuity surface extends to infinity, the assumption of a developable $\bar{q}$-distribution is not permitted. Then, a resultant force determined by the second and third groups of terms in (38) may exist. It follows from (39), since the integral of $\bar{q}'\cdot\bar{n}$ over a closed surface must vanish, that the terms linear in $\bar{q}'$ are determined by the product of $2\bar{c}$ and the integral of $\bar{q}'\times\bar{n}$ which is obviously the same as that of $\bar{q}\times\bar{n}$, that is, the surface circulation according to (15). Thus the following conclusion is reached: *The resultant force of the fluid reaction upon bodies moving at a constant speed* $-\bar{c}$ *through a fluid at rest consists of two parts; the first one, linear in* $\bar{q}'$, *is*

$$\bar{F}_1 = -\rho\bar{c}\times\bar{C} \tag{41}$$

where $\bar{C}$ *is the surface circulation, defined as the left-hand expression in* (15); *the second, quadratic, part is*

$$F_2 = \frac{\rho}{2}\int q'^2\bar{n}'dS \tag{42}$$

where $\bar{n}'$ *is the normal vector reflected on the plane perpendicular to* $\bar{q}'$.

In the two-dimensional case, if there are no discontinuity surfaces, only $\bar{F}_1$ exists and the analogy to (41) is known as the Kutta-Joukowski formula. In the case of flow with separation (Helmholtz flow) $\bar{F}_2$ is prevailing.—Note that in the three-dimensional problem, $\bar{F}_1$ is perpendicular to the velocity vector $\bar{c}$ while, in general, nothing can be said about the direction of $\bar{F}_2$.

In the "regular" case, that is, with $\bar{q}$ developable at infinity, the order of magnitude of $\bar{q}'$ is r^{-3} and thus the surface circulation $\bar{C}$ as well as the integral in (42) vanish.

Let us, finally, compute the first order terms $\bar{M}_1$ (the terms linear in $\bar{q}'$) in the expression for the moment. From the second equation (36) and the second line in (38) we find

$$\begin{aligned} \bar{M} &= -\rho\int\bar{r}\times[(\bar{q}'\cdot\bar{n})\bar{c} + (\bar{c}\cdot\bar{n})\bar{q}' - (\bar{c}\cdot\bar{q}')\bar{n}]dS \\ &= -\rho\int\bar{r}\times(\bar{c}, \bar{q}'; \bar{n})dS. \end{aligned} \tag{43}$$

If $\bar{q}'$ is introduced from (40) with $a_0=0$ and a sphere of radius r with $\bar{n}=\bar{r}/r$ taken for S, one has

$$\bar{q}'\cdot\bar{n} = -2\bar{a}\cdot\bar{r}/r^4, \qquad \bar{r}\times\bar{q}' = \bar{r}\times\bar{a}/r^3, \qquad \bar{r}\times\bar{n} = 0.$$

On the other hand, $\bar{M}_1$ can be identified with the total moment $\bar{M}$ since for an infinite r the quadratic term $\bar{M}_2$ vanishes. Thus

$$\overline{M} = \rho\bar{c} \times \int - 2\frac{\bar{a}\cdot\bar{r}}{r^4}\bar{r}dS + \rho\bar{a} \times \int \frac{\bar{c}\cdot\bar{r}}{r^4}\bar{r}dS.$$

It can be seen easily that for any sphere of radius r

$$(44) \qquad \int (\bar{a}\cdot\bar{r})\bar{r}dS = \frac{4\pi}{3}r^4\bar{a}.$$

Therefore

$$(45) \qquad \overline{M} = (4\pi/3)[-(2\bar{c} \times \bar{a}) + (\bar{a} \times \bar{c})] = 4\pi(\bar{a} \times \bar{c}).$$

Since $\bar{a}$ is proportional to $\bar{q}'$ and, other things being equal, $\bar{q}'$ must be proportional to $\bar{c}$ we have obtained the result—known in the classical theory: *In the regular case, that is, with $\bar{q}'$ developable at infinity, the fluid reaction upon bodies moving at constant speed $-\bar{c}$ has a moment vector perpendicular to $\bar{c}$ and proportional to ρ and c^2.*

A more detailed discussion of this and related questions is beyond the scope of the present note.

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DAVID HILBERT AND HIS MATHEMATICAL WORK

HERMANN WEYL

A great master of mathematics passed away when David Hilbert died in Göttingen on February the 14th, 1943, at the age of eighty-one. In retrospect it seems to us that the era of mathematics upon which he impressed the seal of his spirit and which is now sinking below the horizon achieved a more perfect balance than prevailed before and after, between the mastering of single concrete problems and the formation of general abstract concepts. Hilbert's own work contributed not a little to bringing about this happy equilibrium, and the direction in which we have since proceeded can in many instances be traced back to his impulses. No mathematician of equal stature has risen from our generation.

America owes him much. Many young mathematicians from this country, who later played a considerable role in the development of American mathematics, migrated to Göttingen between 1900 and 1914 to study under Hilbert. But the influence of his problems, his viewpoints, his methods, spread far beyond the circle of those who were directly inspired by his teaching.

Hilbert was singularly free from national and racial prejudices; in all public questions, be they political, social or spiritual, he stood forever on the side of freedom, frequently in isolated opposition against the compact majority of his environment. He kept his head clear and was not afraid to swim against the current, even amidst the violent passions aroused by the first world war that swept so many other scientists off their feet. It was not mere chance that when the Nazis "purged" the German universities in 1933 their hand fell most heavily on the Hilbert school and that Hilbert's most intimate collaborators left Germany either voluntarily or under the pressure of Nazi persecution. He himself was too old, and stayed behind; but the years after 1933 became for him years of ever deepening tragic loneliness.

It was another Germany in which he was born on January 23, 1862, and grew up. Königsberg, the eastern outpost of Prussia, the city of Kant, was his home town. Contrary to the habit of most German students who used to wander from university to university, Hilbert studied at home, and it was in his home university that he climbed the first rungs of the academic ladder, becoming Privatdozent and in due time ausserordentlicher Professor. During his entire life he preserved uncorrupted the characteristic Baltic accent. His reputation as a leading algebraist was well established when on Felix Klein's initiative he was offered a full professorship at Göttingen in 1895. From then on until the end of his life Hilbert remained in Göttingen. He was retired in 1930.

When one inquires into the dominant influences acting upon Hilbert in his formative years one is puzzled by the peculiarly ambivalent character of his relationship to Kronecker: dependent on him, he rebels against him. Kronecker's work is undoubtedly of paramount importance for Hilbert in his algebraic period. But the old gentleman in Berlin, so it seemed to Hilbert, used his power and authority to stretch mathematics upon the Procrustean bed of arbitrary philosophical principles and to suppress such developments as did not conform: Kronecker insisted that existence theorems should be proved by explicit construction, in terms of integers, while Hilbert was an early champion of Georg Cantor's general set-theoretic ideas. Personal reasons added to the bitter feeling.[1] A late echo of this old feud is the polemic against Brouwer's intuitionism with which the sexagenarian Hilbert opens his first article on "Neubegründung der Mathematik" (1922): Hilbert's slashing blows are aimed at Kronecker's ghost whom he sees rising from his grave. But inescapable ambivalence even here —while he fights him he follows him: reasoning along strictly intuitionistic lines is found necessary by him to safeguard non-intuitionistic mathematics.

More decisive than any other influence for the young Hilbert at Königsberg was his friendship with Adolf Hurwitz and Minkowski. He got his thorough mathematical training less from lectures, teachers or books, than from conversation. "During innumerable walks, at times undertaken day after day," writes Hilbert in his obituary on Hurwitz, "we roamed in these eight years through all the corners of mathematical science, and Hurwitz with his extensive, firmly grounded and well ordered knowledge was for us always the leader." Closer and of a very intimate character was Hilbert's lifelong friendship with Minkowski. The Königsberg circle was broken up when Hurwitz in 1892 left for Zürich, soon to be followed by Minkowski. Hilbert first became Hurwitz's successor in Königsberg and then moved on to Göttingen. The year 1902 saw him and Minkowski reunited in Göttingen where a new chair of mathematics had been created for Minkowski. The two friends became the heroes of the great and brilliant period which our science experienced during the following decade in Göttingen, unforgettable to those who lived through it. Klein, for whom mathematical research had ceased to be the central interest, ruled over it as a distant but benevolent god in the clouds. Too soon was this happy constellation dissolved by Minkowski's sudden death in 1909. In a memorial address before the Göttingen Gesellschaft der Wissenschaften, Hilbert spoke thus about his friend: "Our science, which we loved above everything, had brought us together. It appeared to us as a flowering garden. In this garden there are beaten paths where one may look around at leisure and enjoy oneself without effort, especially at the side of a congenial companion. But we also liked to seek out hidden trails and discovered many a novel view, beautiful to behold, so we thought, and when we pointed them out to one another our joy was perfect."

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[1] How Georg Cantor himself in his excitability suffered from Kronecker's opposition is shown by his violent outbursts in letters to Mittag-Leffler; see A. Schoenflies, *Die Krisis in Cantors mathematischem Schaffen*, Acta Math. vol. 50 (1928) pp. 1–23.

I quote these words not only as testimony of a friendship of rare depth and fecundity that was based on common scientific interest, but also because I seem to hear in them from afar the sweet flute of the Pied Piper that Hilbert was, seducing so many rats to follow him into the deep river of mathematics. If examples are wanted let me tell my own story. I came to Göttingen as a country lad of eighteen, having chosen that university mainly because the director of my high school happened to be a cousin of Hilbert's and had given me a letter of recommendation to him. In the fullness of my innocence and ignorance I made bold to take the course Hilbert had announced for that term, on the notion of number and the quadrature of the circle. Most of it went straight over my head. But the doors of a new world swung open for me, and I had not sat long at Hilbert's feet before the resolution formed itself in my young heart that I must by all means read and study whatever this man had written. And after the first year I went home with Hilbert's *Zahlbericht* under my arm, and during the summer vacation I worked my way through it—without any previous knowledge of elementary number theory or Galois theory. These were the happiest months of my life, whose shine, across years burdened with our common share of doubt and failure, still comforts my soul.

The impact of a scientist on his epoch is not directly proportional to the scientific weight of his research. To be sure, Hilbert's mathematical work is of great depth and universality, and yet his tremendous influence is not accounted for by it alone. Gauss and Riemann, to mention two other Göttingers, are certainly of no lesser stature than Hilbert, but they made little stir among their contemporaries and no "school" of devoted followers formed around them. No doubt this is due in part to the changing conditions of time, but the character of the men is probably more decisive. A taste for solitude, even obscurity, is in no way irreconcilable with great creative gifts. But Hilbert's was a nature filled with the zest of living, seeking intercourse with other people, above all with younger scientists, and delighting in the exchange of ideas. Just as he had learned from Hurwitz, so he taught his own pupils, at least those in whom he took a deeper personal interest: on far-flung walks through the woods surrounding Göttingen or, on rainy days, as "peripatetics" in his covered garden walk. His optimism, his spiritual passion, his unshakable faith in the supreme value of science, and his firm confidence in the power of reason to find simple and clear answers to simple and clear questions were irresistibly contagious. If Kant through critique and analysis arrived at the principle of the supremacy of practical reason, Hilbert incorporated, as it were, the supremacy of pure reason—sometimes with laughing arrogance (*arrogancia* in the Spanish sense), sometimes with the ingratiating smile of intellect's spoiled child, but most of the time with the seriousness of a man who believes and must believe in what is the essence of his own life. His enthusiasm was compatible with critical acumen, but not with scepticism. The snobbish attitude of pretended indifference, of "merely fooling around with things," or even of playful cynicism, were unknown in his circle. You had better think twice before you uttered a lie or an empty phrase to him: his directness could be something to be afraid of. He was enormously industrious and liked to quote Lichtenberg's saying: "Genius is industry." Yet for all this there was light and laughter around him. He had great suggestive power; it sometimes lifted even mediocre minds high above their natural level to astonishing, though isolated achievements. I do not remember which mathematician once said to him: "You have forced us all to consider important those problems which you considered important." His vision and experience inspired confidence in the fruitfulness of the hints he dropped. He did not hide his light under a bushel. In his papers one encounters not infrequently utterances of pride in a beautiful or unexpected result, and in his legitimate satisfaction he sometimes did not give to his predecessors on whose ideas he built all the credit they deserved. The problems of mathematics are not isolated problems in a vacuum; there pulses in them the life of ideas which realize themselves *in concreto* through our human endeavors in our historical existence, but forming an indissoluble whole transcend any particular science. Hilbert had the power to evoke this life; against it he measured his individual scientific efforts and felt responsible for it in his own sphere. In this sense he was a philosopher, not in the sense of adhering to one of the established epistemological or metaphysical doctrines. Does not in such personal qualities of the academic teacher, rather than in any objectivities or universally accepted metaphysics, lie the answer to Hutchins's quest for a true ***universitas literarum***?

Were it my aim to give a full picture of Hilbert's personality I should have to touch upon his attitude regarding the great powers in the lives of men: social and political organization, art, religion, morals and manners, family, friendship, love, and I should also probably have to indicate some of the shadows cast by so much light. I wanted merely to sketch the mathematical side of his personality in an attempt to explain, however incompletely, the peculiar charm and the enormous influence which he exerted. In appraising the latter one must not overlook the environmental factor. A German university in a small town like Göttingen, especially in the halcyon days before 1914, was a favorable milieu for the development of a scientific school. The high social prestige of the professors and everything connected with the university created an atmosphere the like of which has hardly ever existed in America. And once a band of disciples had gathered around Hilbert, intent on research and little worried by the chore of teaching, how could they fail to stimulate one another! We have seen the same thing happening here at Princeton during the first years of the Institute for Advanced Study; there is a kind of snowball effect in the formation of such condensation points of scientific research.

Before giving a more detailed account of Hilbert's work, it remains to characterize in a few words the peculiarly Hilbertian brand of mathematical thinking. It is reflected in his literary style which is one of great ***lucidity***. It is as if you were on a swift walk through a sunny open landscape; you look freely around, demarcation lines and connecting roads are pointed out to you, before you must brace yourself to climb the hill; then the path goes straight up, no ambling around, no detours. His style has not the terseness of many of our modern authors in mathematics, which is based on the assumption that printer's labor and paper are costly but the reader's effort and time are not. In carrying out a complete induction Hilbert finds time to develop the first two steps before formulating the general conclusion from n to $n+1$. How many examples illustrate the fundamental theorems of his algebraic papers—examples not constructed ad hoc, but genuine ones worth being studied for their own sake!

In Hilbert's approach to mathematics, ***simplicity*** and ***rigor*** go hand in hand. The generation before him, nay even most analysts of his time, felt the growing demand for rigor imposed upon analysis by the critique of the 19th century, which culminated in Weierstrass, as a heavy yoke that made their steps dragging and awkward. Hilbert did much to change that attitude. In his famous address, *Mathematische Probleme*, delivered before the Paris Congress in 1900, he stresses the importance of great concrete fruitful *problems*. "As long as a branch of science," says he, "affords an abundance of problems, it is full of life; want of problems means death or cessation of independent development. Just as every human enterprise prosecutes final aims, so mathematical research needs problems. Their solution steels the force of the investigator; thus he discovers new methods and viewpoints and widens his horizon." "One who without a definite problem before his eyes searches for methods, will probably search in vain." The *methodical unity of mathematics* was for him a matter of belief and experience. Again I quote his own words: "The question is forced upon us whether mathematics is once to face what other sciences long ago experienced, namely the falling apart into subdivisions whose representatives are hardly able to understand each other and whose connections for this reason will become ever looser. I neither wish nor believe it. The science of mathematics as I see it is an indivisible whole, an organism whose ability to survive rests on the connection between its parts." A characteristic feature of Hilbert's method is a peculiarly *direct attack* on problems, unfettered by algorithms; he always goes back to the questions in their original simplicity. An outstanding example is his salvage of Dirichlet's principle which had fallen a victim of Weierstrass's criticism, but his work abounds in similar examples. His strength, equally disdainful of the convulsion of Herculean efforts and of surprising tricks and ruses, is combined with an uncompromising *purity*.

Hilbert helped the reviewer of his work greatly by seeing to it that it is rather neatly cut into different periods during each of which he was almost exclusively occupied with one particular set of problems. If he was engrossed in integral equations, integral equations seemed everything; dropping a subject, he dropped it for good and turned to something else. It was in this characteristic way that he achieved universality. I discern five main periods: i. Theory of invariants (1885–1893). ii. Theory of algebraic number fields (1893–1898). iii. Foundations, (a) of geometry (1898–1902), (b) of mathematics in general (1922–1930). iv. Integral equations (1902–1912). v. Physics (1910–1922). The headings are a little more specific than they ought to be. Not all of Hilbert's algebraic achievements are directly related to invariants. His papers on calculus of variations are here lumped together with those on integral equations. And of course there are some overlappings and a few stray children who break the rules of time, the most astonishing his proof of Waring's theorem in 1909.

His Paris address on "Mathematical problems" quoted above straddles all fields of our science. Trying to unveil what the future would hold in store for us, he posed and discussed twenty-three unsolved problems which have indeed, as we can now state in retrospect, played an important role during the following forty odd years. A mathematician who had solved one of them thereby passed on to the honors class of the mathematical community.

LITERATURE

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Hilbert is co-author of the following works:

R. Courant and D. Hilbert, *Methoden der mathematischen Physik*, Berlin, vol. 1, 2d ed., 1931, vol. 2, 1937.

D. Hilbert and W. Ackermann, *Grundzüge der theoretischen Logik*, Berlin, 1928.

D. Hilbert and S. Cohn-Vossen, *Anschauliche Geometrie*, Berlin, 1932.

D. Hilbert and P. Bernays, *Grundlagen der Mathematik*, Berlin, vol. 1, 1934, vol. 2, 1939.

The Collected Papers contain articles by B. L. van der Waerden, H. Hasse, A. Schmidt, P. Bernays, and E. Hellinger, on Hilbert's work in algebra, in number theory, on the foundations of geometry and arithmetics, and on integral equations. These articles trace the development after Hilbert, giving ample references. The reader may also consult a number of Die Naturwissenschaften vol. 10 (1922) pp. 65–104, dedicated to Hilbert, which surveys his work prior to 1922, and an article by L. Bieberbach, *Ueber den Einfluss von Hilberts Pariser Vortrag über "Mathematische Probleme" auf die Entwicklung der Mathematik in den letzten dreissig Jahren*, Die Naturwissenschaften vol. 18 (1930) pp. 1101–1111. O. Blumenthal wrote a life of Hilbert (Collected Papers, vol. 3, pp. 388–429).

I omit all quotations of literature covered by these articles.

THEORY OF INVARIANTS

The classical theory of invariants deals with polynomials $J = J(x_1 \cdots x_n)$ depending on the coefficients $x_1, \cdots, x_n$ of one or several ground forms of a given number of arguments $\eta_1, \cdots, \eta_g$. Any linear substitution s of determinant 1 of the g arguments induces a certain linear transformation $U(s)$ of the variable coefficients

$x_1, \cdots, x_n$, $x \rightarrow x' = U(s)x$, whereby $J = J(x_1 \cdots x_n)$ changes into a new form $J(x_1' \cdots x_n') = J^s(x_1 \cdots x_n)$. J is an invariant if $J^s = J$ for every s. (The restriction to *unimodular* transformations s enables one to avoid the more involved concept of *relative* invariants and to remove the restriction to homogeneous polynomials, with the convenient consequence that the invariants form a *ring*.) The classical problem is a special case of the general problem of invariants in which s ranges over an arbitrary given abstract group Γ and $s \rightarrow U(s)$ is any representation of that group (that is, a law according to which every element s of Γ induces a linear transformation $U(s)$ of the n variables $x_1, \cdots, x_n$ such that the composition of group elements is reflected in composition of the induced transformations). The development before Hilbert had led up to two main theorems, which however had been proved in very special cases only. The first states that *the invariants have a finite integrity basis*, or that we can pick a finite number among them, say $i_1, \cdots, i_m$, such that every invariant J is expressible as a polynomial in $i_1, \cdots, i_m$. An identical relation between the basic invariants $i_1, \cdots, i_m$ is a polynomial $F(z_1 \cdots z_m)$ of m independent variables $z_1, \cdots, z_m$ which vanishes identically by virtue of the substitution

$$z_1 = i_1(x_1 \cdots x_n), \cdots, z_m = i_m(x_1 \cdots x_n).$$

The second main theorem asserts that *the relations have a finite ideal basis*, or that one can pick a finite number among them, say $F_1, \cdots, F_h$, such that every relation F is expressible in the form

$$F = Q_1F_1 + \cdots + Q_hF_h, \tag{1}$$

the Q_i being polynomials of the variables $z_1, \cdots, z_m$.

I venture the guess that Hilbert first succeeded in proving the *second* theorem. The relations F form a subset within the ring $k[z_1 \cdots z_m]$ of all polynomials of $z_1, \cdots, z_m$ the coefficients of which lie in a given field k. When Hilbert found his simple proof he could not fail to notice that it applied to any set of polynomials Σ whatsoever and he thus discovered one of the most fundamental theorems of algebra, which was instrumental in ushering in our modern abstract approach, namely:

(A) *Every subset Σ of the polynomial ring $k[z_1 \cdots z_m]$ has a finite ideal basis.*

Is it bad metaphysics to add that his proof turned out so simple *because* the proposition holds in this generality? The proof proceeds by the adjoining of one variable z_i after the other, the individual step being taken care of by the statement: If a given *ring* r satisfies the condition (P): that every subset of r has a finite ideal basis, then the ring $r[z]$ of polynomials of a single variable z with coefficients in r satisfies the same condition (P). Once this is established one gets not only (A) but also an arithmetic refinement discussed by Hilbert in which the field k of rational numbers is replaced by the ring of rational integers.

The subset Σ of relations to which Hilbert applies his theorem (A) is itself an *ideal*, and thus the ideal $\{F_1, \cdots, F_h\}$, that is, the totality of all elements of the form (1), $Q_i \varepsilon k[z_1 \cdots z_m]$, not only contains, but coincides with, Σ. The proof, however, works even if Σ is not an ideal, and yields at one stroke (1) the enveloping ideal $\{\Sigma\}$ of Σ and (2) the reduction of that ideal to a finite basis, $\{\Sigma\} = \{F_1, \cdots, F_h\}$.

Construction of a full set of relations $F_1, \cdots, F_h$ would finish the investigation of the algebraic structure of the ring of invariants were it true that any relation F can be represented in the form (1) *in one way only*. But since, generally speaking, this is not so, we must ask for the "vectors of polynomials" $M = (M_1, \cdots, M_h)$ for which $M_1F_1 + \cdots + M_hF_h$ vanishes identically in z (syzygy of first order). These linear relations M between $F_1, \cdots, F_h$ again form an ideal to which Theorem (A) is applicable, the basis of the M thus obtained giving rise to syzygies of the second order. To the first two main theorems Hilbert adds a third to the effect that if redundance is avoided, the chain of syzygies breaks off after at most m steps.

All this hangs in the air unless we can establish the *first main theorem*, which is of an altogether different character because it asks for an integrity, not an ideal basis. Discussing invariants we operate in the ring $k_x = k[x_1 \cdots x_n]$ of polynomials of $x_1, \cdots, x_n$ in a given field k. Hilbert applies his Theorem (A) to the totality $\mathfrak{J}$ of all *invariants J for which* $J(0, \cdots, 0) = 0$ (a subring of k_x which, by the way, is not an ideal!) and thus determines an ideal basis $i_1, \cdots, i_m$ of $\mathfrak{J}$. Each of the invariants $i = i_r$ may be decomposed into a sum $i = i^{(1)} + i^{(2)} + \cdots$ of homogeneous forms of degree 1, 2, $\cdots$, and as the summands are themselves invariants, the i_r may be assumed to be homogeneous forms of degrees $\nu_r \geqq 1$. Hilbert then claims that the $i_1, \cdots, i_m$ constitute an integrity basis for all invariants. I use a finite group Γ consisting of N elements s (although this case of the general problem of invariants was never envisaged by Hilbert himself) in order to illustrate the idea by which the transition is made. Every invariant J is representable in the form

$$J = c + L_1 i_1 + \cdots + L_m i_m \qquad (L_r \varepsilon k_x) \tag{2}$$

where c is the constant $J(0)$. If J is of degree ν one may lop off in L_r all terms of higher degree than $\nu-\nu_r$ without destroying the equation. If it were possible by some process to change the coefficients L in (2) into invariants, the desired result would follow by induction with respect to the degree of J. In the case of a finite group such a process is readily found: the process of averaging. The linear transformation $U(s)$ of the variables $x_1, \cdots, x_n$ induced by s carries (2) into

$$J = c + L_1^s \cdot i_1 + \cdots + L_m^s \cdot i_m.$$

Summation over s and subsequent division by the number N yields the relation

$$J = c + L_1^* i_1 + \cdots + L_m^* i_m$$

where

$$L_r^* = \frac{1}{N} \cdot \sum_s L_r^s.$$

It is of the same nature as (2), except for the decisive fact that according to their formation the new coefficients L^* are invariants.[3]

Actually Hilbert had to do, not with a finite group but with the classical problem in which the group Γ consists of all linear transformations s of g variables $\eta_1, \cdots, \eta_g$, and instead of the averaging process he had to resort to a differentiation process invented by Cayley, the so-called Cayley Ω-process, which he skillfully adapted for his end. (It is essential in Cayley's process that the g^2 components of the matrix s are independent variables; instead of the absolutely invariant polynomials J one has to consider relatively invariant homogeneous *forms* each of which has a definite degree and weight.)

Hilbert's theorem (A) is the foundation stone of the general theory of *algebraic manifolds*. Let us now think of k more specifically as the field of all complex numbers. It seems natural to define an algebraic manifold in the space of n coordinates $x_1, \cdots, x_n$ by a number of simultaneous algebraic equations $f_1=0, \cdots, f_h=0$ $(f_i \varepsilon k_x)$. According to Theorem (A), nothing would be gained by admitting an infinite number of equations. Let us denote by $Z(f_1, \cdots, f_h)$ the set of points $x=(x_1, \cdots, x_n)$ where $f_1, \cdots, f_h$ and hence all elements of the ideal $\mathfrak{J}=\{f_1, \cdots, f_h\}$ vanish simultaneously. g vanishes on $Z(f_1, \cdots, f_h)$ whenever $g \varepsilon \{f_1, \cdots, f_h\}$, but the converse is not generally true. For instance x_1 vanishes wherever x_1^3 does, and yet x_1 is not of the form $x_1^3 \cdot q(x_1 \cdots x_n)$. The language of the algebraic geometers distinguishes here between the simple plane $x_1 = 0$ and the triple plane, although the point set is the same in both cases. Hence what they actually mean by an algebraic manifold is the polynomial ideal and not the point set of its zeros. But even if one cannot expect that every polynomial g vanishing on $Z(f_1, \cdots, f_k) = Z(\mathfrak{J})$ is contained in the ideal $\mathfrak{J} = \{f_1, \cdots, f_k\}$ one hopes that at least some *power* of g will be. Hilbert's "*Nullstellensatz*" states that this is true, at least if k is the field of complex numbers. It holds in an arbitrary coefficient field k provided one admits points x the coordinates x_i of which are taken from *k or any algebraic extension of k*. Clearly this Nullstellensatz goes to the root of the very concept of algebraic manifolds.[3]

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[3] The example of finite groups is used here as an illustration only. Indeed, a direct elementary proof of the first main theorem for finite groups that makes no use of Hilbert's principle (A) has been given by E. Noether, Math. Ann. vol. 77 (1916) p. 89. In dividing by N we have assumed the field k to be of characteristic zero.

Actually Hilbert conceived it as a tool for the investigation of invariants. As we are now dealing with the full linear group let us consider homogeneous invariants only and drop the adjective homogeneous. Exclude the constants (the invariants of degree 0). Suppose we have ascertained μ non-constant invariants $J_1, \cdots, J_\mu$ such that every non-constant invariant vanishes wherever they vanish simultaneously. An ideal basis of the set $\mathfrak{J}$ of all non-constant invariants certainly meets the demand, but a system $J_1, \cdots, J_\mu$ may be had much more cheaply. Indeed, by a beautiful combination of ideas Hilbert proves that if for a given point $x = x^0$ there exists at all an invariant which neither vanishes for $x = x^0$ nor reduces to a constant, then there exists such an invariant whose weight does not exceed a certain a priori limit W (for example, $W = 9n(3n+1)^8$ for a ternary ground form of order n). Hence the $J_1, \cdots, J_\mu$ may be chosen from the invariants of weight not greater than W, and they thus come within the grasp of explicit algebraic construction.

When Hilbert published his proof for the existence of a finite ideal basis, Gordan the formalist, at that time looked upon as the king of invariants, cried out: "This is not mathematics, it is theology!" Hilbert remonstrated then, as he did all his life, against the disparagement of existential arguments as "theology," but we see how, by digging deeper, he was able to meet Gordan's constructive demands. By combining the Nullstellensatz with the Cayley process he further showed that every invariant J is an integral *algebraic* (though not an integral *rational*) function of $J_1, \cdots, J_\mu$, satisfying an equation

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[3] B. L. van der Waerden's book *Moderne Algebra*, vol. 2, 2nd ed., 1940, gives on pp. 1–72 an excellent account of the general algebraic concepts and facts with which we are here concerned.

$$J^{e} + G_1 J^{e-1} + \cdots + G_e = 0$$

in which the G's are polynomials of $J_1, \cdots, J_\mu$. Hence it must be possible by suitable algebraic extensions to pass from $J_1, \cdots, J_\mu$ to a full integrity basis. From there on familiar algebraic patterns such as were developed by Kronecker and as are amenable to explicit construction may be followed.

After the formal investigations from Cayley and Sylvester to Gordan, Hilbert inaugurated a new epoch in the theory of invariants. Indeed, by discovering new ideas and introducing new powerful methods he not only brought the subject up to the new level set for algebra by Kronecker and Dedekind, but made such a thorough job of it that he all but finished it, at least as far as the full linear group is concerned. With justifiable pride he concludes his paper, *Ueber die vollen Invariantensysteme*, with the words: "Thus I believe the most important goals of the theory of the function fields generated by invariants have been attained," and therewith quits the scene.[4]

Of later developments which took place after Hilbert quit, two main lines seem to me the most important: (1) The averaging process, which we applied above to finite groups, carries over to continuous compact groups. By this transcendental process of integration over the group manifold, Adolf Hurwitz treated the real orthogonal group. The method has been of great fertility. The simple remark that invariants for the real orthogonal group are *eo ipso* also invariant under the full complex orthogonal group indicates how the results can be transferred even to non-compact groups, in particular, as it turns out, to all semi-simple Lie groups. (2) Today the theory of invariants for arbitrary groups has taken its natural place within the frame of the theory of representations of groups by linear substitutions, a development which owes its greatest impulse to G. Frobenius.

Although the first main theorem has been proved for wide classes of groups Γ we do not yet know whether it holds for every group. Such attempts as have been made to establish it in this generality were soon discovered to have failed. A promising line for an algebraic attack is outlined in item 14 of Hilbert's Paris list of Mathematical Problems.

Having dwelt in such detail on Hilbert's theory of invariants, we must be brief with regard to his other, more isolated, contributions to algebra. The first paper in which the young algebraist showed his real mettle concerns the conditions under which a form with real coefficients is representable as a sum of squares of such forms, in particular with the question whether the obviously necessary condition that the form be positive for all real values of its arguments is sufficient. By ingenious continuity arguments and algebraic constructions Hilbert finds three special cases for which the answer is affirmative, among them of course the positive definite quadratic forms, but counterexamples for all other cases. Similar methods recur in two papers dealing with the attractive problem of the maximum number of real ovals of an algebraic curve or surface and their mutual position. Hilbert conjectured that, irrespective of the number of variables, every *rational function* with real (or rational) coefficients is a sum of squares of such functions provided its values are positive for real values of the arguments; and in his *Grundlagen der Geometrie* he pointed out the role of this fact for the geometric constructions with ruler and "Eichmass." Later O. Veblen conceived, as the basis of the distinction between positive and negative in any field, the axiom that no square sum equals zero. Independently of him, E. Artin and O. Schreier developed a detailed theory of such "real fields," and by means of it Artin succeeded in proving Hilbert's conjecture.[5]

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[4] I recommend to the reader's attention a brief résumé of his invariant-theoretic work which Hilbert himself wrote for the International Mathematical Congress held at Chicago in conjunction with the World Fair in 1893; Collected Papers, vol. 2, item 23.

In passing I mention Hilbert's irreducibility theorem according to which one may substitute in an irreducible polynomial suitable integers for all of the variables but one without destroying the irreducibility of the polynomial, and his paper on the solution of the equation of ninth degree by functions with a minimum number of arguments. They became points of departure for much recent algebraic work (E. Noether, N. Tschebotareff and others). Finally, it ought to be recorded that on the foundations laid by Hilbert a detailed theory of polynomial ideals was erected by E. Lasker and F. S. Macaulay which in turn gave rise to Emmy Noether's general axiomatic theory of ideals. Thus in the field of algebra, as in all other fields, Hilbert's conceptions proved of great consequence for the further development.

ALGEBRAIC NUMBER FIELDS

When Hilbert, after finishing off the invariants, turned to the theory of *algebraic number fields*, the ground had been laid by Dirichlet's analysis of the group of units more than forty years before, and by Kummer's, Dedekind's and Kronecker's introduction of ideal divisors. The theory deals with an algebraic field κ over the field k

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[5] O. Veblen, Trans. Amer. Math. Soc. vol. 7 (1906) pp. 197–199. E. Artin and O. Schreier, Abh. Math. Sem. Hamburgischen Univ. vol. 5 (1926) pp. 85–99; E. Artin, ibid. pp. 100–115.

of rational numbers. One of the most important general results beyond the foundations had been discovered by Dedekind, who showed that the rational prime divisors of the discriminant of κ are at the same time those primes whose ideal prime factors in κ are not all distinct (ramified primes). l being a rational prime, the adjunction to κ of the lth root of a number α in κ yields a relative cyclic field $K=\kappa(\alpha^{1/l})$ of degree l over κ, *provided* κ *contains the lth root of unity* $\zeta=e^{2\pi i/l}$ (and according to Lagrange, the most general relative cyclic field of degree l over κ is obtained in this fashion). It may be said that it was this circumstance which forced Kummer, as he tried to prove Fermat's theorem of the impossibility of the equation $\alpha^l+\beta^l=\gamma^l$, to pass from the rational ground field k to the cyclotomic field $\kappa_l=k(\zeta)$ and then to conceive his ideal numbers in κ_l and to investigate whether the number of classes of equivalent ideal numbers in κ_l is prime to l. Hilbert sharpened his tools in resuming Kummer's study of the relative cyclic fields of degree l over κ_l, which he christened "Kummer fields."

His own first important contribution was a theory of *relative Galois fields* K over a given algebraic number field κ. His main concern is the relation of the Galois group Γ of K/κ to the way in which the prime ideals of κ decompose in K. Given a prime ideal $\mathfrak{P}$ in K of relative degree f, those substitutions s of Γ for which $s\mathfrak{P}=\mathfrak{P}$ form the splitting group. As always in Galois theory one constructs the corresponding subfield of K/κ (splitting field), to which a number of K belongs if it is invariant under all substitutions of the splitting group. The substitutions t which carry every integer A in K into one, tA, that is congruent to A mod $\mathfrak{P}$ form an invariant subgroup of the splitting group of index f, called the inertial group, and the corresponding field (inertial field) is sandwiched in between the splitting field and K. Let $\mathfrak{p}$ be the prime ideal in κ into which $\mathfrak{P}$ goes, and $\mathfrak{P}^e$ the exact power of $\mathfrak{P}$ by which $\mathfrak{p}$ is divisible. I indicate the nature of Hilbert's results by the following central theorem of his: In the splitting field of $\mathfrak{P}$ the prime ideal $\mathfrak{p}$ in κ splits off the prime factor $\mathfrak{p}^*=\mathfrak{P}^e$ of degree 1 (therefore the name!); in passing from the splitting to the inertial field $\mathfrak{p}^*$ stays prime but its degree increases to f; in passing from the inertial to the full field K, $\mathfrak{p}^*$ breaks up into e equal prime factors $\mathfrak{P}$ of the same degree f. For later application I add the following remarks. If $\mathfrak{P}$ goes into $\mathfrak{p}$ in the first power only, $e=1$ (which is necessarily so provided $\mathfrak{p}$ is not a divisor of the relative discriminant of K/κ), then the inertial group consists of the identity only. In that case the theory of Galois's strictly finite fields shows that the splitting group is cyclic of order f and that its elements $1, s, s^2, \cdots, s^{f-1}$ are uniquely determined by the congruences

$$sA \equiv A^{P}, \quad s^{2}A \equiv A^{P^{2}}, \cdots \pmod{\mathfrak{P}}$$

holding for every integer A. Here P is the number of residues in κ modulo $\mathfrak{p}$ and thus P^{f} the number of residues in K modulo $\mathfrak{P}$. Today we call $s=\sigma(\mathfrak{P})$ the Frobenius substitution of $\mathfrak{P}$; it is of paramount importance that one particular generating substitution of the splitting group may thus be distinguished among all others. One readily sees that for any substitution u of the Galois group $\sigma(u\mathfrak{P})=u^{-1}\cdot\sigma(\mathfrak{P})\cdot u$. Thus if the Galois field K/κ is *Abelian*, the substitution $\sigma(\mathfrak{P})=\sigma(u\mathfrak{P})$ depends on $\mathfrak{p}$ only and may be denoted by $(\frac{K}{\mathfrak{p}})$.

In 1893 the Deutsche Mathematiker-Vereinigung asked Hilbert and Minkowski to submit within two years a report on number theory. Minkowski dropped out after a while. Hilbert's monumental report *Die Theorie der algebraischen Zahlkörper* appeared in the Jahresberichte of 1896 (the preface is dated April 1897). What Hilbert accomplished is infinitely more than the Vereinigung could have expected. Indeed, his report is a jewel of mathematical literature. Even today, after almost fifty years, a study of this book is indispensable for anybody who wishes to master the theory of algebraic numbers. Filling the gaps by a number of original investigations, Hilbert welded the theory into an imposing unified body. The proofs of all known theorems he weighed carefully before he decided in favor of those "the principles of which are capable of generalization and the most useful for further research." But before such a selection could be made, that "further research" had to be carried out! Meticulous care was given to the notations, with the result that they have been universally adopted (including, to the American printer's dismay, the German letters for ideals!) He greatly simplified Kummer's theory, which rested on very complicated calculations, and he introduced those concepts and proved a number of those theorems in which we see today the foundations of a general theory of relative Abelian fields. The most important concept is the norm residue symbol, a pivotal theorem on relative cyclic fields, his famous Satz 90 (Collected Works, vol. I, p. 149). From the preface in which he describes the general character of number theory, and the topics covered by his report in particular, let me quote one paragraph:

"The theory of number fields is an edifice of rare beauty and harmony. The most richly executed part of this building, as it appears to me, is the theory of Abelian fields which Kummer by his work on the higher laws of reciprocity, and Kronecker by his investigations on the complex multiplication of elliptic functions, have opened up to us. The deep glimpses into the theory which the work of these two mathematicians affords reveals at the same time that there still lies an abundance of priceless treasures hidden in this domain, beckoning as a rich reward to the explorer who knows the value of such treasures and with love pursues the art to win them."

Hilbert himself was the miner who during the following two years brought to light much of the hidden ore. The analogy with the corresponding problems in the realm of algebraic functions of one variable where Riemann's powerful instruments of topology and Abelian integrals are available was for him a guiding principle throughout (cf. his remarks in item 12 of his Paris Problems). It is a great pleasure to watch how, step by step, advancing from the special to the general, Hilbert evolves the adequate concepts and methods, and the essential conclusions emerge. I mention his great paper dealing with the relative quadratic fields, and his last and most important *Ueber die Theorie der relativ Abelschen Zahlkörper*. On the basis of the examples he carried through in detail, he conceived as by divination and formulated the basic facts about the so-called class fields. Whereas Hilbert's work on invariants was an end, his work on algebraic numbers was a beginning. Most of the labor of such number theorists of the last decades, as Furtwängler, Takagi, Hasse, Artin, Chevalley, has been devoted to proving the results anticipated by Hilbert. By deriving from the ζ-function the existence of certain auxiliary prime ideals, Hilbert had leaned heavily on transcendental arguments. The subsequent development has gradually eliminated these transcendental methods and shown that though they are the fitting and powerful tool for the investigation of the distribution of prime ideals they are alien to the problem of class fields. In attempting to describe the main issues I shall not ignore the progress and simplification due to this later development.

Hilbert's theory of norm residues is based on the following discoveries of his own: (1) he conceived the basic idea and defined the norm residue symbol for all non-exceptional prime spots; (2) he realized the necessity of introducing infinite prime spots; (3) he formulated the general law of reciprocity in terms of the norm symbol; (4) he saw that by means of that law one can extend the definition of the norm symbol to the exceptional prime spots where the really interesting things happen.—It was an essential progress when E. Artin later (5) replaced the roots of unity by the elements of the Galois group as values of the residue symbol. In sketching Hilbert's problems I shall make use of this idea of Artin's and also of the abbreviating language of (6) Hensel's p-adic numbers and (7) Chevalley's *idèles*.[6]

As everybody knows an integer a indivisible by the prime $p\neq 2$ is said to be a quadratic residue if the congruence $x^2\equiv a \pmod{p}$ is solvable. Gauss introduced the symbol $(\frac{a}{p})$ which has the value $+1$ or -1 according to whether a is a quadratic residue or non-residue mod p, and observed that it is a character, $(\frac{a}{p})\cdot(\frac{a'}{p})=(\frac{aa'}{p})$. Indeed, the p residues modulo p—as whose representatives one may take $0, 1, \cdots, (p-1)$—form a *field*, and after exclusion of 0 a *group* in which the quadratic residues form a subgroup of index 2. Let $K=k(b^{1/2})$ be a quadratic field which arises from the rational ground field k by adjunction of the square root of the rational number b. An integer $a\neq 0$ is called by Hilbert *a p-adic norm* in K if modulo any given power of p it is congruent to the norm of a suitable integer in K. He sets $(\frac{a,K}{p})=+1$ if a is p-adic norm, -1 in the opposite case, and finds that this p-adic norm symbol again is a character. The investigation of numbers modulo arbitrarily high powers of p was systematized by K. Hensel in the form of his p-adic numbers, and I repeat Hilbert's definition in this language: "The rational number $a\neq 0$, or more generally the p-adic number $a_p\neq 0$, is a p-adic norm in K if the equation

$$a_p = \mathrm{Nm}\,(x + yb^{1/2}) = x^2 - by^2$$

has a p-adic solution $x=x_p$, $y=y_p$; the norm symbol (a_p, K) equals $+1$ or -1 according to whether or not a_p is (p-adic) norm in K." The p-adic numbers form a field $k(p)$ and after exclusion of 0 a multiplicative group G_p in which, according to Hilbert's result, the p-adic norms in K form a subgroup of index 2 or 1. The cyclic nature of the factor group is the salient point. One easily finds that the p-adic squares form a subgroup G_p^2 of index 4 if $p\neq 2$, of index 8 if $p=2$, and thus the factor group G_p/G_p^2 is not cyclic and could not be described by a single character. Of course every p-adic square is a p-adic norm in K. Both steps, the substitution of K-norms for squares and the passage from the modulus p to arbitrarily high powers of p; the first step amounting to a relaxation, the second to a sharpening of Gauss's condition for quadratic residues; are equally significant for the success of Hilbert's definition.

Every p-adic number $a_p\neq 0$ is of the form $p^h\cdot e_p$ where e_p is a p-adic unit, and thus a_p is of a definite *order* h (at p). An ordinary rational number a coincides with a definite p-adic number $I_p(a)=a_p$. Here I_p symbolizes a homomorphic projection of k into $k(p)$:

$$I_p(a + a') = I_p(a) + I_p(a'), \qquad I_p(aa') = I_p(a)\cdot I_p(a').$$

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[6] The latest account of the theory is C. Chevalley's paper *La théorie du corps de classes*, Ann. of Math. vol. 41 (1940) pp. 394–418.

The character $(\frac{a,K}{p})$ is identical with $(I_p(a), K)$.

We come to Hilbert's second discovery: he realized that simple laws will not result unless one adds to the "finite prime spots" p one infinite prime spot q. By definition the q-adic numbers are the real numbers and $I_q(a)$ is the real number with which the rational number a coincides. Hence the real number a_q is a q-adic norm in K if the equation $a_q = x^2 - by^2$ has a solution in real numbers x, y. Clearly if $b > 0$ or K is real, this is the case for every a_q; if however $b < 0$ or K is imaginary, only the positive numbers a_q are q-adic norms. Hence

$$(a_q, K) = 1 \text{ if } K \text{ real}; \qquad (a_q, K) = \operatorname{sgn} a_q \text{ if } K \text{ imaginary}.$$

The fact that the norm symbol is a character is thus much more easily verified for the infinite prime spot than for the finite ones.

Hilbert's third observation is to the effect that Gauss's reciprocity law with its two supplements may be condensed into the elegant formula

$$\prod_p (I_p(a), K) = \prod_p \left(\frac{a, K}{p}\right) = 1, \tag{3}$$

the product extending over the infinite and all finite prime spots p. There is no difficulty in forming this product because almost all factors (that is, all factors with but a finite number of exceptions) equal unity. Indeed, if the prime p does not go into the discriminant of K, then $(a_p, K) = 1$ for every p-adic unit a_p. Formula (3) is the first real vindication for the norm residue idea, which must have given Hilbert the assurance that the higher reciprocity laws had to be formulated in terms of norm residues.

A given rational number a assigns to every prime spot p a p-adic number $a_p = I_p(a)$. On which features of this assignment does one rely in forming the product (3)? The obvious answer is given by Chevalley's notion of *idèle*: an idèle $\mathbf{a}$ is a function assigning to every prime spot p a p-adic number $a_p \neq 0$ which is a p-adic unit for almost all prime spots p. The idèles form a multiplicative group J_k. By virtue of the assignment $a_p = I_p(a)$ every rational number $a \neq 0$ gives rise to an idèle, called the *principal idèle* $\mathbf{a}$. With the idèles $\mathbf{a}$ at our disposal we might as well return to the notation $(\frac{\mathbf{a},K}{p})$ for (a_p, K). The formula

$$\phi_K(\mathbf{a}) = (\mathbf{a}, K) = \prod_p (a_p, K) = \prod_p \left(\frac{\mathbf{a}, K}{p}\right) \tag{4}$$

defines a character ϕ_K, the norm character, in the group J_k of all idèles. The reciprocity law in Hilbert's form (3) maintains that

$$(a, K) = 1 \tag{5}$$

if a is principal. By the very definition of the norm symbol (a_p, K) the same equation holds if a is a norm in K, that is, if a_p is a p-adic norm in K for every prime spot p. Two idèles $\mathbf{a}, \mathbf{a}'$ are said to be equivalent, $\mathbf{a}\sim\mathbf{a}'$, if their quotient $\mathbf{a}'\mathbf{a}^{-1}$ is principal. Let us denote by $\mathrm{Nm}J_K$ the group of all idèles which are equivalent to norms in K. Then (5) holds for all idèles $\mathbf{a}$ of $\mathrm{Nm}J_K$; it would be good to know that it holds for no other idèles, or, in other words, that $\mathrm{Nm}J_K$ is a subgroup of J_k of index 2.

The stage is now reached where the experiences gathered for a quadratic field K over the rational ground field k may be generalized to any relative Abelian field K over a given algebraic number field $\kappa=k(\theta)$. First a word about the infinite prime spots of κ. The defining equation $f(\theta)=0$, an irreducible equation in k of some degree m, has m distinct roots $\theta', \theta'', \cdots, \theta^{(m)}$ in the continuum of complex numbers. Suppose that r of them are real, say $\theta', \cdots, \theta^{(r)}$. Then each element α of κ has its r real conjugates $\alpha', \cdots, \alpha^{(r)}$, and $\alpha^{(t)}$ arises from α by a homomorphic projection $I^{(t)}$ of κ into the field of all real numbers,

$$\alpha \to \alpha^{(t)} = I^{(t)}(\alpha) \qquad (t = 1, \cdots, r).$$

We therefore speak of r real infinite prime spots $\mathfrak{q}', \cdots, \mathfrak{q}^{(r)}$ with the corresponding projections $I'=I_{\mathfrak{q}'}, \cdots, I^{(r)}$; the fields $\kappa(\mathfrak{q}'), \cdots, \kappa(\mathfrak{q}^{(r)})$ are identical with the field of all real numbers. Thus α is an nth $\mathfrak{q}'$-adic power if the equation $\alpha'=\xi'^n$ has a real solution ξ'. One sees that this imposes a condition only if n is even, and then requires α' to be positive. (In the complex domain the equation is always solvable whether n be even or odd, and that is the reason why we ignore the complex infinite prime spots altogether.)

The finite prime spots are the prime ideals $\mathfrak{p}$ of κ. In studying a Galois field K/κ of relative degree n we first exclude the ramified ideals $\mathfrak{p}$ which go into the relative discriminant of K/κ. An unramified ideal $\mathfrak{p}$ of κ factors in K into a number g of distinct prime ideals $\mathfrak{P}_1 \cdots \mathfrak{P}_g$ of relative degree f, $fg=n$. It is easily seen that a $\mathfrak{p}$-adic number $\alpha_\mathfrak{p}\neq 0$ is a $\mathfrak{p}$-adic norm in K if and only if its order (at $\mathfrak{p}$) is a multiple of f. In particular, the $\mathfrak{p}$-adic units are norms. Thus we encounter a situation which is essentially simpler than the one taken care of by Gauss's quadratic residue symbol: the norm character of $\alpha_\mathfrak{p}$ depends only on the order i at $\mathfrak{p}$ of $\alpha_\mathfrak{p}$. It is now clear how to proceed: we choose a primitive fth root of unity ζ and define $(\alpha_\mathfrak{p}, K)=\zeta^i$ if $\alpha_\mathfrak{p}$ is of order i. This function of $\alpha_\mathfrak{p}\neq 0$ is a character which assumes the value 1 for the norms and the norms only. But here is the rub: there is no algebraic property distinguishing the several primitive fth roots of unity from one another. Thus the choice of ζ among them remains arbitrary. One could put up with this if one dealt with one prime ideal only. But when one has to take all prime spots simultaneously into account, as is necessary in forming products of the type (4), then the arbitrariness involved in the choice of ζ for each $\mathfrak{p}$ will destroy all hope of obtaining a simple reciprocity law like (5). I shall forego describing the devices by which Eisenstein, Kummer, Hilbert, extricated themselves from this entanglement. By far the best solution was found by Artin: if K/κ is Abelian, then the Frobenius substitution $(\frac{K}{\mathfrak{p}})$ is uniquely determined by K and $\mathfrak{p}$ and is an element of order f of the Galois group Γ of K/κ. Let this element of the Galois group replace ζ in our final definition of the $\mathfrak{p}$-adic norm symbol:

$$(6) \qquad (\alpha_{\mathfrak{p}}, K) = \left(\frac{\alpha,\, K}{\mathfrak{p}}\right) = \left(\frac{K}{\mathfrak{p}}\right)^{i} \text{ if } \alpha_{\mathfrak{p}} \text{ is of order } i \text{ at } \mathfrak{p}.$$

We could now form for any idèle α the product

$$\prod_{\mathfrak{p}} (\alpha_{\mathfrak{p}}, K) = \prod_{\mathfrak{p}} \left(\frac{\alpha,\, K}{\mathfrak{p}}\right) = (\alpha, K)$$

extending over all finite and (real) infinite prime spots $\mathfrak{p}$ and formulate the reciprocity law asserting that $(\alpha, K) = 1$ for any principal idèle α—had we not excluded certain exceptional prime spots in our definition of $(\alpha_{\mathfrak{p}}, K)$, namely the infinite prime spots and the ramified prime ideals. In the special case he investigated Kummer had succeeded in obtaining the correct value of $(\alpha_{\mathfrak{p}}, K)$ for the exceptional $\mathfrak{p}$ by extremely complicated calculations. Hilbert's fourth discovery is a simple and ingenious method of circumventing this formidable obstacle which blocked the road to further progress. Let us first restrict ourselves to idèles α which are nth powers at our exceptional prime spots; in other words, we assume that the equation $\alpha_{\mathfrak{p}} = \xi_{\mathfrak{p}}^{n}$ is solvable for the $\mathfrak{p}$-adic values $\alpha_{\mathfrak{p}}$ of α at this finite number of prime spots. There is no difficulty in defining (α, K) under this restriction:

$$(\alpha, K) = \prod_{\mathfrak{p}}{}' (\alpha_{\mathfrak{p}}, K),$$

the product extending, as indicated by the accent, over the non-exceptional prime spots only, for which we know what $(\alpha_{\mathfrak{p}}, K)$ means. Under the same restriction we prove (with Artin) the reciprocity law

$$(7) \qquad (\alpha, K) = 1 \text{ if } \alpha \text{ is principal,}$$

and observe that by its very definition $(\alpha, K) = 1$ if α is norm. We now return to an *arbitrary* idèle α. It is easily shown that there exists an equivalent idèle $\alpha^*\sim\alpha$ which is an nth power at all exceptional prime spots, but of course there will be plenty of them. However, the restricted law of reciprocity insures that

$$(\alpha^*, K) = \prod_{\mathfrak{p}}{}'(\alpha^*_{\mathfrak{p}}, K)$$

has the same value for every one of the α^*'s, and it is this value which we now denote by (α, K). This definition adopted, the reciprocity law (7) and the statement that $(\alpha, K)=1$ for every norm α follow at once *without restriction*. Thus the reciprocity law itself is made the tool for getting the exceptional prime spots under control!

Once (α, K) is known for every idèle α we can compute $(\alpha_{\mathfrak{p}}, K)$ for a given prime spot $\mathfrak{p}$ and a given $\mathfrak{p}$-adic number $\alpha_{\mathfrak{p}}\neq 0$ as the value of (α, K) for that "primary" idèle, also denoted by $\alpha_{\mathfrak{p}}$, which equals $\alpha_{\mathfrak{p}}$ at $\mathfrak{p}$ and 1 at any other prime spot. (The idèle α is the product of its primary components, $\alpha=\prod_{\mathfrak{p}}\alpha_{\mathfrak{p}}$.) One expects the following two propositions to hold:

I. $(\alpha_{\mathfrak{p}}, K)=1$ *if and only if* $\alpha_{\mathfrak{p}}$ *is a* $\mathfrak{p}$*-adic norm.*

II. *Given a prime ideal* $\mathfrak{p}$, $(\alpha_{\mathfrak{p}}, K)=1$ *for every* $\mathfrak{p}$*-adic unit* $\alpha_{\mathfrak{p}}$ *if and only if* $\mathfrak{p}$ *is unramified.*

The direct parts of I and II:

(I_0) $\alpha_{\mathfrak{p}}=$norm implies $(\alpha_{\mathfrak{p}}, K)=1$;

(II_0) $\mathfrak{p}$ unramified implies $(\alpha_{\mathfrak{p}}, K)=1$ for every $\mathfrak{p}$-adic unit $\alpha_{\mathfrak{p}}$, were settled above. The converse statement of I_0 is trivial for the non-exceptional prime spots; but owing to the indirect definition of the norm symbol for the exceptional prime spots, the proofs of the converse of I_0 for the exceptional spots and of the converse of II_0 are rather intricate. From II we learn that for none of the ramified prime ideals $\mathfrak{p}$ does the norm character of $\alpha_{\mathfrak{p}}$ depend on the order of $\alpha_{\mathfrak{p}}$ only: this simple feature which makes the definition (6) possible is limited to non-ramified $\mathfrak{p}$. One would also expect:

III. *If the principal idèle* α *is an idèle norm in* K, *then the number* α *is norm of a number in* K.

This is true for *cyclic* fields K/κ, but in general not for Abelian fields.

Let us again denote by $\mathrm{Nm}J_K$ the subgroup in J_κ of the idèles which are equivalent to norms. Then the norm symbol $\phi_K(\alpha)=(\alpha, K)$ determines a homomorphic mapping of the factor group $J_\kappa/\mathrm{Nm}J_K$ into the Galois group of K/κ. One would expect that this mapping is one-to-one:

IV. *By means of the norm symbol the factor group* $J_\kappa/\mathrm{Nm} J_K$ *is isomorphically mapped onto the Galois group of* K/κ.

I, II, III_c (the subscript c indicating restriction to cyclical fields) and IV are the main propositions of what one might call the *norm theory of relative Abelian fields.* They refer to a *given* field K/κ.

There is a second part of the theory, the *class field theory proper,* which is concerned with the manner in which *all possible* relative Abelian fields K over κ are reflected in the structure of the group J_κ of idèles in κ. Each such field K determines, as we have seen, a subgroup $\mathrm{Nm} J_K$ of J_κ of finite index. The question arises *which* subgroups J_κ^* of J_κ are generated in this way by Abelian fields K/κ. Clearly the following conditions are necessary:

(1) Every principal idèle is in J_κ^*.

(2) There is a natural number n such that every nth power of an idèle is in J_κ^*.

(3) There is a finite set S of prime spots such that α is in J_κ^* provided α is a unit at every prime spot and equals 1 at the prime spots of S.

The main theorem concerning class fields states that these conditions are also sufficient.

V. *Given a subgroup* J_κ^* *of* J_κ *fulfilling the above three conditions* (and therefore, as one readily verifies, of finite index), *there exists a uniquely determined Abelian field* K/κ *such that* $J_\kappa^* = \mathrm{Nm} J_K$.

We divide the idèles of κ into *classes* by throwing two idèles into the same class if their quotient is in J_κ^*. Then J_κ/J_κ^* is the class group and K is called the corresponding *class field.* The most important example results if one lets J_κ^* consist of the *unit idèles* α whose values $\alpha_\mathfrak{p}$ are $\mathfrak{p}$-adic units at every prime spot $\mathfrak{p}$.[7] Then the classes may be described as the familiar classes of *ideals*: two ideals are put in the same class if their quotient is a principal ideal (α) springing from a number α positive at all real infinite prime spots. The corresponding class field K, the so-called absolute class field, is of relative discriminant 1, and the largest unramified Abelian field over κ (Theorem II). Its degree n over κ is the class number of ideals, its Galois group isomorphic to the class group of ideals in κ (Theorem IV). f being the least power of $\mathfrak{p}$ which lies in the principal class, $\mathfrak{p}$ decomposes into n/f distinct prime ideals in K, each of relative degree f. This last statement does nothing but repeat the norm definition of the class field. Hence the way in which $\mathfrak{p}$ factors in K depends only on the class to which $\mathfrak{p}$ belongs. The easiest way of extending the theory from the case with no ramified prime ideals, which was preponderant in Hilbert's thought, to Takagi's ramified case is by substituting idèles for ideals. Hilbert also stated that every ideal in κ becomes a principal ideal in the absolute class field. It is today possible to show that this is so, by arguments, however, which are far from being fully understood, because this question transcends the domain of Abelian fields.

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[7] At the (real) infinite prime spots the positive numbers are considered the units.

As was stated above, Hilbert did not prove these theorems in their full generality, but taking his departure from Gauss's theory of genera in quadratic fields and Kummer's investigations he worked his way gradually up from the simplest cases, developing as he went along the necessary machinery of new concepts and propositions about them until he could survey the whole landscape of class fields. We cannot attempt here to give an idea of the highly involved proofs. The completion of the work he left to his successors. The day is probably still far off when we shall have a theory of relative *Galois* number fields of comparable completeness.

Kronecker had shown, and Hilbert found a simpler proof for the fact, that Abelian fields over the rational ground field k are necessarily subfields of the cyclotomic fields, and are thus obtained from the transcendental function $e^{2\pi i x}$ by substituting rational values for the argument x. For Abelian fields over an imaginary quadratic field the so-called complex multiplication of the elliptic and modular functions plays a similar role ("Kronecker's Jugendtraum"). While Heinrich Weber following in Kronecker's footsteps, and R. Fuëter under Hilbert's guidance, made this dream come true, Hilbert himself began to play with modular functions of several variables which are defined by means of algebraic number fields, and to study their arithmetical implications. He never published these investigations, but O. Blumenthal, and later E. Hecke, used his notes and developed his ideas. The results are provocative, but still far from complete. It is indicative of the fertility of Hilbert's mind at this most productive period of his life that he handed over to his pupils a complex of problems of such fascination as that of the relation between number theory and modular functions.[8]

There remain to be mentioned a particularly simple proof of the transcendence of e and π with which Hilbert opened the series of his arithmetical papers, and the 1909 paper settling Waring's century-old conjecture. I should classify the latter paper among his most original ones, but we can forego considering it more closely because a decade later Hardy and Littlewood found a different approach which yields asymptotic formulas for the number of representations, and it is the Hardy-Littlewood "circle method" which has given rise in recent times to a considerable literature on this and related subjects.[9]

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[8] R. Fuéter, *Singuläre Moduln und complexe Multiplication*, 2 vols., Leipzig, 19[illegible]4, 1927; cf. also H. Hasse, J. Reine Angew. Math. vol. 157 (1927) pp. 115–1[illegible]9. O. Blumenthal, Math. Ann. vol. 56 (1903) pp. 509–548, vol. 58 (1904) pp. 497–5[illegible]7. E. Hecke, Math. Ann. vol. 71 (1912) pp. 1–37, vol. 74 (1913) pp. 465–510.

AXIOMATICS

There could not have been a more complete break than the one dividing Hilbert's last paper on the theory of number fields from his classical book, *Grundlagen der Geometrie*, published in 1899. Its only forerunner is a note of the year 1895 on the straight line as the shortest way. But O. Blumenthal records that as early as 1891 Hilbert, discussing a paper on the role of Desargues's and Pappus's theorems read by H. Wiener at a mathematical meeting, made a remark which contains the axiomatic standpoint in a nutshell: "It must be possible to replace in all geometric statements the words *point*, *line*, *plane*, by *table*, *chair*, *mug*."

The Greeks had conceived of geometry as a deductive science which proceeds by purely logical processes once the few axioms have been established. Both Euclid and Hilbert carry out this program. However, Euclid's list of axioms was still far from being complete; Hilbert's list is complete and there are no gaps in the deductions. Euclid tried to give a descriptive definition of the basic spatial objects and relations with which the axioms deal; Hilbert abstains from such an attempt. All that we must know about those basic concepts is contained in the axioms. The axioms are, as it were, their implicit (necessarily incomplete) definitions. Euclid believed the axioms to be evident; his concern is the real space of the physical universe. But in the deductive system of geometry the evidence, even the truth of the axioms, is irrelevant; they figure rather as hypotheses of which one sets out to develop the logical consequences. Indeed there are many different material interpretations of the basic concepts for which the axioms become true. For instance, the axioms of n-dimensional Euclidean vector geometry hold if a distribution of direct current in a given electric circuit, the n branches of which connect in certain branching points, is called a vector, and Joule's heat produced per unit time by the current is considered the square of the vector's length. In building up geometry on the foundation of its axioms one will attempt to economize as much as possible and thus illuminate the role of the several groups of axioms. Arranged in their natural hierarchy they are the axioms of incidence, order, congruence, parallelism, and continuity. For instance, if the theory of geometric proportions or of the areas of polygons can be established without resorting to the axioms of continuity, this ought to be done.

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[9] It must suffice here to quote the first paper in this line: G. H. Hardy and J. E. Littlewood, Quart. J. Math. vol. 48 (1919) pp. 272–293, and its latest successor which carries Waring's theorem over to arbitrary algebraic fields: C. L. Siegel, Amer. J. Math. vol. 66 (1944) pp. 122–136.

In all this, though the execution shows the hand of a master, Hilbert is not unique. An outstanding figure among his predecessors is M. Pasch, who had indeed traveled a long way from Euclid when he brought to light the hidden axioms of order and with methodical clarity carried out the deductive program for projective geometry (1882). Others in Germany (F. Schur) and a flourishing school of Italian geometers (Peano, Veronese) had taken up the discussion. With respect to the economy of concepts, Hilbert is more conservative than the Italians: quite deliberately he clings to the Euclidean tradition with its three classes of undefined elements, points, lines, planes, and its relations of incidence, order and congruence of segments and angles. This gives his book a peculiar charm, as if one looked into a face thoroughly familiar and yet sublimely transfigured.

It is one thing to build up geometry on sure foundations, another to inquire into the logical structure of the edifice thus erected. If I am not mistaken, Hilbert is the first who moves freely on this higher "metageometric" level: systematically he studies the mutual independence of his axioms and settles the question of independence from certain limited groups of axioms for some of the most fundamental geometric theorems. His method is the *construction of models*: the model is shown to disagree with one and to satisfy all other axioms; hence the one cannot be a consequence of the others. One outstanding example of this method had been known for a considerable time, the Cayley-Klein model of non-Euclidean geometry. For Veronese's non-Archimedean geometry Levi-Civita (shortly before Hilbert) had constructed a satisfactory arithmetical model. The question of *consistency* is closely related to that of independence. The general ideas appear to us almost banal today, so thoroughgoing has been their influence upon our mathematical thinking. Hilbert stated them in clear and unmistakable language, and embodied them in a work that is like a crystal: an unbreakable whole with many facets. Its artistic qualities have undoubtedly contributed to its success as a masterpiece of science.

In the construction of his models Hilbert displays an amazing wealth of invention. The most interesting examples seem to me the one by which he shows that Desargues's theorem does not follow from the plane incidence axioms, but that the plane incidence axioms combined with Desargues's theorem enable one to embed the plane in a higher dimensional space in which all incidence axioms hold; and then the other example by which he decides whether the Archimedean axiom of continuity is necessary to restore the full congruence axioms after having curtailed them by the exclusion of reflections.

What is the building material for the models? Klein's model of non-Euclidean geometry could be interpreted as showing that he who accepts Euclidean geometry with its points and lines, and so on, can by mere change of nomenclature also get the non-Euclidean geometry. Klein himself preferred another interpretation in terms of projective space. However, Descartes's analytic geometry had long provided a more general and satisfactory answer, of which Riemann, Klein and many others must have been aware: All that we need for our construction is the field of real *numbers*. Hence any contradiction in Euclidean geometry must show up as a contradiction in the arithmetical axioms on which our operations with real numbers are based. Nobody had said that quite clearly before Hilbert. He formulates a complete and simple set of axioms for real numbers. The system of arithmetical axioms will have its exchangeable parts just as the geometric system has. From a purely algebraic standpoint the most important axioms are those characterizing a (commutative or non-commutative) *field*. Any such abstract number field may serve as a basis for the construction of corresponding geometries. *Vice versa*, one may introduce numbers and their operations in terms of a space satisfying certain axioms; Hilbert's Desarguesian *Streckenrechnung* is a fine example. In general this reverse process is the more difficult one. The Chicago school under E. H. Moore took up Hilbert's investigations, and in particular O. Veblen did much to reveal the perfect correspondence between the projective spaces obeying a set of simple incidence axioms (and no axioms of order), and the abstractly defined number fields.[10]

What the question of independence literally asks is to make sure that a certain proposition cannot be deduced from other propositions. It seems to require that we make the propositions, rather than the things of which they speak, the object of our investigation, and that as a preliminary we fully analyze the logical mechanism of deduction. The method of models is a wonderful trick to avoid that sort of logical investigations. It pays, however, a heavy price for thus shirking the fundamental issue: it merely reduces everything to the question of consistency for the arithmetical axioms, which is left unanswered. In the same manner *completeness*, which literally means that every general proposition about the objects with which the axioms deal can be decided by inference from the axioms, is replaced by *categoricity* (Veblen), which asserts that any imaginable model is isomorphic to the one model by which consistency is established. In this sense Hilbert proves that there is but "one," the Cartesian geometry, which fulfills all his axioms. Only in the case of G. Fano's and O. Veblen's finite projective spaces, for example, of the projective plane consisting of seven points, the model is a purely combinatorial scheme, and the questions of consistency, independence and completeness can be answered in the absolute sense. Hilbert never seems to have thought of illustrating his conception of the axiomatic method by purely combinatorial schemes, and yet they provide by far the simplest examples.

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[10] Among later contributions to this question I mention W. Schwan, *Streckenrechnung und Gruppentheorie*, Math. Zeit. vol. 3 (1919) pp. 11–28. A complete bibliography of geometric axiomatics since Hilbert would probably cover many pages. I refrain from citing a list of names.

An approach to the foundations of geometry entirely different from the one followed in his book is pursued by Hilbert in a paper which is one of the earliest documents of set-theoretic topology. From the standpoint of mechanics the central task which geometry ought to perform is that of describing the mobility of a solid. This was the viewpoint of Helmholtz, who succeeded in characterizing the group of motions in Euclidean space by a few simple axioms. The question had been taken up by Sophus Lie in the light of his general theory of continuous groups. Lie's theory depends on certain assumptions of differentiability; to get rid of them is one of Hilbert's Paris Problems. In the paper just mentioned he *does* get rid of them as far as Helmholtz's problem in the plane is concerned. The proof is difficult and laborious; naturally continuity is now the foundation—and not the keystone of the building as it had been in his Grundlagen book. Other authors, R. L. Moore, N. J. Lennes, W. Süss, B. v. Kérékjarto, carried the problem further along these topological lines. A half-personal reminiscence may be of interest. Hilbert defines a two-dimensional manifold by means of neighborhoods, and requires that a class of "admissible" one-to-one mappings of a neighborhood upon Jordan domains in an x, y-plane be designated, any two of which are connected by continuous transformations. When I gave a course on Riemann surfaces at Göttingen in 1912, I consulted Hilbert's paper and noticed that the neighborhoods themselves could be used to characterize that class. The ensuing definition was given its final touch by F. Hausdorff; the Hausdorff axioms have become a byword in topology.[11] (However, when it comes to explaining what a *differenti-*

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[11] A parallel development, with E. H. Moore as the chief prompter, must have taken place in this country. As I have to write from memory mainly, it is inevitable that my account should be colored by the local Gottingen tradition.

able manifold is, we are to this day bound to Hilbert's roundabout way; cf. Veblen and Whitehead, *The foundations of differential geometry*, Cambridge, 1932.)

The fundamental issue of an *absolute proof of consistency* for the axioms which should include the whole of mathematical analysis, nay even Cantor's set theory in its wildest generality, remained in Hilbert's mind, as a paper read before the International Congress at Heidelberg in 1904 testifies. It shows him on the way, but still far from the goal. Then came the time when integral equations and later physics became his all-absorbing interest. One hears a loud rumbling of the old problem in his Zürich address, *Axiomatisches Denken*, of 1917. Meanwhile the difficulties concerning the foundations of mathematics had reached a critical stage, and the situation cried for repair. Under the impact of undeniable antinomies in set theory, Dedekind and Frege had revoked their own work on the nature of numbers and arithmetical propositions, Bertrand Russell had pointed out the hierarchy of types which, unless one decides to "reduce" them by sheer force, undermine the arithmetical theory of the continuum; and finally L. E. J. Brouwer by his intuitionism had opened our eyes and made us see how far generally accepted mathematics goes beyond such statements as can claim real meaning and truth founded on evidence. I regret that in his opposition to Brouwer, Hilbert never openly acknowledged the profound debt which he, as well as all other mathematicians, owe Brouwer for this revelation.

Hilbert was not willing to make the heavy sacrifices which Brouwer's standpoint demanded, and he saw, at least in outline, a way by which the cruel mutilation could be avoided. At the same time he was alarmed by signs of wavering loyalty within the ranks of mathematicians some of whom openly sided with Brouwer. My own article on the *Grundlagenkrise* in Math. Zeit. vol. 10 (1921), written in the excitement of the first postwar years in Europe, is indicative of the mood. Thus Hilbert returns to the problem of foundations in earnest. He is convinced that complete certainty can be restored without "committing treason to our science." There is anger and determination in his voice when he proposes "die Grundlagenfragen einfürallemal aus der Welt zu schaffen." "Forbidding a mathematician to make use of the principle of excluded middle," says he, "is like forbidding an astronomer his telescope or a boxer the use of his fists."

Hilbert realized that the mathematical statements themselves could not be made the subject of a mathematical investigation whose aim is to answer the question of their consistency in its primitive sense, lest they be first reduced to mere *formulas*. Algebraic formulas like $a+b=b+a$ are the most familiar examples. The process of deduction by which formulas previously obtained give rise to new formulas must be described without reference to any meaning of the formulas. The deduction starts from certain initial formulas, the axioms, which must be written out explicitly. Whereas in his *Grundlagen der Geometrie* the meaning of the geometric terms had become irrelevant, but the meaning of logical terms, as "and," "not," "if then," had still to be understood, now every trace of meaning is obliterated. As a consequence, logical symbols like → in $a \to b$, read: a implies b, enter into the formulas. Hilbert fully agrees with Brouwer in that the great majority of mathematical propositions are not "real" ones conveying a definite meaning verifiable in the light of evidence. But he insists that the non-real, the "ideal propositions" are indispensable in order to give our mathematical system "completeness." Thus he parries Brouwer, who had asked us to give up what is meaningless, by relinquishing the pretension of meaning altogether, and what he tries to establish is not *truth* of the individual mathematical proposition, but *consistency* of the system. The game of deduction when played according to rules, he maintains, will never lead to the formula $0 \neq 0$. In this sense, and in this sense only, he promises to salvage our cherished classical mathematics in its entirety.

For those who accuse him of degrading mathematics to a mere game he points first to the introduction of ideal elements for the sake of completeness as a common method in all mathematics—for example, of the ideal points outside an accessible portion of space, without which space would be incomplete—; secondly, to the neighboring science of physics where likewise not the individual statement is verifiable by experiment, but in principle only the system as a whole can be confronted with experience.

But how to make sure that the "game of deduction" never leads to a contradiction? Shall we prove this by the same mathematical method the validity of which stands in question, namely by deduction from axioms? This would clearly involve a regress *ad infinitum*. It must have been hard on Hilbert, the axiomatist, to acknowledge that the insight of consistency is rather to be attained by *intuitive reasoning* which is based on evidence and not on axioms. But after all, it is not surprising that ultimately the mind's seeing eye must come in. Already in communicating the rules of the game we must count on understanding. The game is played in silence, but the rules must be *told* and any reasoning about it, in particular about its consistency, communicated by *words*. Incidentally, in describing the indispensable intuitive basis for his *Beweistheorie* Hilbert shows himself an accomplished master of that, alas, so ambiguous medium of communication, language. With regard to what he accepts as evident in this "metamathematicial" reasoning, Hilbert is more papal than the pope, more exacting than either Kronecker or Brouwer. But it cannot be helped that our reasoning in following a hypothetic sequence of formulas leading up to the formula $0\neq 0$ is carried on in hypothetic generality and uses that type of evidence which a formalist would be tempted to brand as application of the principle of complete induction. Elementary arithmetics can be founded on such intuitive reasoning as Hilbert himself describes, but we need the formal apparatus of variables and "quantifiers" to invest the infinite with the all important part that it plays in higher mathematics. Hence Hilbert prefers to make a clear cut: he becomes strict formalist in mathematics, strict intuitionist in metamathematics.

It is perhaps possible to indicate briefly how Hilbert's formalism restores the *principle of the excluded middle* which was the main target of Brouwer's criticism. Consider the infinite sequence of numbers $0, 1, 2, \cdots$. Any property A of numbers (for example, "being prime") may be represented by a propositional function $A(x)$ ("x is prime"), from which a definite proposition $A(b)$ arises by substituting a concrete number b for the variable x ("6 is prime"). Accepting the principle which Brouwer denies and to which Hilbert wishes to hold on, that (i) either there exists a number x for which $A(x)$ holds, or (ii) $A(x)$ holds for no x, we can find a "representative" r for the property A, a number such that $A(b)$ implies $A(r)$ whatever the number b, $A(b)\rightarrow A(r)$. Indeed, in the alternative (i) we choose r as one of the numbers x for which $A(x)$ holds, in the alternative (ii) at random. Thus Aristides is the representative of honesty; for, as the Athenians said, if there is any honest man it is Aristides. Assuming that we know the representative we can decide the question whether there is an honest man or whether all are dishonest by merely looking at *him*: if he is dishonest everybody is. In the realm of numbers we may even make the choice of the representative unique, in case (i) choosing $x=r$ as the *least* number for which $A(x)$ holds, and $r=0$ in the opposite case (ii). Then r arises from A by a certain operator ρ_x, $r=\rho_x A(x)$, applicable to every imaginable property A. A propositional function may contain other variables $y, z, \cdots$ besides x. Therefore it is necessary to attach an index x to ρ, just as in integrating one must indicate with respect to which variable one integrates. ρ_x eliminates the variable x; for instance $\rho_x A(x, y)$ is a function of y alone. The word quantifier is in use for this sort of operator. Hence we write our axiom as follows:

$$A(b) \rightarrow A(\rho_x A(x)). \tag{8}$$

It is immaterial whether we fix the representative in the unique manner described above; our specific rule would not fit anyhow unless x ranges over the numbers 0, 1, 2, $\cdots$. Instead we imagine a quantifier ρ_x of universal applicability which, as it were, selects the representative for us. Zermelo's axiom of choice is thus woven into the principle of the excluded middle. It is a bold step; but the bolder the better, as long as it can be shown that we keep within the bounds of consistency!

In the formalism, propositional functions are replaced by *formulas* the handling of which must be described without reference to their meaning. In general, variables $x, y, \cdots$ will occur among the symbols of a formula $\mathfrak{A}$. We say that the symbol ρ_x *binds* the variable x in the formula $\mathfrak{A}$ which follows the symbol[12] and that x occurs *free* in a formula wherever it is not bound by a quantifier with index x. $x, y, \rightarrow, \rho_x$ are symbols entering into the formulas; the German letters are no such symbols, but are used for communication. It is more natural to describe our critical axiom (8) as a rule for the formation of axioms. It says: take any formula $\mathfrak{A}$ in which only the variable x occurs free, and any formula $\mathfrak{b}$ without free variables, and by means of them build the formula

$$\mathfrak{A}(\mathfrak{b}) \rightarrow \mathfrak{A}(\rho_x \mathfrak{A}). \tag{9}$$

Here $\mathfrak{A}(\mathfrak{b})$ stands for the formula derived from $\mathfrak{A}$ by putting in the entire formula $\mathfrak{b}$ for the variable x *wherever x occurs free.*

In this way formulas may be *obtained* as *axioms* according to certain rules. *Deduction* proceeds by the rule of syllogism: From two formulas $\mathfrak{a}$ and $\mathfrak{a} \rightarrow \mathfrak{b}$ *previously obtained*, in the second of which the first formula reappears at the left of the symbol $\rightarrow$, one *obtains* the formula $\mathfrak{b}$.

How does Hilbert propose to show that the game of deduction will never lead to the formula $0 \neq 0$? Here is the basic idea of his procedure. As long as one deals with "finite" formulas only, formulas from which the quantifiers $\rho_x, \rho_y, \cdots$ are absent, one can decide whether they are true or false by merely looking at them. With the entrance of ρ such a descriptive valuation of formulas becomes impossible: evidence ceases to work. But a concretely given deduction is a sequence of formulas in which only a limited number of instances of the axiomatic rule (9) will turn up. Let us assume that the only quantifier which occurs is ρ_x and wherever it occurs it is followed by the *same finite* formula $\mathfrak{A}$, so that the instances of (9) are of the form

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[12] If we wish the rule that ρ_x binds x in all that comes after to be taken literally, we must write $\mathfrak{a} \rightarrow \mathfrak{b}$ in the form $\rightarrow\left\{\begin{smallmatrix}\mathfrak{a}\\ \mathfrak{b}\end{smallmatrix}\right.$. The formulas will then look like genealogical trees.

$$\mathfrak{A}(\mathfrak{b}_1) \rightarrow \mathfrak{A}(\rho_x\mathfrak{A}), \cdots, \mathfrak{A}(\mathfrak{b}_h) \rightarrow \mathfrak{A}(\rho_x\mathfrak{A}). \tag{10}$$

Assume, moreover, $\mathfrak{b}_1, \cdots, \mathfrak{b}_h$ to be finite. We then carry out a *reduction*, replacing $\rho_x\mathfrak{A}$ by a certain finite $\mathfrak{r}$ wherever it occurs as part of a formula in our sequence. In particular, the formulas (10) will change into

$$\mathfrak{A}(\mathfrak{b}_1) \rightarrow \mathfrak{A}(\mathfrak{r}), \cdots, \mathfrak{A}(\mathfrak{b}_h) \rightarrow \mathfrak{A}(\mathfrak{r}). \tag{11}$$

We now see how to choose $\mathfrak{r}$: if by examining the finite formulas $\mathfrak{A}(\mathfrak{b}_1), \cdots, \mathfrak{A}(\mathfrak{b}_h)$ one after the other, we find one that is true, say $\mathfrak{A}(\mathfrak{b}_3)$, then we take $\mathfrak{b}_3$ for $\mathfrak{r}$. If every one of them turns out to be false, we choose $\mathfrak{r}$ at random. Then the h reduced formulas (11) are "true" and our hypothesis that the deduction leads to the false formula $0 \neq 0$ is carried *ad absurdum*. The salient point is that a concretely given deduction makes use of a limited number of explicitly exhibited individuals $\mathfrak{b}_1, \cdots, \mathfrak{b}_h$ only. If we make a wrong choice, for example, by choosing Alcibiades rather than Aristides as the representative of incorruptibility, our mistake will do no harm as long as the few people (out of the infinite Athenian crowd) with whom we actually deal are all corruptible.

A slightly more complicated case arises when we permit the $\mathfrak{b}_1, \cdots, \mathfrak{b}_h$ to contain ρ_x, but always followed by the same $\mathfrak{A}$. Then we first make a *tentative* reduction replacing $\rho_x\mathfrak{A}$ by the number 0, say. The formulas $\mathfrak{b}_1, \cdots, \mathfrak{b}_h$ are thus changed into reduced finite formulas $\mathfrak{b}_1^0, \cdots, \mathfrak{b}_h^0$ and (10) into

$$\mathfrak{A}(\overset{0}{\mathfrak{b}}_1) \rightarrow \mathfrak{A}(0), \cdots, \mathfrak{A}(\overset{0}{\mathfrak{b}}_h) \rightarrow \mathfrak{A}(0).$$

This reduction will do unless $\mathfrak{A}(0)$ is false and at the same time one of the $\mathfrak{A}(\mathfrak{b}_1^0), \cdots, \mathfrak{A}(\mathfrak{b}_h^0)$, say $\mathfrak{A}(\mathfrak{b}_3^0)$, is true. But then we have in $\mathfrak{b}_3^0$ a perfectly legitimate representative of $\mathfrak{A}$, and a second reduction which replaces $\rho_x\mathfrak{A}$ by $\mathfrak{b}_3^0$ will work out all right.

However, this is only a modest beginning of the complications awaiting us. Quantifiers $\rho_x, \rho_y, \cdots$ with different variables and applied to different formulas will be piled one upon the other. We make a tentative reduction; it will go wrong in certain places and from that failure we learn how to correct it. But the corrected reduction will probably go wrong at other places. We seem to be driven around in a vicious circle, and the problem is to direct our consecutive corrections in such a manner as to obtain assurance that finally a reduction will result that makes good at all places in our given sequence of formulas. Nothing has contributed more to revealing the circle-like character of the usual transfinite arguments of mathematics than these attempts to make sure of consistency in spite of all circles.

The symbolism for the formalization of mathematics as well as the general layout and first steps of the proof of consistency are due to Hilbert himself. The program was further advanced by younger collaborators, P. Bernays, W. Ackermann, J. von Neumann. The last two proved the consistency of "arithmetics," of that part in which the dangerous axiom about the conversion of predicates into sets is not yet admitted. A gap remained which seemed harmless at the time, but already detailed plans were drawn up for the invasion of analysis. Then came a catastrophe: assuming that consistency is established, K. Gödel showed how to construct arithmetical propositions which are evidently true and yet not deducible within the formalism. His method applies to Hilbert's as well as any other not too limited formalism. Of the two fields, the field of formulas obtainable in Hilbert's formalism and the field of real propositions that are evidently true, neither contains the other (provided consistency of the formalism can be made evident). Obviously *completeness* of a formalism in the absolute sense in which Hilbert had envisaged it was now out of the question. When G. Gentzen later closed the gap in the consistency proof for arithmetics, which Göbel's discovery had revealed to be serious indeed, he succeeded in doing so only by substantially lowering Hilbert's standard of evidence.[18] The boundary line of what is intuitively trustworthy once more became vague. As all hands were needed to defend the homeland of arithmetics, the invasion of analysis never came off, to say nothing of general set theory.

This is where the problem now stands; no final solution is in sight. But whatever the future may bring, there is no doubt that Brouwer and Hilbert raised the problem of the foundations of mathematics to a new level. A return to the standpoint of Russell-Whitehead's *Principia Mathematica* is unthinkable.

Hilbert is the champion of axiomatics. The axiomatic attitude seemed to him one of universal significance, not only for mathematics, but for all sciences. His investigations in the field of physics are conceived in the axiomatic spirit. In his lectures he liked to illustrate the method by examples taken from biology, economics, and so on. The modern epistemological interpretation of science has been profoundly influenced by him. Sometimes when he praised the axiomatic method he seemed to imply that it was destined to obliterate completely the constructive or genetic method. I am certain that, at least in later life, this was not his true opinion. For whereas he deals with the primary mathematical objects by means of the axioms of his symbolic system, the formulas are constructed in the most explicit and finite manner. In recent times the axiomatic method has spread from the roots to all branches of the mathematical tree. Algebra, for one, is permeated from top to bottom by the axiomatic spirit. One may describe the role of axioms here as the subservient one of fixing the range of variables entering into the explicit constructions. But it would not be too difficult to retouch the picture so as to make the axioms appear as the masters. An impartial attitude will do justice to both sides; not a little of the attractiveness of modern mathematical research is due to a happy blending of axiomatic and genetic procedures.

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[18] G. Gentzen, Math. Ann. vol. 112 (1936) pp. 493–565.

INTEGRAL EQUATIONS

Between the two periods during which Hilbert's efforts were concentrated on the foundations, first of geometry, then of mathematics in general, there lie twenty long years devoted to analysis and physics.

In the winter of 1900–1901 the Swedish mathematician E. Holmgren reported in Hilbert's seminar on Fredholm's first publications on integral equations, and it seems that Hilbert caught fire at once. The subject has a long and tortuous history, beginning with Daniel Bernoulli. The mathematicians' efforts to solve the (mechanical, acoustical, optical, electromagnetical) problem of the oscillations of a continuum and the related boundary value problems of potential theory span a period of two centuries. Fourier's *Théorie analytique de la chaleur* (1822) is a landmark. H. A. Schwarz proved for the first time (1885) the existence of a proper oscillation in two and more dimensions by constructing the fundamental frequency of a membrane. The last decade of the nineteenth century saw Poincaré on his way to the development of powerful function-theoretic methods; C. Neumann and he came to grips with the harmonic boundary problem; Volterra studied that type of integral equations which now bears his name, and for linear equations with infinitely many unknowns Helge von Koch developed the infinite determinants. Most scientific discoveries are made when "their time is fulfilled"; sometimes, but seldom, a genius lifts the veil decades earlier than could have been expected. Fredholm's discovery has always seemed to me one that was long overdue when it came. What could be more natural than the idea that a set of linear equations connected with a discrete set of mass points gives way to an integral equation when one passes to the limit of a continuum? But the fact that in the simpler cases a differential rather than an integral equation results in the limit riveted the mathematicians' attention for two hundred years on differential equations!

It must be said, however, that the simplicity of Fredholm's results is due to the particular form of his equation, on which it was hard to hit without the guidance of the problems of mathematical physics to which he applied it:

$$x(s) - \int_0^1 K(s, t)x(t)dt = f(s) \qquad (0 \leqq s \leqq 1).$$

Indeed the linear operator which in the left member operates on the unknown function x producing a given f, $(E-K)x=f$, consists of two parts, the identity E and the integral operator K, which in a certain sense is weak compared to E. Fredholm proved that for this type of integral equation the two main facts about n linear equations with the same number n of unknowns hold: (1) The homogeneous equation $[f(s)=0]$ has a finite number of linearly independent solutions $x(s)=\phi_1(s), \cdots, \phi_h(s)$, and the homogeneous equation with the transposed kernel $K'(s, t)=K(t, s)$ has the same number of solutions, $\psi_1(s), \cdots, \psi_h(s)$. (2) The nonhomogeneous equation is solvable if and only if the given f satisfies the h linear conditions

$$\int_0^1 f(s)\psi_i(s)ds = 0 \qquad (i = 1, \cdots, h).$$

Following an artifice used by Poincaré, Fredholm introduces a parameter λ replacing K by λK and obtains a solution in the form familiar from finite linear equations, namely as a quotient of two determinants of H. v. Koch's type, either of which is an entire function of the parameter λ.

Hilbert saw two things: (1) after having constructed Green's function K for a given region G and for the potential equation $\Delta u=0$ by means of a Fredholm equation on the boundary, the differential equation of the oscillating membrane $\Delta\phi+\lambda\phi=0$ changes into a homogeneous integral equation

$$\phi(s) - \lambda \int K(s, t)\phi(t)dt = 0$$

with the symmetric kernel K, $K(t, s)=K(s, t)$ (in which the parameter λ is no longer artificial but of the very essence of the problem); (2) the problem of ascertaining the "eigen values" λ and "eigen functions" $\phi(s)$ of this integral equation is the analogue for integrals of the transformation of a quadratic form of n variables onto principal axes. Hence the corresponding theorem for the quadratic integral form

$$\int_0^1 \int_0^1 K(s, t)x(s)x(t)dsdt \tag{12}$$

with an *arbitrary symmetric kernel* K must provide the general foundation for the theory of oscillations of a continuous medium. If others saw the same, Hilbert saw it at least that much more clearly that he bent all his energy on proving that proposition, and he succeeded by the same direct method which about 1730 Bernoulli had applied to the oscillations of a string: passage to the limit from the algebraic problem. In carrying out the limiting process he had to make use of the Koch-Fredholm determinant. He finds that there is a sequence of eigen values $\lambda_1, \lambda_2, \cdots$ tending to infinity, $\lambda_n \rightarrow \infty$ for $n \rightarrow \infty$, and an orthonormal set of corresponding eigen functions $\phi_n(s)$,

$$\phi_n(s) - \lambda_n \int_0^1 K(s, t)\phi_n(t)dt = 0,$$

$$\int_0^1 \phi_m(s)\phi_n(s)ds = \delta_{mn},$$

such that

$$\int_0^1 \int_0^1 K(s, t)x(s)x(t)dsdt = \sum \xi_n^2/\lambda_n,$$

ξ_n being the Fourier coefficient $\int_0^1 x(s)\phi_n(s)ds$. The theory implies that every function of the form

$$y(s) = \int_0^1 K(s, t)x(t)dt$$

may be expanded into a uniformly convergent Fourier series in terms of the eigen functions ϕ_n,

$$y(s) = \sum \eta_n\phi_n(s), \qquad \eta_n = \int_0^1 y(s)\phi_n(s)ds.$$

Hilbert's passage to the limit is laborious. Soon afterwards E. Schmidt in a Göttingen thesis found a simpler and more constructive proof for these results by adapting H. A. Schwarz's method invented twenty years before to the needs of integral equations.

From finite forms the road leads either to integrals or to infinite series. Therefore Hilbert considered the same problem of orthogonal transformation of a given quadratic form

$$\sum K_{mn}x_m x_n \tag{13}$$

into a form of the special type

$$\kappa_1\xi_1^2 + \kappa_2\xi_2^2 + \cdots \qquad (\kappa_n = 1/\lambda_n \to 0) \tag{14}$$

also for infinitely many (real) variables $(x_1, x_2, \cdots)$ or vectors x in a space of a denumerable infinity of dimensions. Only such vectors are admitted as have a finite length $|x|$,

$$|x|^2 = x_1^2 + x_2^2 + \cdots ;$$

they constitute what we now call the Hilbert space. The advantage of Hilbert space over the "space" of all continuous functions $x(s)$ lies in a certain property of completeness, and due to this property one can establish "complete continuity" as the necessary and sufficient condition for the transformability of a given quadratic form K, (13), into (14), by following an argument well known in the algebraic case: one determines $\kappa_1, \kappa_2, \cdots$ as the consecutive maxima of K on the "sphere" $|x|^2=1$.

As suggested by the theorem concerning a quadratic integral form, the link between the space of functions $x(s)$ and the Hilbert space of vectors $(x_1, x_2, \cdots)$ is provided by an arbitrary *complete* orthonormal system $u_1(s)$, $u_2(s)$, $\cdots$ and expressed by the equations

$$x_n = \int_0^1 x(s)u_n(s)ds.$$

Bessel's inequality states that the square sum of the Fourier coefficients x_n is less than or equal to the square integral of $x(s)$. The relation of completeness, first introduced by A. Hurwitz and studied in detail by W. Stekloff, requires that in this inequality the *equality* sign prevail. Thus the theorem on quadratic forms of infinitely many variables at once gives the corresponding results about the eigen values and eigen functions of symmetric kernels $K(s, t)$—or would do so if one could count on the uniform convergence of $\sum x_n u_n(s)$ for any given vector $(x_1, x_2, \cdots)$ in Hilbert space. In the special case of an eigen vector of that quadratic form (13) which corresponds to the integral form (12),

$$x_n = \lambda \sum_m K_{nm}x_m,$$

Hilbert settles this point by forming the uniformly convergent series

$$\lambda \sum_m x_m \int_0^1 K(s, t) u_m(t) dt$$

which indeed yields a continuous function $\phi(s)$ with the nth Fourier coefficient

$$\lambda \sum K_{nm} x_m = x_n,$$

and thus obtains the eigen function of $K(s, t)$ for the eigen value λ. Soon afterwards, under the stimulus of Hilbert's investigations. E. Fischer and F. Riesz proved their well known theorem that the space of all functions $x(s)$ the square of which has a finite Lebesgue integral enjoys the same property of completeness as Hilbert space, and hence one is mapped isomorphically upon the other in a one-to-one fashion by means of a complete orthonormal system $u_n(s)$. I mention these details because the historic order of events may have fallen into oblivion with many of our younger mathematicians, for whom Hilbert space has assumed that abstract connotation which no longer distinguishes between the two realizations, the square integrable functions $x(s)$ and the square summable sequences $(x_1, x_2, \cdots)$. I think Hilbert was wise to keep within the bounds of continuous functions when there was no actual need for introducing Lebesgue's general concepts.

Perhaps Hilbert's greatest accomplishment in the field of integral equations is his extension of the theory of spectral decomposition from the completely continuous to the so-called *bounded* quadratic forms. He finds that then the point spectrum will in general have condensation points and a continuous spectrum will appear beside the point spectrum. Again he proceeds by directly carrying out the transition to the limit, letting the number of variables $x_1, x_2, \cdots$ increase *ad infinitum*. Again, not long afterwards, simpler proofs for his results were found.

While thus advancing the boundaries of the general theory, he did not lose sight of the ordinary and partial differential equations from which it had sprung. Simultaneously with the young Italian mathematician Eugenio Elia Levi he developed the parametrix method as a bridge between differential and integral equations. For a given elliptic differential operator Δ^* of the second order, the parametrix $K(s, t)$ is a sort of qualitative approximation of Green's function, depending like the latter on an argument point s and a parameter point t. It is supposed to possess the right kind of singularity for $s = t$ so that the nonhomogeneous equation $\Delta^* u = f$ for

$$u = K\rho, \qquad u(s) = \int K(s, t)\rho(t)dt$$

gives rise to the integral equation $\rho+L\rho=f$ for the density ρ, with a kernel $L(s, t)=\Delta_s^* K(s, t)$ regular enough at $s=t$ for Fredholm's theory to be applicable. It is important to give up the assumption that K satisfies the equation $\Delta^* K=0$, because in general a fundamental solution will not be known for the given differential operator Δ^*. In order not to be bothered by boundary conditions, Hilbert assumes the domain of integration to be a *compact* manifold, like the surface of a sphere, and finds that the method works if the parametrix, besides having the right kind of singularity, is symmetric with respect to argument and parameter.

What has been said should be enough to make clear that in the terrain of analysis a rich vein of gold had been struck, comparatively easy to exploit and not soon to be exhausted. The linear equations of infinitely many unknowns had to be investigated further (E. Schmidt, F. Riesz, O. Toeplitz, E. Hellinger, and others); the continuous spectrum and its appearance in integral equations with "singular" kernels awaited closer analysis (E. Hellinger, T. Carleman); ordinary differential equations, with regular or singular boundaries, of second or of higher order, received their due share of attention (A. Kneser, E. Hilb, G. D. Birkhoff, M. Bôcher, J. D. Tamarkin, and many others).[14] It became possible to develop such asymptotic laws for the distribution of eigen values as were required by the thermodynamics of radiation (H. Weyl, R. Courant). Expansions in terms of orthogonal functions were studied independently of their origin in differential or integral equations. New light fell upon Stieltjes's continued fractions and the problem of momentum. The most ambitious began to attack nonlinear integral equations. A large international school of young mathematicians gathered around Hilbert and integral equations became the fashion of the day, not only in Germany, but also in France where great masters like E. Picard and Goursat paid their tributes, in Italy and on this side of the Atlantic. Many good papers were written, and many mediocre ones. But the total effect was an appreciable change in the aspect of analysis.

Remarkable are the applications of integral equations outside the field for which they were invented. Among them I mention the following three: (1) Riemann's problem of determining n analytic functions $f_1(z), \cdots, f_n(z)$, regular except at a finite number of points, which by analytic continuation around these points suffer preassigned linear transformations. The problem was solved by Hilbert himself, and subsequently in a simpler and more complete form by J. Plemelj. (A very special case of it is the existence of algebraic functions on a Riemann surface if that surface is given as a covering surface of the complex z-plane.) Investigations by G. D. Birkhoff on matrices of analytic functions lie in the same line. (2) Proof for the completeness of the irreducible representations of a compact continuous group. This is an indispensable tool for the approach to the general theory of invariants by means of Adolf Hurwitz's integration method, and with its refinements and generalizations plays an important role in modern group-theoretic research, including H. Bohr's theory of almost periodic functions.[15] Contact is thus made with Hilbert's old friend, the theory of invariants. (3) Quite recently Hilbert's parametrix method has served to establish the central existence theorem in W. V. D. Hodge's theory of harmonic integrals in compact Riemannian spaces.[16]

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[14] For later literature and systems of differential equations see Axel Schur, Math. Ann. vol. 82 (1921) pp. 213–239; G. A. Bliss, Trans. Amer. Math. Soc. vol. 28 (1926) pp. 561–584; W. T. Reid, ibid. vol. 44 (1938) pp. 508–521.

The story would be dramatic enough had it ended here. But then a sort of miracle happened: the spectrum theory in Hilbert space was discovered to be the adequate mathematical instrument of the new quantum physics inaugurated by Heisenberg and Schrödinger in 1923. This latter impulse led to a reexamination of the entire complex of problems with refined means (J. von Neumann, A. Wintner, M. H. Stone, K. Friedrichs). As J. von Neumann was Hilbert's collaborator toward the close of that epoch when his interest was divided between quantum physics and foundations, the historic continuity with Hilbert's own scientific activities is unbroken, even for this later phase. What has become of the theory of abstract spaces and their linear operators in our times lies beyond the bounds of this report.

A picture of Hilbert's "analytic" period would be incomplete without mentioning a second motif, *calculus of variations*, which crossed the dominating one of integral equations. The "theorem of independence" with which he concludes his Paris survey of mathematical problems (1900) is an important contribution to the formal apparatus of that calculus. But of much greater consequence was his audacious direct assault on the functional maxima and minima problems. The whole finely wrought machinery of the calculus of variations is here consciously set aside. He proposes instead to construct the minimizing function as the limit of a sequence of functions for which the value of the integral under investigation tends to its minimum value. The classical example is Dirichlet's integral in a two-dimensional region G,

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[15] H. Weyl and F. Peter, Math. Ann. vol. 97 (1927) pp. 737–755. A. Haar, Ann. of Math. vol. 34 (1933) pp. 147–169. J. von Neumann, Trans. Amer. Math. Soc. vol. 36 (1934) pp. 445–492. Cf. also L. Pontrjagin, *Topological groups*, Princeton, 1939.

[16] W. V. D. Hodge, *The theory and applications of harmonic integrals*, Cambridge, 1941. H. Weyl, Ann. of Math. vol. 44 (1943) pp. 1–6.

$$D[u] \doteq \int\int_G \left\{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right\} dxdy.$$

Admitted are all functions u with continuous derivatives which have given boundary values. d being the lower limit of $D[u]$ for admissible u, one can ascertain a sequence of admissible functions u_n such that $D[u_n] \rightarrow d$ with $n \rightarrow \infty$. One cannot expect the u_n themselves to converge; rather they have to be prepared for convergence by the smoothing process of integration. As the limit function will be harmonic and the value of the harmonic function at any point P equals its mean value over any circle K around P, it seems best to replace $u_n(P)$ by its mean value in K, with the expectation that this mean value will converge toward a number $u(P)$ which is independent of the circle and in its dependence on P solves the minimum problem. Besides integration Hilbert uses the process of sifting a suitable subsequence from the u_n before passing to the limit. Owing to the simple inequality

$$\{D[u_m - u_n]\}^{1/2} \leqq \{D[u_n] - d\}^{1/2} + \{D[u_m] - d\}^{1/2}$$

discovered by S. Zaremba this second step is unnecessary.

Hilbert's method is even better suited for problems in which the boundary does not figure so prominently as in the boundary value problems. By a slight modification one is able to include point singularities, and Hilbert thus solved the fundamental problem for flows on Riemann surfaces, providing thereby the necessary foundation for Riemann's own approach to the theory of Abelian integrals, and he further showed that Poincaré's and Koebe's fundamental theorems on uniformization could be established in the same way. We should be much better off in number theory if methods were known which are as powerful for the construction of relative Abelian and Galois fields over given algebraic number fields as the Riemann-Hilbert transcendental method proves to be for the analogous problems in the fields of algebraic functions! Its wide application in the theory of conformal mapping and of minimal surfaces is revealed by the work of the man who was Hilbert's closest collaborator in the direction of mathematical affairs at Göttingen for many years, Richard Courant.[17]

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[17] A book by Courant on the Dirichlet principle is in preparation.

Of a more indirect character, but of considerable vigor, is the influence of Hilbert's ideas upon the whole trend of the modern development of the calculus of variations; in Europe Carathéodory, Lebesgue, Tonelli could be mentioned among others, in this country the chain reaches from O. Bolza's early to M. Morse's most recent work.

PHYSICS

Already before Minkowski's death in 1909, Hilbert had begun a systematic study of theoretical physics, in close collaboration with his friend who had always kept in touch with the neighboring science. Minkowski's work on relativity theory was the first fruit of these joint studies. Hilbert continued them through the years, and between 1910 and 1930 often lectured and conducted seminars on topics of physics. He greatly enjoyed this widening of his horizon and his contact with physicists, whom he could meet on their own ground. The harvest however can hardly be compared with his achievements in pure mathematics. The maze of experimental facts which the physicist has to take into account is too manifold, their expansion too fast, and their aspect and relative weight too changeable for the axiomatic method to find a firm enough foothold, except in the thoroughly consolidated parts of our physical knowledge. Men like Einstein or Niels Bohr grope their way in the dark toward their conceptions of general relativity or atomic structure by another type of experience and imagination than those of the mathematician, although no doubt mathematics is an essential ingredient. Thus Hilbert's vast plans in physics never matured.

But his application of integral equations to kinetic gas theory and to the elementary theory of radiation were notable contributions. In particular, his asymptotic solution of Maxwell-Boltzmann's fundamental equation in kinetic gas theory, which is an integral equation of the second order, clearly separated the two layers of phenomenological physical laws to which the theory leads; it has been carried out in more detail by the physicists and applied to several concrete problems. In his investigations on general relativity Hilbert combined Einstein's theory of gravitation with G. Mie's program of pure field physics. For the development of the theory of general relativity at that stage, Einstein's more sober procedure, which did not couple the theory with Mie's highly speculative program, proved the more fertile. Hilbert's endeavors must be looked upon as a forerunner of a unified field theory of gravitation and electromagnetism. However, there was still much too much arbitrariness involved in Hilbert's Hamiltonian function; subsequent attempts (by Weyl, Eddington, Einstein himself, and others) aimed to reduce it. Hopes in the Hilbert circle ran high at that time; the dream of a universal law accounting both for the structure of the cosmos as a whole, and of all the atomic nuclei, seemed near fulfillment. But the problem of a unified field theory stands to this day as an unsolved problem; it is almost certain that a satisfactory solution will have to include the material waves (the Schrödinger-Dirac ψ for the electron, and similar field quantities for the other nuclear particles) besides gravitation and electromagnetism, and that its mathematical frame will not be a simple enlargement of that of Einstein's now classical theory of gravitation.

Hilbert was not only a great scholar, but also a great teacher. Witnesses are his many pupils and assistants, whom he taught the handicraft of mathematical research by letting them share in his own work and its overflow, and then his lectures, the notes of many of which have found their way from Göttingen into public and private mathematical libraries. They covered an extremely wide range. The book he published with S. Cohn-Vossen on *Anschauliche Geometrie* is an outgrowth of his teaching activities. Going over the impressive list attached to his Collected Papers (vol. 3, p. 430) one is struck by the considerable number of courses on general topics like "Knowledge and Thinking," "On the Infinite," "Nature and Mathematics." His speech was fairly fluent, not as hesitant as Minkowski's, and far from monotonous. He had no difficulty in finding the pregnant words, and liked to emphasize short pivotal phrases by repeating them several times. On the whole, his lectures were a faithful reflection of his spirit; direct, intense; how could they fail to be inspiring?

BOOK REVIEW

Meromorphic functions and analytic curves. By Hermann Weyl and Joachim Weyl. (Annals of Mathematics Studies, no. 12.) Princeton University Press; London, Humphrey Milford and Oxford University Press, 1943. 9+269 pp. $3.50.

The theory of meromorphic functions began seriously with the now classical investigations of Poincaré, Hadamard, and Picard. After being continued by various other mathematicians it culminated in a paper by R. Nevanlinna in 1925, which was (according to the present author[1]) "one of the few great mathematical events in our century."

Nevanlinna associates with every meromorphic function $f(z)$ in the complex z-plane two non-negative real functions $N(r, a)$ and $m(r, a)$; the former counts how often f takes the value a in $|z| < r$, the latter measures the average closeness of f to a in $|z| < r$. Two of the principal results of Nevanlinna are these: (1) $N(r, a) + m(r, a) = T(r) + O(1)$, where $T(r)$ does not depend on a. (2) If any different complex numbers $a_1, \cdots, a_n$ and $\epsilon > 0$ are given, then for all $r > 0$ except for a set of finite measure, $\sum_i m(r, a_i) < (2+\epsilon)T(r)$. This implies in particular $N(r, a) \equiv 0$ for at most two a, that is, Picard's Theorem.

H. and J. Weyl and Ahlfors generalized these results in two respects. The underlying space is not any longer the z-plane, but any Riemann surface $\mathfrak{R}$, and the function f is replaced by $n+1$ functions on $\mathfrak{R}$ whose ratios define an analytic curve in the n-dimensional complex projective space P^n. Special cases are the algebraic curves (if $\mathfrak{R}$ is compact) and the meromorphic curves (if $\mathfrak{R}$ is the z-plane).

The first chapter provides the foundations for this theory and starts by introducing the Plücker coordinates of a $(p-1)$-dimensional linear subspace (p-spread) in P^n. The p-spreads appear as special vectors in the $C_{n,p}$-dimensional vector space of all p-ads. This space is metrized with the help of an Hermitian form in P^n. Next analytic curves are defined. An analytic curve C_1 determines for every p, $1 < p \leqq n$, the locus C_p of its osculating p-spreads in the space of all p-ads. The methods of Hensel-Landsberg allow to treat the stationary indices of these curves and their relations to each other. This investigation leads for algebraic curves to the general Plücker relations between the orders and the stationary indices of the C_p.

The analogue to Nevanlinna's result (1) is the subject of chapter II. The function $N(r, a)$ is replaced by a function $N_p(R, A)$, $1 \leqq p \leqq n$, which counts the number of intersections of C_p with the hyperplane A (in the space of the p-ads) for all points of C_p corresponding to values $|z| < R$. The function $m(r, a)$ becomes a function $m_p(R, A)$ which measures the average (unitary) distance of the same points from A. Then, independently of the unitary metric in P^n,

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[1] The preface states that the book, while based on the joint work of the two authors, was actually written by H. Weyl.

$$N_p(R, A) + m_p(R, A) = T_p(R) + O(1) \quad \text{(first main theorem).}$$

Meromorphic functions and exponential curves are treated as examples. The theorems of Poincaré and Hadamard on meromorphic functions of finite order conclude the chapter.

The following chapter (III) treats the relations between the order functions $T_p(R)$, $1 \leqq p \leqq n$, and the stationary indices of a given curve C_1, and yields thus an extension of the Plücker formulas. The result (second main theorem) is indeed surprisingly similar to these formulas. In addition we find several inequalities relating these numbers. One of the main tools is expressing the $T_p(R)$ as averages, for instance $T_1(R)$ turns out to be the average of $N_1(R, A)$ over all hyperplanes A, of P^n.

In chapter IV the first and second main theorems are generalized to analytic curves. A system of bounded open sets G containing a fixed compact nucleus K_0 and exhausting the underlying Riemann surface R replaces the circles $|z| < R$. (In the case of meromorphic curves a fixed circle $|z| \leqq r_0$ plays the role of K_0; its influence is concealed in the $O(1)$ used by the reviewer.) The functions $N_p(G, A)$ are formed with the help of a harmonic function in $G - K_0$ which has the values 1 and 0 on the boundaries of K_0 and G respectively. The first main theorem holds except that $O(1)$ must be replaced by an expression $m^0(G, A)$ which depends on G and K_0 but whose order of magnitude can be estimated. A similar function $\eta(G, K_0)$ (and the genus of R) enters the second main theorem.

The last chapter deals with Picard's Theorem, more generally with Nevanlinna's defect relations. These results can be extended to analytic curves for which the above mentioned term $\eta(G, K_0)$ is small with respect to $T_1(G)$ (Hypothesis H). The implications of this assumption are rather difficult to understand. Whereas Picard's theorem for meromorphic functions on Riemann surfaces is readily obtained, the general defect relations are involved and require a great deal of technique. But they lead finally to the following beautiful result: If H holds then "for almost all" $G \supset K_0$ (compare the formulation of (2) above)

$$\sum_i m_p(G, A_i^p) < [C_{n,p} + \epsilon] T_p(G)$$

for any finite number of contravariant p-ads A_i^p in general position. This implies in particular that an analytic curve in P^n intersects at least one of any $n+1$ hyperplanes in general position.

The book begins with an allegoric preface and a historical introduction. The style is spirited and permeated by that elegant technique which the author displays in all his books. The reader sees always a wide mathematical landscape, and never just a one-dimensional approach to a given problem. Only in one instance, namely the construction of the harmonic function in chapter IV, §4, does it seem disputable whether the author chose the most efficient method. The reviewer has seen lecture notes on a course given by Ahlfors in Harvard which convinced him that subharmonic functions furnish a smoother approach, because they allow operating in the large.

It seems difficult to enumerate the exact prerequisites for the book. The reader must be familiar with various mathematical tools; this applies in particular to Riemann surfaces, since the indications in chapter I, §9, are not sufficient as a first introduction. The subject matter is involved and interrelated with several fields. The book is therefore not easy to read. But as always, the author rewards the reader by a wealth of information and mathematical technique which is useful far beyond the immediate scope of the book.

H. BUSEMANN

NOTES

Professor E. A. Milne of the University of Oxford has been elected president of the Royal Astronomical Society.

Professor G. D. Birkhoff of Harvard University has been awarded the honorary doctorate of science by Illinois Institute of Technology.

Dr. R. v. Mises of Harvard University and Professor John von Neumann of the Institute for Advanced Study have been elected to membership in the American Academy of Arts and Sciences.

Professor Oscar Zariski of Johns Hopkins University has been elected to membership in the National Academy of Sciences.

Professor A. L. Nilson of Wayne University has been elected president of the Michigan Academy of Science, Arts and Letters.

Professor Richard Morris of Rutgers University has retired with the title professor emeritus.

Professor Gertrude Smith of Vassar College has retired with the title professor emeritus.

Assistant Professor Robert Schatten of the University of Vermont has been appointed National Research Fellow for the academic year 1944–1945. He will study at the Institute for Advanced Study.

Assistant Professor J. M. Barbour of Michigan State College has been promoted to an associate professorship.

Dr. A. E. Basch of the College of the City of New York has been appointed to an assistant professorship at Rensselaer Polytechnic Institute.

Associate Professor W. D. Baten of Michigan State College has been promoted to a professorship.

Dr. Helen P. Beard of Newcomb College has been promoted to an assistant professorship.

Dr. R. A. Beaumont of the University of Washington has been promoted to an assistant professorship.

Assistant Professor E. M. Beesley of the University of Nevada has been promoted to an associate professorship.

Mr. D. R. Belcher has been appointed treasurer of the American Telephone and Telegraph Company.

Associate Professor L. M. Blumenthal of the University of Missouri has been promoted to a professorship.

Assistant Professor J. R. Britton of the University of Colorado has been promoted to an associate professorship.

Dr. R. H. Bruck of the University of Wisconsin has been promoted to an assistant professorship.

Sister M. Roberdette Burns of Clarke College, Dubuque, Iowa, has been promoted to an assistant professorship.

Dr. Nancy Cole of Sweet Briar College has been appointed to an assistant professorship at Connecticut College.

Dr. Esther Comegys of the University of Maine has been promoted to an assistant professorship.

Mr. W. S. Dawkins has been appointed to a professorship at Loyola College, Baltimore, Maryland.

Assistant Professor L. A. Dye of The Citadel, Charleston, South Carolina, has been appointed to an associate professorship.

Assistant Professor G. M. Ewing of the University of Missouri has been promoted to an associate professorship.

Dr. Marciano García of the University of Richmond has been appointed to an assistant professorship at the College of Agriculture and Mechanic Arts, University of Puerto Rico, Mayagüez, Puerto Rico.

Dr. Hilda P. Geiringer of Bryn Mawr College has been appointed to a professorship at Wheaton College, Norton, Massachusetts.

Mr. E. L. Godfrey of Fenn College has been appointed to an assistant professorship at Northwest Missouri State Teachers College, Maryville, Missouri.

Dr. Michael Golomb of Purdue University has been promoted to an assistant professorship.

Professor A. J. Hoare of the University of Wichita has retired.

Assistant Professor S. A. Jennings of the University of British Columbia has been promoted to an associate professorship.

Dr. Wilfred Kaplan of the University of Michigan has been promoted to an assistant professorship.

Professor N. J. Lennes of Montana State University has retired.

Assistant Professor Norman Levinson of Massachusetts Institute of Technology has been promoted to an associate professorship.

Dr. Mark Lotkin of Carleton College has been appointed to an assistant professorship at Wabash College.

Assistant Professor Eugene Lukacs of Illinois College has been appointed to an associate professorship at Berea College.

Assistant Professor M. S. Macphail of Acadia University has been promoted to an associate professorship.

Professor A. L. McCarty of San Francisco Junior College has retired.

Dr. C. T. McCormick of Fort Hays Kansas State College, Hays, Kansas, has been appointed to an associate professorship at the Illinois State Normal University, Normal, Illinois.

Assistant Professor W. C. McDaniel of Southern Illinois State Normal University has been promoted to an associate professorship.

Professor W. G. McGavock of Davidson College is on leave and at Duke University as visiting assistant professor.

Assistant Professor A. N. Milgram of the University of Notre Dame has been promoted to an associate professorship.

Dr. W. L. Mitchell of Michigan State College has been promoted to an assistant professorship.

Assistant Professor Harriet F. Montague of the University of Buffalo has been promoted to an associate professorship.

Assistant Professor David Moskovitz of the Carnegie Institute of Technology has been promoted to an associate professorship.

Assistant Professor D. C. Murdoch of the University of Saskatchewan has been appointed to an associate professorship at the University of British Columbia.

Assistant Professor Abba V. Newton of Smith College has been appointed to an assistant professorship at Vassar College.

Dr. K. L. Nielsen of Louisiana State University has been promoted to an assistant professorship.

Assistant Professor E. L. Post of the College of the City of New York has been promoted to an associate professorship.

Associate Professor J. E. Powell of Michigan State College has been promoted to a professorship.

Associate Professor W. T. Reid of the University of Chicago has been appointed to a professorship at Northwestern University.

Associate Professor L. V. Robinson of Marshall College has been appointed to an associate professorship at Emory University, Emory University, Georgia.

Assistant Professor W. J. Robinson of Allegheny College, Meadville, Pennsylvania, has been appointed to a professorship at Centre College of Kentucky.

Dr. Louise J. Rosenbaum of Reed College has been promoted to an assistant professorship.

Assistant Professor R. G. Sanger of the University of Chicago has been appointed dean of students.

Professor G. W. Starcher of Ohio University has been appointed acting dean of the graduate college and the college of arts and sciences.

Associate Professor E. P. Starke of Rutgers University has been promoted to a professorship.

Associate Professor F. H. Steen of Allegheny College has been promoted to a professorship.

Dr. R. G. Sturm has been appointed to a professorship of engineering mechanics at Purdue University.

Assistant Professor E. G. Swafford of Fort Hays Kansas State College is on leave of absence. He has accepted a temporary appointment to an assistant professorship at Park College, Parkville, Missouri.

Professor T. Y. Thomas of the University of California at Los Angeles has been appointed to a professorship at Indiana University.

Associate Professor H. S. Wall of Northwestern University has been appointed visiting lecturer at Illinois Institute of Technology.

Assistant Professor J. F. Wardwell of Colgate University has been promoted to an associate professorship.

Professor Agnes E. Wells of Indiana University has retired.

Mr. A. H. Wheeler has been appointed professor affiliate at Clark University.

Assistant Professor P. A. White of Louisiana State University has been appointed to a visiting assistant professorship at Tulane University.

Professor W. M. Whyburn of the University of California at Los Angeles has been appointed to the presidency of Texas Technological College.

Dr. František Wolf of the University of California has been promoted to an assistant professorship.

Dr. Alexander Wundheiler of the College of the City of New York has been appointed to an assistant professorship in physics at the University of Rochester.

The following appointments to instructorships are announced: Berea College: Mrs. Elizabeth C. Lukacs; Brooklyn College: Dr. Meyer Karlin; Central College, Fayette, Missouri: Mr. T. E. Rodd; Columbia University: Mr. P. B. Norman; Duke University: Miss Eva A. Pirkle; George Williams College, Chicago, Illinois: Dr. Ruth M. Ballard; Hollins College, Hollins College, Virginia, Instructor in Education: Miss Frances E. Falvey; Michigan State College: Miss Marian Michmerhuizen, Miss Elaine Van Aken; University of Pennsylvania: Dr. W. H. Gottschalk; Rice Institute: Dr. Albert Newhouse; University of Rochester: Mr. W. N. Huff; Russell Sage College, Troy, New York: Dr. Helene Reschovsky; Sarah Lawrence College: Mrs. Ruth C. Strodt; Tulane University: Dr. R. L. Swain, Dr. Bernard Vinograde; Wellesley College: Miss Ellen E. Fedder; Williams College: Dr. Abraham Seidenberg.

Professor Emeritus W. E. H. Berwick of University College of North Wales died May 13, 1944, at the age of fifty-five years.

Professor J. R. Wilton of the University of Adelaide, Adelaide, Australia, died in April, 1944.

Mr. Nathan Hayward of Philadelphia died June 21, 1944.

Professor Emeritus F. G. Reynolds of the College of the City of New York died June 8, 1944, at the age of seventy-two years. He had been a member of the Society since 1903.

It has been reported that Dr. Ivor Schilansky of the University of Michigan was killed in action over Germany in October, 1943.

Professor Emeritus D. E. Smith of Teachers College, Columbia University, died July 29, 1944, at the age of eighty-four years. He had been a member of the Society since 1893.

ABSTRACTS OF PAPERS

SUBMITTED FOR PRESENTATION TO THE SOCIETY

The following papers have been submitted to the Secretary and the Associate Secretaries of the Society for presentation at meetings of the Society. They are numbered serially throughout this volume. Cross references to them in the reports of the meetings will give the number of this volume, the number of this issue, and the serial number of the abstract.

ALGEBRA AND THEORY OF NUMBERS

198. Warren Ambrose: *Structure theorems for a special class of normed rings.*

Normed rings (in the sense of Gelfand, but without assumptions of commutativity or of a unit) are considered which satisfy the following two conditions: (1) the underlying Banach space is a Hilbert space, (2) to each x in the ring there is an "adjoint," x^*, with the properties that the linear transformations defined by $y \to x^*y$, and $y \to yx^*$ are the adjoint operators of $y \to xy$ and $y \to yx$. The structure of such rings is completely determined. It is shown that such a ring is always a direct sum of simple rings and that a simple ring is always a full matrix ring, of either finite or infinite dimension. By a full matrix ring of infinite dimension is meant here the set of all functions of two variables (complex-valued) for which the sum of the squares of the absolute values of the elements converges, with the expected definitions of the operations in the ring. (Received July 6, 1944.)

199. E. T. Bell: *Separable diophantine equations.*

A monomial in the independent indeterminates $x_1, \cdots, x_n$ is called elementary if at least one of the indeterminates occurs only to the first power. A sum of elementary monomials with integer coefficients, equated to zero, is called an extended multiplicative equation. Systems of extended equations are called separable if their complete integer solution is reducible to that of a multiplicative system (methods for the complete solution of which are known) and a system of linear diophantine equations, or to either of these alone. Among the applications are the complete integer solution of $S_m = S_n$, where S_m is a sum of m squares with arbitrary integer coefficients. The methods used apply to diophantine equations in any unique factorization domain, in particular to an integral domain with a euclidean algorithm. The paper will appear in the Transactions. (Received July 10, 1944.)

200. L. M. Blumenthal: *Note concerning an extension of the notion of matrix rank.*

Investigations concerning the imbedding of metric spaces in elliptic spaces suggested an extension of the notion of matrix rank in terms of which certain metric properties of the space could be algebraically formulated and described in a better way than by the use of classical concepts. If $\mathcal{A} = (a_{ij})$, $\mathcal{E} = (\epsilon_{ij})$ are two $m \times n$ matrices, $\mathcal{A}$ is said to have $\mathcal{E}$-relative rank r provided the matrix $(a_{ij}\epsilon_{ij})$ $(i = 1, 2, \cdots, m;$

$j=1, 2, \cdots, n$) has rank r. The applications of matrix-relative rank so far deal only with the case in which $\mathcal{A}$ and $\mathcal{E}$ are symmetric (square) matrices, with $e_{ii}=1$, $e_{ij}=e_{ji}=\pm 1$. A fundamental problem is that of extending to matrix-relative rank the theorem of Frobenius connecting the ordinary rank of a symmetric matrix with the ranks of certain of its principal minors. Among the theorems of this note, it is shown that the symmetric matrix $\mathcal{A}=(a_{ij})$, $a_{ii}=1$, $0 \leqq a_{ij} < 1$, $i \neq j$ $(i, j=1, 2, \cdots, m)$, has an $\mathcal{E}$-relative rank 2 provided each fourth-order principal minor has an $\mathcal{E}$-relative rank 2 (where $\mathcal{E}$ represents generically any of the class $\{\mathcal{E}\}$ of symmetric matrices with $e_{ii}=1$, $e_{ij}=\pm 1$), while if each $a_{ij}>0$ and $n>4$, then $\mathcal{A}$ has an $\mathcal{E}$-relative rank 2 whenever each *third-order* principal minor has, *unless every element of* $\mathcal{A}$ *is either* $1/2$ *or* $3^{1/2}/2$. (Received July 6, 1944.)

201. H. W. Eves: *Some associated theorems on matrices and determinants.*

In offering a vector-matrix treatment of projective geometry it was found that certain portions of the subject can be elegantly handled with the use of a small group of associated theorems concerning vectors, matrices, and determinants. These theorems are interesting in themselves and are presented here isolated from their geometrical application. (Received June 10, 1944.)

202. Nathan Jacobson: *Structure theory of simple rings without finiteness assumptions.*

In the first part of the paper the structure is obtained of arbitrary simple associative rings that either (1) contain a minimal right ideal or (2) contain a maximal right ideal. Representations of these rings are found as certain types of rings of linear transformations in vector spaces over division rings. The question of the invariance of the representation is discussed in the case (1). In the second part of the paper simple algebras, including simple non-associative algebras, with a possibly infinite basis are considered. A definition of the extended center and of central simple algebra for the infinite dimensional case is given. It is proved that a central simple algebra remains simple when the underlying field is arbitrarily extended. (Received June 9, 1944.)

203. B. W. Jones: *A canonical quadratic form for the ring of 2-adic integers.*

Types of transformations are given by which any form in the ring of 2-adic integers may be reduced to a unique canonical form. This establishes a means of ascertaining the equivalence or non-equivalence of two given forms which is superior to Minkowski's criterion involving concomitant forms. The results are similar to unpublished results obtained in a different way by Gordon Pall. There are applications to problems involving genera of forms. (Received July 7, 1944.)

204. Gordon Overholtzer: *A new application of the Schur derivate.*

I. Schur introduced the derivate of a sequence $\{a_n\}$ with respect to the number p, $\bar{\Delta} a_n=(a_{n+1}-a_n)/p^{n+1}$, to generalize Fermat's theorem of elementary number theory. Fermat's theorem yields that Δa^{p^n}, the first Schur derivate of $\{a^{p^n}\}$, is integral; Schur proved the first $p-1$ derivates integral. Let $S[n, g(x)]=\sum\nolimits'^{p^n}_{i=1} g(i)/p^n$, where the summation is over numbers prime to p, and p is a rational prime. An established number theory result is that $S[n, x^k]$ (k an integer) is integral if $p-1$ does not divide

k and has exponent of p of -1 if $p-1$ divides k. This paper generalizes this result, proving that all the Schur derivates $\Delta^m S[n, x^k]$ are p-adically bounded with exponent of p not less than $-2m-1-m/(p-1)$ and hence p-adically convergent. Formulas for $\lim_{n\to\infty} \Delta^m S[n, x^k]$ are given in terms of $\lim_{n\to\infty} S[n, x^{k-m}]$. For positive k, $\lim_{n\to\infty} S[n, x^{2k+1}]=0$ and $\lim_{n\to\infty} S[n, x^{2k}]=(-1)^{k-1} B_k(1-p^{2k-1})$, where B_i is the ith Bernouillian number. The Schur derivates of $\{S[n, f(x)]\}$, where $f(x)=\sum_{i=1}^{\infty} a_i x^i$ and the valuation of $a_i \to 0$ as $i\to\infty$, are p-adically bounded and convergent; moreover $\lim_{n\to\infty} S[n, f(x)]=\sum_{i=1}^{\infty} a_i \lim_{n\to\infty} S[n, x^i]$. (Received June 29, 1944.)

205. Gordon Pall: *Note on factorization in quadratic fields.*

It is proved that if the quadratic integer $x_0+x_1\omega$ is primitive, that is x_0 and x_1 are coprime, then the divisors of $x_0+x_1\omega$ of a given norm are uniquely determined up to a unit factor. Conditions are obtained for the existence of factors of a given norm. It is claimed that the necessity for the introduction of ideals should be based not on the statement that factorization is not unique, but rather that factors do not exist. Thus in the arithmetic of ordinary quaternions, factorization of imprimitive quaternions is not unique, but that of primitive quaternions is both possible and unique; and ideals are in that case unnecessary. (Received July 10, 1944.)

206. R. R. Stoll: *Primitive semigroups.*

Let F denote the class of semigroups S each of whose elements s satisfies an equation of the form $s^n=s^m$ $(n>m)$. A semigroup $S\in F$ is called primitive if for each idempotent $e\in S$ there exists no idempotent $f\neq e$ such that $ef=fe=f$. Examples of such semigroups are (a) semigroups of F which contain only one idempotent and (b) semigroups containing a zero and such that each element is nilpotent (nil semigroups). The following structure theorem is proved for primitive semigroups. A primitive semigroup S contains a unique minimal ideal M with these properties: it is a completely simple semigroup without zero (Rees, Proc. Cambridge Philos. Soc. vol. 36 (1940) pp. 387–400), and the difference semigroup of S modulo M is a nil semigroup. Conversely, a semigroup $S\in F$ with this structure is primitive. (Received July 10, 1944.)

ANALYSIS

207. R. P. Agnew: *Abel transforms of Tauberian series.*

Let $\rho_1=.9680448\cdots$; the constant is Euler's constant plus log log 2 minus $2Ei(-\log 2)$. The following assertion is true when $\rho\geqq\rho_1$ and false when $\rho<\rho_1$. Let $u_0+u_1+\cdots$ be a series satisfying the Tauberian condition $n|u_n|<K$. Let L be the set of limit points of the sequence of partial sums of $\sum u_n$. Let $\sigma(t)=\sum t^k u_k$ be the Abel transform of $\sum u_n$. Let L_A denote the set of limit points of $\sigma(t)$; $s''\in L_A$ if there is a sequence t_n such that $0<t_n<1$, $t_n\to 1$, and $\sigma(t_n)\to s''$. To each $s'\in L$ corresponds a $s''\in L_A$ such that $|s'-s''|\leqq\rho \limsup n|u_n|$. (Received July 19, 1944.)

208. R. P. Agnew: *A genesis for Cesàro methods.*

The family C_r of Cesàro methods of summability, $r\neq 0, -1, -2, \cdots$, is and can be defined as the unique class of methods of summability whose members are simultaneously Nörlund methods and Hurwitz-Silverman-Hausdorff methods. The only methods simultaneously Riesz methods and Hurwitz-Silverman-Hausdorff methods are methods Γ_r closely related to the methods C_r. (Received June 16, 1944.)

209. R. P. Agnew: *Criteria for completeness of orthonormal sets and summability of Fourier series.*

Let $\phi_n(x)$ be an orthonormal set over Euclidean space of one or more dimensions or a measurable subset of such a space. Let G be a convergence-factor method of summability determined by functions $G_n(t)$ defined over a set T having a limit point t_0 not in T and satisfying the conditions (1) $\sum |G_n(t)|^2 < \infty$ for each $t \in T$; (2) $|G_n(t)| < K$ when $t \in T$ and $n=0, 1, \cdots$; and (3) for each n, $G_n(t) \to 1$ as $t \to t_0$. Several criteria, involving the kernel $K(x, y, t)$ defined as the limit in mean of the partial sums of the series $\sum G_n(t)\phi_n(x)\overline{\phi_n(y)}$, for completeness of the set $\phi_n(x)$ are obtained. It is shown that Fourier series of functions in L_2 are essentially summable by many non-regular as well as regular methods of summability. (Received June 5, 1944.)

210. E. F. Beckenbach and R. H. Bing: *Concerning the vertex mean-value property of harmonic polynomials.*

Results concerning harmonic polynomials and vertex mean-values, by Walsh (Bull. Amer. Math. Soc. vol. 42 (1936) pp. 923–930) for all regular n-gons in the domain of definition, and by Beckenbach and Reade (Trans. Amer. Math. Soc. vol. 53 (1943) pp. 230–238) for oriented regular n-gons, are given under weakened hypotheses. For example it is shown that if $f(x, y)$ is defined in the square S: $0<x<1$, $0<y<1$, if for each regular n-gon in S the value of $f(x, y)$ at the center of the n-gon is equal to the average of the values of $f(x, y)$ on the vertices, and if there is an M such that the exterior measure of the set of values of (x, y) on which $f(x, y)$ is greater than M is less than 1, then $f(x, y)$ is a harmonic polynomial of degree at most $n-1$. (Received June 26, 1944.)

211. Stefan Bergman: *A class of nonlinear partial differential equations and their properties.*

If one assumes that the thermal conductivity, μ, changes according to the law $\mu=\chi(U)Q(x, y)$ where χ is a function of temperature, U, alone then the heat equation for a steady, two-dimensional heat flow assumes the form $N(U)=U_{z\bar z}+aU_z+bU_{\bar z}+\chi^{-1}\chi_U U_z U_{\bar z}=0$. The function $\Phi(U)$, where $\Phi(U)=c_1\int_0^U \chi dU+c_2$, c_1 and c_2 being constants, satisfies a linear equation $L(\Phi)=\Phi_{z\bar z}+a\Phi_z+b\Phi_{\bar z}=0$. Complex solutions ϕ of $L(\phi)=0$, $\Phi=\mathrm{Re}(\phi)$, were introduced in the paper in Trans. Amer. Math. Soc. vol. 53 (1943) pp. 130–155 and other papers cited there, and it was shown that many results of the theory of analytic functions of a complex variable may be generalized to the case of such functions ϕ. Let $U=\Lambda(\Phi)$ be the function inverse to $\Phi=\Phi(U)$. The author introduces complex solutions $u=\Lambda(\phi)$ of $N(\phi)=0$ and shows that certain theorems of the theory of analytic functions hold (in a conveniently altered form) for the u's. In particular if $\chi=1/U$ then every u which is "meromorphic" in the domain B can be represented there by $u=[\prod n_B(z, \nu_k)/\prod n_B(z, \mu_k)]u_1$, ν_k and μ_k being the zeros and poles of u. Here u_1, $N(u_1)=0$, is regular and nonvanishing in B; and n_B, where $N(n_B)=0$, vanishes only at one point and has boundary values 1 on the boundary of B. (Received June 30, 1944.)

212. R. H. Cameron and W. T. Martin: *Evaluation of various Wiener integrals by use of certain Sturm-Liouville differential equations.*

In this paper the authors evaluate a number of Wiener integrals whose integrands involve exponentials of integrals of the square of the variable function. These evaluations are based on the following theorem. Let $p(t)$ be continuous and positive on $0\leqq t\leqq 1$, let λ_0 be the least characteristic value of $f''(t)+\lambda p(t)f(t)=0$, subject to the boundary conditions $f(0)=f'(1)=0$, and let $f_\lambda(t)$ denote a non-trivial solution of the above differential equation satisfying $f'(1)=0$. Then if $-\infty<\lambda<\lambda_0$ and $F(x)$ is a functional which makes either side of the following equation exist, one has $\int_C^W F(x)\exp[\lambda\int_0^1 p(t)x^2(t)dt]d_Wx = [f_\lambda(1)/f_\lambda(0)]^{1/2}\int_C^W F[y(\cdot)+f_\lambda(\cdot)\int_0^{(\cdot)} f_\lambda'(s)[f_\lambda(s)]^{-2}\cdot y(s)ds]d_Wy$. (Received June 3, 1944.)

213. R. H. Cameron and W. T. Martin: *The Wiener measure of Hilbert neighborhoods in the space of real continuous functions.*

Consider the space C consisting of all real functions $x(t)$ continuous on $0\leqq t\leqq 1$ and vanishing at $t=0$. In an earlier paper (*An expression for the solution of a class of non-linear integral equations*, Amer. J. Math. vol. 46 (1944) pp. 281–298) the authors showed that for every positive number R the subset of C for which $\int_0^1[x'(t)]^2dt<R^2$ has positive Wiener measure. In the present paper the authors evaluate the measure of this set and in addition evaluate Wiener integrals of functionals $F[\int_0^1 x'(t)dt]$ over the space C for a general class of functions $F(u)$. The evaluation is in terms of the theta-function of the first kind. Integrals are also evaluated for the case of functions $x(t)$ not necessarily vanishing at $t=0$. (Received June 2, 1944.)

214. J. J. Dennis: *Some points in the theory of positive definite J-fractions.*

Wall and Wetzel ([1] Trans. Amer. Math. Soc. vol. 55 (1944) pp. 373–392; [2] Duke Math. J. vol. 11 (1944) pp. 89–102) have extended a considerable part of the Stieltjes theory to "positive definite J-fractions" (1) $1/(b_1+z)-a_1^2/(b_2+z)-a_2^2/(b_3+z)-\cdots$, characterized by the condition that the quadratic forms (2) $\sum_{p=1}^{n}(I(b_p)+y)x_p^2-2\sum_{p=1}^{n-1}I(a_p)x_px_{p+1}$ are positive definite for $y>0$. In [1] they build upon the determinant inequalities $D_n(y)>0$, where $D_n(y)$ is the discriminant of (2). In [2] they formulate the condition of positive definiteness without determinants as (3) $I(b_p)\supseteq 0$, $|a_p^2|-R(a_p^2)\leqq 2I(b_p)I(b_{p+1})(1-g_{p-1})g_p$, $0\leqq g_{p-1}\leqq 1$, $p=1, 2, 3, \cdots$, and are able to connect the theory with other work on continued fractions. The present paper is concerned with the problem: To simplify the theory in [1] by building upon the inequalities (3). The formulation (3) is proved without determinants; and an extension of a certain minimum property of (2) [2, §3] is given. The "nest of circles" is obtained and their properties established in a very simple way. Extensions of certain theorems of [2] are given, for instance, a "best" extension of Szász' theorem. Certain parts of the theory are carried over to (1′) $1/(b_1+z_1)-a_1^2/(b_2+z_2)-a_2^2/(b_3+z_3)-\cdots$. (Received July 5, 1944.)

215. Nelson Dunford and Einar Hille: *The differentiability and uniqueness of continuous solutions of addition formulas.*

Let $G(u, v)$ be a single-valued analytic function of u, v. If $f(\lambda)$ is continuous for $0\leqq\lambda\leqq\alpha$ and has its values in a commutative normed ring and satisfies the equation $f(\lambda+\mu)=G(f(\lambda), f(\mu))$ and if $G_1(f(0), f(0))$, where $G_1=\partial G/\partial u$, has an inverse then $f(\lambda)$ has derivatives of all orders and any such solution is determined by $f(0), f'(0)$. Analogous questions are discussed in the case that λ, μ are elements of a normed ring and also in the case where f has its values in an operator ring and is continuous in the strong topology. (Received July 8, 1944.)

216. Paul Erdös: *Converse of Fabry's gap theorem.*

Let $n_1 < n_2 < \cdots < n_k < \cdots$ be a sequence of integers with $\lim (n_k/k) = \infty$. The gap theorem of Fabry states that if $\sum a_k z^{n_k}$ is a power series whose circle of convergence is the unit circle then the unit circle is the natural boundary. Pólya proved the following converse of this theorem: Let $n_1 < n_2 < \cdots < n_k < \cdots$ be a sequence such that $\liminf (n_k/k) < \infty$, then there exists a power series $\sum a_k z^{n_k}$ whose circle of convergence is the unit circle and the unit circle is not the natural boundary. For this theorem of Pólya a simple and elementary proof has been obtained. (Received August 1, 1944.)

217. B. M. Ingersoll: *On singularities of solutions of linear partial differential equations.* II.

In continuation of his previous investigation (see abstract 50-3-60), the author considers the problem of locating singularities of real solutions $u = \sum_{m,n=0}^{\infty} D_{mn} z^m \bar{z}^n$, $D_{mn} = \overline{D}_{nm}$, and of complex solutions $u = \sum_{m,n=0}^{\infty} A_{mn} z^m \bar{z}^n$, $z = x + iy$, $\bar{z} = x - iy$, of the equation $L(U) = \Delta U/4 + A^{(1)} U_x + A^{(2)} U_y + A^{(3)} U = 0$ where $A^{(k)} = \sum_{m,n=0}^{\infty} A_{mn}^{(k)} x^m y^n$ are entire functions of x and y. Using a result of Hadamard he determines the location of singularities of U in terms of a subsequence $\{D_{mk}\}$, k fixed, $m = 0, 1, 2, \cdots$ and a finite number of derivatives of $A^{(k)}$, $k = 1, 2, 3$. Further, employing an integral representation of Bergman for the solutions he obtains a majorant $X(z, \bar{z})$ of $|U(z, \bar{z})|$, $U(z, \bar{z}) = \mathrm{Re}[u(z, \bar{z})]$. X depends on a sequence $\{D_{mk}\}$, k fixed, $m = 0, 1, 2, \cdots$, and the upper bounds of the coefficients $A^{(k)}$ of L in any finite region of the plane. He also investigates the character of certain types of singularities of U and the rate of growth of U in the neighborhood of these points. (Received July 10, 1944.)

218. W. H. Ingram: *On the integral equations of continuous dynamical systems.*

The dynamical systems are the one-dimensional ones considered recently (Philosophical Magazine, 1940, Bull. Amer. Math. Soc. vol. 48) and are governed by the equation $\gamma(x, t): = \int_0^1 \mathcal{K}(x, \xi)\dot{\gamma}(\xi, t):d\xi$. The substitution $\gamma_i(x, t) = W_i(x)T_i(t)$ leads to the equation $\mathcal{B}\dot{T}: = T:$ having the solution $T: = e^{\mu t}C:$ with $C:$ and μ satisfying the *subsidiary modal equation* $\mu\mathcal{B}C: = C:$. The elements of the $n \times n$ matrix $\mathcal{B}$ are obtained by an iterative process. An expansion for $\mathcal{K}(x, y)$ is assumed in establishing convergence. Evidence is cited for the existence of a wide but undefined class of kernels the members of which, or their iterates, may be represented by this expansion. The method for the polynomial approximation of modal functions explained in the Philosophical Magazine and somewhat more general formulas for the algebraic determination of the modal numbers are shown to be valid in a sub-class of practical importance. If $\Phi(x):$ is any modal vector then $\mathcal{N}\Phi(x):$ is also with $\mathcal{N}$ the diagonal form of any of the solutions of $\mu\mathcal{B}N: = N:$ and with the corresponding μ of this equation the modal number belonging to $\mathcal{N}\Phi(x):$. It is found finally that the abstract theory of Hilbert space has no contact with the differential equations of dissipative elastokinetics and therefore little if any application in the present or any related problem. (Received June 6, 1944.)

219. R. L. Jeffery and D. S. Miller: *Convergence factors for generalized integrals.*

In a generalization of the derivative, J. C. Burkill (Proc. London Math. Soc. (2) vol. 34 (1932) pp. 314–322) used the Cesàro mean of a function $F(x)$ defined by

$C_r(F, x, x+h)=(r/h^r)\int_x^{x+h}(x+h-t)F(t)dt$. For the first mean of an interval (a, b), $C_1(F, a, b)=(b-a)^{-1}\int_a^b F(t)dt=(a-b)^{-1}\int_b^a F(t)dt=C_1(F, b, a)$. For $r>1$, $C_r(F, a, b)$ is in general not equal to $C_r(F, b, a)$. Consequently, the methods used for inverting derivatives of the first order could not be used for derivatives of a higher order, and this led to the following consideration. The expression $(x+h-t)^{r-1}$ serves as a convergence factor in $C_r(F, x, x+h)$. Would it be possible to replace this by a convergence factor defined in such a way that means corresponding to $C_r(F, a, b)$ and $C_r(F, b, a)$ are equal? The present paper is an answer to this question. It turns out that there is a wide class of suitable factors. The results parallel those of Burkill, but owing to the general form of the convergence factors, it was necessary to devise new methods of proof. (Received July 24, 1944.)

220. D. S. Miller: *Some properties of Carathéodory linearly measurable plane point sets.*

In 1920 H. Steinhaus (see Fund. Math. vol. 1 (1920) pp. 93–104) proved that the set of distances between the points of a measurable set along the line of positive measure fills up an interval about the origin. An application of Fubini's theorem at once establishes the same result in the plane when ordinary Lebesgue measure is used. In this paper plane sets measurable in the sense of Carathéodory are considered. It will be shown that nonzero measurability in the Carathéodory sense is not enough to assure that the set of distances between the points fills up an interval. Also some consideration will be given to the Δ-density function first invented by Besicovitch (see Math. Ann. vol. 98 (1927) pp. 422–464). It will be shown that $3^{1/2}/2\leqq\overline{\Delta}(A, p)\leqq 1$ for almost all points of the set A. Furthermore it will turn out that this is the best possible relation. (Received June 20, 1944.)

221. C. N. Moore: *Convergence factors in general analysis*, I.

In several previous papers (Proc. Nat. Acad. Sci. U.S.A. vol. 8 (1922) pp. 288–293, Trans. Amer. Math. Soc. vol. 24 (1922) pp. 79–88, vol. 25 (1923) pp. 459–468) the author has developed the basis of a branch of general analysis which would include the results common to the theory of infinite series and the theory of infinite integrals. In these papers the usefulness of the general theory was illustrated by proving a theorem in it which included as special cases the Knopp-Schnee-Ford theorem with regard to the equivalence of the Cesàro and Hölder means in summing divergent series, the analogous theorem of Landau concerning summable integrals, and a further new theorem concerning the equivalence of generalized derivatives of the Cesàro and Hölder type. In the present paper certain theorems are proved concerning the introduction of convergence factors into operations of the type $(J\eta)(\sigma)$, defined in the papers referred to above. These theorems include theorems concerning infinite series in vol. 22 of the Society's Colloquium Publications and analogous theorems concerning infinite integrals. Further theorems of the type indicated will be considered in later papers. (Received July 5, 1944.)

222. E. N. Nilson and J. L. Walsh: *On functions analytic in a region: approximation in the sense of least pth powers.*

Let R be a finite region bounded by a finite sum of Jordan curves C_1, and let S be a closed set interior to R with boundary C_0 a finite sum of Jordan curves which separates no point of $R-S$ from C_1. Let $f(z)$ be a function analytic on S but not throughout R. Let $\mathfrak{F}_M(z)$ be a function analytic in R such that the integral mean of

$|\mathfrak{F}_M(z)|^q$ $(q>0)$ on C_1, with respect to the conjugate of the harmonic function equal to zero and unity on C_0 resp. C_1, is not greater than M and such that the integral mean of $|f(z)-\mathfrak{F}_M(z)|^p$ $(p>0)$ on C_0 is least. By means of a generalization of the Hardy Mean-Value Theorem, the authors consider the asymptotic behavior of $\{\mathfrak{F}_M(z)\}$ as $M\rightarrow\infty$, extending the results of their earlier paper (Trans. Amer. Math Soc. vol. 55 (1944) pp. 53–67). These functions of best approximation are characterized for the special case of circular regions and least squares. (Received June 5, 1944.)

223. Harry Pollard: *One-sided boundedness as a condition for the unique solution of certain heat equations.*

The mathematical problem (1) $u_{xx}=u_t$ $(-\infty<x<\infty, 0<t<c)$, (2) $u(x, 0+)=f(x)$, for heat flow in an infinite rod with prescribed initial temperature, is not uniquely determined. For if $u(x, t)$ is a solution, so is $u+xt^{-3/2}\exp(-x^2/4t)$. Various additional conditions on $|u(x, t)|$ which guarantee uniqueness are known. A new, one-sided condition is suggested by the phenomenon of absolute zero. Mathematically this may be stated as: (3) $u(x, t)\geqq -M$. It is proved in this paper that solution of problem (1)-(2)-(3), if it exists, is unique. For $f(x)\equiv 0$ and $M=0$ this result is due to Widder. A result of similar character is obtained for functions harmonic in a half-plane. (Received July 6, 1944).

224. P. C. Rosenbloom: *Interpolation and extremal problems for absolutely monotonic functions.*

Let F be the class of functions $f(x)=\sum_{n=0}^{\infty}a_nx^n$ which are absolutely monotonic in the interval $0\leqq x\leqq 1$ and satisfy the conditions $f(x_i)=m_i$, $i=1, \cdots, k-1$, $f(1)\leqq m_k$, where $-1<x_1<\cdots<x_{k-1}<1$, and $x_1, \cdots, x_{k-1}, m_1, \cdots, m_k$ are given numbers. Let $L(f)=\sum_{n=0}^{\infty}c_na_n$, where $c_n\rightarrow 0$. Then $L(f)$ attains its maximum in the class F for a polynomial with at most k nonvanishing coefficients. A necessary and sufficient condition that F be non-empty is that it contain such a polynomial. As an application, the following problem is solved. Let $g(z)$ be analytic in $|z|<r_3$, and let its mean square be prescribed on the circles $|z|=r_1$ and $|z|=r_3$, $0<r_1<r_3$. For a given circle $|z|=r_2$, $r_1<r_2<r_3$, what is the greatest possible value of the mean square of $g(z)$? The analogous problem for the maximum modulus was studied by Heins in a recent paper, Trans. Amer. Math. Soc. vol. 55 (1944). Banach space methods are used in the proofs. (Received July 7, 1944.)

225. Otto Szász: *On the generalized jump of a function and Gibbs' phenomenon.*

An odd function $f(t)$ is said to have a generalized jump of index r at the origin if its Cesàro mean of order r has a limit not equal to 0, as t decreases to zero. In this paper, generalizing earlier results, formulae are given to determine the jump by means of the Fourier coefficients, when $r<3$. Furthermore in this case a Gibbs' phenomenon is established. Certain trigonometric transforms and their relation to Cesàro summability are employed. (Received June 28, 1944.)

226. W. J. Trjitzinsky: *Singular elliptic and hyperbolic partial differential equations.*

In this paper the author studies the elliptic equation $F(u)=f(x)$, where (E) $F(u)=\sum_i\sum_j\partial(A_{ij}(x)(\partial u/\partial x_i))/\partial x_j+C(x)u$ $(i, j=1, \cdots, m; x=(x_1, \cdots, x_m))$ and the related hyperbolic equation of normal type (H) $F(u)=\rho(x)\partial^2u/\partial t^2+\rho(x)f(x, t)$

$(\rho(x)>0;\ -\infty<t\leqq t_0)$. The coefficients are continuous and suitably differentiable in $x_1, \cdots, x_m$ for x in a bounded open domain D; they are allowed to become infinite in the neighborhood of the frontier $F(D)$ of D; $F(D)$ may be irregular. (E), (H) are transformed into an integral equation, whose kernel $K(x, s)$ is fundamental for our theory. When $K(x, s)$ is L_1 in a certain one of the variables, while $F(u)$ is self-adjoint, (E), (H) can be effectively studied by methods of spectral theory; when $K(x, s)$ is L_2 in (x, s), Fredholm's theory of integral equations is applicable even when $F(u)$ is not self-adjoint. In the first case it is said the problems are of type (S); in the second—of type (F). Explicit conditions on the coefficients in (E), (H) are found under which the problems are of types (S) or (F). The spectral theory then is developed, yielding various results on existence of solutions, their properties, conditions for their uniqueness, and so on. This work is related to the author's and T. Carleman's earlier works on elliptic partial differential equations. (Received July 8, 1944.)

227. H. S. Wall: *Note on the expansion of a power series into a continued fraction.*

This paper contains an algorithm for expanding a power series into a continued fraction which is based upon the fact that the process for constructing a sequence of orthogonal polynomials can be so arranged that it gives simultaneously a continued fraction expansion for a power series. (Received June 5, 1944.)

228. H. S. Wall: *The convergence of a positive definite J-fraction in the limit-circle case.*

Theorems 4.1, 4.2, 4.3 and 4.4 of the paper *Contributions to the analytic theory of continued fractions and infinite matrices* by E. D. Hellinger and H. S. Wall, Ann. of Math. vol. 44 (1943) pp. 103–127, are extended, with appropriate modifications in the series (4.7), (4.8) and in the polynomials (4.12), to general positive definite J-fractions. Thus, in the limit-circle case, a positive definite J-fraction must either converge over the whole plane to a meromorphic limit-function, or else diverge for every value of the variable. (Received July 13, 1944.)

APPLIED MATHEMATICS

229. Nathaniel Coburn: *The Kármán-Tsien pressure-volume relation in the two-dimensional supersonic flow of compressible fluids.*

First, the applicability of the Kármán-Tsien idea in the supersonic range is discussed. Secondly, it is shown that when the Kármán-Tsien relation can be used, the characteristics form a Tschebyscheff net. Further, if the diagonal curves of these characteristics are drawn so as to correspond to equi-intervaled values of the arc length parameter along these characteristics, then these diagonal curves will be the families of equipotentials and stream lines. Analytically, this last result means that the determination of the stream lines depends upon two arbitrary functions of a real variable. The conditions satisfied by these functions are discussed in the case where the given data is of Dirichlet type (two known stream lines as in the jet problem). In particular, if one of the known stream lines coincides with the x-axis, it is shown that only one arbitrary function enters into the problem of determining the stream lines. (Received June 2, 1944.)

230. A. J. Coleman: *Phase space in Eddington's theory.*

The set of 4-way symmetric unitary matrices constitutes a 10-dimensional continuum which plays a large role, under the name *phase space*, in Sir Arthur Eddington's *Relativity theory of protons and electrons*. Eddington introduces a metric into phase space and asserts that it is important for his theory that the total volume of phase space be *finite*. If one denotes the transpose of A by A', then any point, P, may be written $P=ADA'$ where D is unitary and diagonal and A is unitary and satisfies $AA'=I$. By choosing the ten parameters upon which A and D depend as coordinates in phase space, it is shown in the present paper that the volume is *infinite*. (Received July 6, 1944.)

231. A. J. Coleman: *Red-shift in the sun.*

The observed shift of the lines of the sun's spectrum, including the Limb effect, is too complex to be explained by Einstein's classic discussion which makes no assumption about the nature of the source of light. In this paper a correction of the Coulomb force due to a point charge is derived by solving Maxwell's equations in a gravitational field. For the sun, the correction is of the same order of magnitude as Einstein's but depends on the orientation of the atom in such a way as to suggest an explanation of the Limb effect. The correct choice of quantum equations for an atom in a strong gravitational field being still in doubt, it has not yet been possible to compare the consequences of the above result with experiment. (Received July 6, 1944.)

232. Eric Reissner: *On the theory of bending of elastic plates.*

In this paper a system of differential equations of the sixth order is established for the linear problem of bending of thin plates. Results obtained by the application of these equations coincide with the results of the classical theory of thin plates, except for narrow edge zones. Three boundary conditions may now be satisfied while, as is well known, the classical theory leads to two boundary conditions only. The significance of the problem has recently been discussed by J. J. Stoker (Bull. Amer. Math. Soc. vol. 48 (1942) pp. 247–261). (Received June 24, 1944.)

233. Moses Richardson: *The pressure distribution on a body in shear flow.*

If the undisturbed shear flow is given by the velocity field $v_x=U(1+ky)$, $v_y=0$, where U and k are constants, then there exists a stream function $\psi(x, y)$ satisfying a Poisson equation $\nabla^2\psi=kU$ with certain boundary conditions on the smooth contour C of the cross section of an infinite cylindrical body immersed in the flow. Instead of attacking the boundary value problem for ψ directly, an integral equation is derived for the velocity distribution $v(s)$ on C, which may be solved by approximative methods. Knowledge of $v(s)$ implies knowledge of the pressure distribution on C. (Received July 7, 1944.)

234. Andrew Vazsonyi: *An existence theorem in the theory of compressible fluids*. Preliminary report.

Let ψ be the streamfunction of a *steady*, *adiabatic*, *irrotational*, *compressible* flow, q the velocity, ρ the density, $q^2\rho^2=\psi_x^2+\psi_y^2=\tau$ and $F(\tau)=\int^\tau d\tau/\rho(\tau)$. The Euler equation of the variational problem $\iint F(\psi_x^2+\psi_y^2)dxdy=\min$ is given by $\partial(\psi_x dF/d\tau)/\partial x+\partial(\psi_y dF/d\tau)/\partial y=\partial(\rho^{-1}\psi_x)/\partial x+\partial(\rho^{-1}\psi_y)/\partial y=-\omega=0$ which shows that minimizing the integral $\iint F dx dy$ is equivalent to the condition of irrotationality. Hilbert conjectured that every regular variational problem is solvable. A. Haar showed (Math. Ann. vol. 97, p. 124) that this is indeed the case for the integral $\iint f(\psi_x, \psi_y)$. Haar's theorem cannot be applied directly for our case because ρ is a double valued function of τ. Furthermore τ has a maximum value τ_s at the speed of sound. However one can introduce a hypothetical "semi-compressible" fluid such that (a) for $\tau<\tau_s-\epsilon$, $p\rho^{-k}$ is (approximately) constant; (b) for $\tau>\tau_s-\epsilon$ the density is (approximately) constant; (c) the associated variational problem is regular. As a consequence of Haar's theorem *the interior boundary value problem is always solvable for such a semi-compressible fluid. The necessary and sufficient condition for the existence of a subsonic solution of a real compressible fluid is that for the* (*always existing*) *"semi-compressible" solution* $\tau<\tau_s$. It is believed that the theorem can be extended for the exterior problem too. (Received June 27, 1944.)

GEOMETRY

235. Herbert Busemann: *Local metric geometry.*

The author studies systematically (not necessarily symmetric) metric spaces in which extremals exist. The local properties of such spaces are investigated first, for instance, segments are constructed which may replace the line elements of differential geometry. Convergence of extremals as point sets, sets of line elements, and curves is analyzed. Then a theory of parallelism between infinite rays in unbounded spaces is developed which is new also for the differentiable case. The theory of locally isometric spaces proves to be analogous to the topological theory of covering spaces with one noteworthy exception: compact and locally isometric spaces are congruent. Finally some of the fundamental theorems of differential geometry on spaces with constant curvature are derived without any differentiability assumptions. (Received June 17, 1944.)

236. Claude Chevalley: *Intersections of algebraic and algebroid varieties.*

The paper contains a local theory of intersections for algebroid and algebraic varieties. The definition of intersection multiplicities is based on the notion of multiplicities of a local ring with respect to a system of parameters as introduced in the author's paper on *The theory of local rings*, Ann. of Math. vol. 44, no. 4. (Received July 1, 1944.)

237. John DeCicco: *Conformal maps with isothermal systems of scale curves.*

Let a surface Σ be mapped conformally upon a plane Π. The scale function $\sigma=ds/dS$ depends only upon the position of the point. A scale curve is the locus of a point along which the scale σ does not vary. Under any conformal map of Σ upon Π there are ∞^1 scale curves (whereas in the nonconformal case there are ∞^2 scale curves). Any family (isothermal or not) of ∞^1 curves may represent the scale curves of a conformal map of some surface Σ upon the plane Π. The author considers the surfaces Σ of non-constant gaussian curvature which are applicable upon a surface of revolution for which there exists a conformal map of Σ upon Π with isothermal families of scale curves. Of course, any such surface Σ may be so mapped. In that case, the scale curves must be parallel straight lines or concentric circles. It is proved that there are essentially seven types of surfaces Σ applicable upon a surface of revolution for which there exist conformal maps upon a plane Π with isothermal systems of scale curves which are neither parallel lines nor concentric circles. (Received July 5, 1944.)

238. H. W. Eves: *Applications of some new matrix products to geometry.*

A symbolic product, such as the various products of vectors and of matrices, is useful to the geometer only when the combination defining the product occurs with fair frequency in geometrical investigation, and this degree of frequency is, to the geometer, the gauge justifying the product's existence. This paper partially exploits as new symbolic products certain combinations of the elements of pairs of matrices which are quite frequent in an analytical treatment of the projective geometry of conics and in the differential invariant theory of surfaces in 3-space. Only the former applications, however, are considered in this paper. The products are closely associated with the invariant and covariant theory of conics. After briefly developing the needed algebra of the new products certain analytical criteria, in terms of the products, are obtained for some projective relationships, and several classical results are proved with these products as new tools. (Received June 10, 1944.)

239. H. W. Eves: *Skew curves setting up a null system in space.*

An n-curve is defined as a skew curve such that (A) the points of osculation of all osculating planes through a point are coplanar with the point, (B) the osculating planes of a set of coplanar points of the curve are copunctual at a point on this plane. The author develops, by vector methods, a number of projective and metrical properties of n-curves. Some necessary and sufficient conditions for a curve to be an n-curve are given, and the relation of n-curves to linear complexes is dealt with. It is shown that every n-curve satisfies a differential equation of the form $|r, x, x'| = s \cdot x'$, where r, x, s are vectors. This differential equation is integrated and special n-curves, such as the twisted cubic, the circular helix, and so on, are noted. (Received June 21, 1944.)

240. C. C. Hsiung: *A study on the theory of conjugate nets.*

Let an analytic non-ruled surface S in ordinary space be referred to a conjugate parametric net N_x, and C_λ be any curve on S through a general point P_x. As P_x moves along C_λ the parametric tangents u, v generate respectively two non-developable ruled surfaces $R^{(u)}$, $R^{(v)}$. In this paper the author first discusses the correspondence between the line joining any two points T, $\overline{T}$ on the parametric tangents u, v through P_x and the intersection of the tangent planes of $R^{(u)}$, $R^{(v)}$ at T, $\overline{T}$, respectively; and then introduces two quadrics, one has contact of the second order with $R^{(u)}$ at T and contact of the first order with $R^{(v)}$ at $\overline{T}$, and the other has similar properties with the roles of u, v interchanged. Finally, by means of the projections of the two parametric curves from any point on a line through P_x onto their osculating planes at P_x and a one-parameter family of quadrics associated with P_x of N_x obtained by the author in a previous paper (*Theory of conjugate nets on a surface*, to appear in the American Journal of Mathematics) he derives two one-parameter families of lines, of which two may be regarded as generalizations of the canonical edges of Green of the asymptotic net of a surface to a conjugate net. (Received June 8, 1944.)

241. C. C. Hsiung: *New geometrical characterizations of some conjugate nets.*

Let N_x be a conjugate net with parameters u, v on an analytic proper surface S in ordinary space, and let x_{-1}, x_1 be respectively the ray-points, or Laplace transformed points, of a general point P_x of S with respect to the v-curve and the u-curve of N_x. As u, v vary the loci of x_{-1}, x_1 are two surfaces S_{-1}, S_1 on which the parametric curves form two conjugate nets N_{-1}, N_1, which are called the minus-first and first Laplace transformed nets of N_x, respectively. The purpose of this paper is to give new geometrical characterizations of the conjugate net N_x in some special cases by introducing two quadrics, one of which has contact of the second order with S_{-1} at x_{-1} and contact of the first order with S_1 at x_1, and the other has similar properties with the roles of u, v interchanged. (Received July 20, 1944.)

242. C. C. Hsiung: *The contact of conics and quadrics with two surfaces in space of n dimensions.*

In this paper the author establishes, among other things, the following theorem: Let S, S^* be two proper analytic surfaces with general relative position in a linear space S_n of $n(>2)$ dimensions, and O_1, O_2 be two ordinary points on them respectively. Let F, F^* be respectively the sections of S, S^* made by a variable linear space S_ν of ν $(n>\nu\geqq 2)$ dimensions through a given linear space $S_{\nu-1}$ of $\nu-1$ dimensions which contains O_1, O_2. As S_ν varies through $S_{\nu-1}$ the locus of the quadric in S_ν with second order contact with F at O_1 and first order contact with F^* at O_2 is a unique quadric in S_n which has contact of the second order with S at O_1 and contact of the first order with S^* at O_2. (Received June 8, 1944.)

243. Edward Kasner: *Conformal symmetry and satellite theory for algebraic curves.*

Schwarz defined symmetry with respect to any real analytic curve, and this concept is used for analytic prolongation. The author gave the theory a purely intrinsic geometric formulation (Proceedings of the Fifth International Congress of Mathematicians, Cambridge, 1912). In the present paper he studies the case where the base curve is any algebraic curve (real or imaginary) in the complete complex (4-dimensional) plane. Symmetry is then many-valued. The image of the given curve is in general reducible and the new component is called the satellite curve. The satellite of a conic is a conic. The satellite of a potential algebraic curve is itself. Use is made of the induced satellite transformation. (Received June 19, 1944.)

244. Sister Ingonda von Mezynski: *Projective description of some plane sextic curves derived from conics as base curves.*

Following a general method described by Pettit (Tôhoku Math. J. vol. 28 (1927) pp. 72–79), corresponding rays of two pencils A and C meet a single ray of a pencil B in points of two conics M_A, M_B. The two pencils, A and C, through this correspondence, generate an octive curve. By a suitable choice of the positions of points A, B and C relative to the base conics M_A, M_B, the octic may be made to degenerate into two straight lines and a sextic. Thus 159 types of sextics with one quadruple point and three double points, 143 with one quadruple point and four double points, 32 with nine double points and 20 having ten double points were derived. (Received July 6, 1944.)

245. Peter Scherk: *On the number of certain singularities of differentiable curves of order $n+1$ in real projective n-space.*

The order of a curve in real projective n-space R_n is the maximum number of its points of intersection with a linear $(n-1)$-space. A K^{n+1} is a closed curve of order $n+1$ in R_n which satisfies certain differentiability assumptions. The singular points of a K^{n+1} can be described and provided with multiplicities. Let N_ν^n be the number of $(n-\nu)$-fold singular points and let $N_{\mu 0}^n$ be the number of osculating μ-spaces of the K^{n+1} that meet it again. In a previous paper it was proved that $\sum_0^{n-1}(n-\nu)N_\nu^n \leqq n+1$ and that the $N_{\mu 0}^n$ are finite and even bounded for a given n $(0 \leqq \mu \leqq n-2)$. Its methods would yield the estimate $\sum_0^{n-2} N_{\mu 0}^n \leqq \exp(O(n))$. In this paper, the following estimate is proved: $\sum_0^{n-1}(n-\nu)N_\nu^n + \sum_0^{n-2}(n-\mu-1)N_{\mu 0}^n \leqq n^2/4+3/2n$. (Received June 23, 1944.)

246. Y. C. Wong: *Family of totally umbilical hypersurfaces in Einstein 4-space.*

A U_3 (or $\overline{U}_3$) is a Riemannian 3-space V_3 imbeddable in an Einstein 4-space as a member of a family of totally umbilical (or totally geodesic) hypersurfaces. The purpose of this paper is to study those U_3 and $\overline{U}_3$ for which the elementary divisors of the Ricci and fundamental tensors are all simple. Let ω_1, ω_2, ω_3 be the Ricci invariants, then there are three cases. (1) $\omega_1=\omega_2=\omega_3$. Every V_3 of constant curvature is a U_3 and a $\overline{U}_3$, and conversely. (2) $\omega_1=\omega_2\neq\omega_3$. The existence of some simple equations characterizing both the U_3 and the $\overline{U}_3$ of this type proves that these U_3 and $\overline{U}_3$ are identical. These equations have an immediate geometric meaning and lead to the fundamental forms of the U_3 and $\overline{U}_3$ in certain privileged coordinates. A particular case was previously considered by Levi-Civita. (3) ω_1, ω_2, $\omega_3\neq$. Only the particular case where the ω's are all constant is considered. It is proved that no such U_3 exists and that the only $\overline{U}_3$ of this class is imbedded in an Einstein 4-space previously obtained by Kasner in another connection. A consequence of this is the non-existence of U_3 and $\overline{U}_3$ with non-null geodesic Ricci congruences. (Received July 5, 1944.)

247. Y. C. Wong: *Quasi-orthogonal ennuple of congruences in Riemannian space.*

In the investigation of those problems in a Riemannian n-space V_n with fundamental tensor g_{ij} where the principal congruences of a certain symmetric tensor h_{ij} play a part, it is customary to suppose that the elementary divisors of these two tensors are all simple, so that there exists an orthogonal ennuple of unit principal congruences and Ricci's method of the coefficients of rotation may be used. When the elementary divisors of h_{ij} and g_{ij} are not all simple, however, null principal congruences appear; for this case Ricci's method is powerless and a deadlock is reached. In this paper it is shown that this difficulty can be overcome by using a special ennuple of congruences, called a *quasi-orthogonal* ennuple, which consists partly of null and partly of non-null congruences. First, some general observations are made on this ennuple, including its use to express a symmetric tensor in normal form. Then an invariant theory of the ennuple is developed, analogous to that of Ricci's coefficients of rotation. Finally, the quasi-orthogonal ennuple in a V_3 is studied in some detail; and, as an application, a complete solution is given to the problem of finding pairs of V_3's with corresponding geodesics. (Received July 5, 1944.)

STATISTICS AND PROBABILITY

248. C. W. Churchman and Benjamin Epstein: *Statistics of sensitivity data.* II. Preliminary report.

In this paper a study is made of the distribution of the first two moments of sensitivity data as functions of the sample size. The chief results are briefly these: (a) The distributions of $\bar{x}$ and σ^2 (for definition of these functions, see *On the statistics of sensitivity data* by the authors in the Annals of Mathematical Statistics vol. 15 (1944)) approach normality rapidly as functions of the sample size; (b) $\bar{x}$ and $\sigma_{\bar{x}}^2$ are "almost" independent even for small sample sizes, thus justifying the use of Student's ratio in tests of significance for differences between two sample means. (Received July 1, 1944.)

249. E. J. Gumbel: *Ranges and midranges.*

The mth range w_m and the mth midrange v_m are defined as the difference and as the sum of the mth extreme value taken in descending magnitude ("from above") and the mth extreme value taken in ascending magnitude ("from below"). The semi-invariant generating functions $L_m(t)$ and ${}_mL(t)$ of the mth extreme values from above and from below are simple generalizations of the semi-invariant generating functions of the largest and of the smallest value which have been given by R. A. Fisher and L. H. C. Tippett. If the sample size is large enough the two mth extreme values may be considered as independent variates. Then the semi-invariant generating functions $L_w(t, m)$ and $L_v(t, m)$ of the mth range and of the mth midrange are $L_w(t, m) = L_m(t) + {}_mL(-t)$; $L_v(t, m) = L_m(t) + {}_mL(t)$. If the initial distribution is symmetrical the semi-invariant generating function of the mth range is twice the semi-invariant generating function of the mth extreme value from above. The distribution of the mth range is skew, whereas the distribution of the mth midrange is of the generalized, symmetrical, logistic type. The even semi-invariants of the mth midrange are equal to the even semi-invariants of the mth range. For increasing indices m the distributions of the mth extremes, of the mth ranges, and of the mth midranges converge toward normality. (Received July 14, 1944.)

250. P. L. Hsu: *The approximate distribution of the mean and of the variance of independent variates.*

Let X_k be mutually independent random variables with the same cumulative distribution function; let $E(X_k) = 0$, $E(X_k^2) = 1$ and $E(X_k^4) = \delta$. Finally put $S = n^{-1} \sum_{k=1}^{n} X_k$ and $\eta = n^{-1} \sum_{k=1}^{n} (X_k - S)^2$. The author first gives a new derivation of H. Cramér's well known asymptotic expansions for Pr $(n^{1/2}S \leq x)$. The proof is much more elementary and avoids in particular the use of M. Riesz' singular integrals. Instead a considerably simpler Cesàro-type kernel is used, which has first been introduced by A. C. Berry (Trans. Amer. Math. Soc. vol. 49 (1941) pp. 122–136). The same method is then used to derive similar asymptotic expansions for Pr $(n^{1/2}(\eta - 1) \leq (\delta - 1)^{1/2} x)$. The method can be extended to the case of unequal components and also for the study of other functions encountered in mathematical statistics. (Received July 3, 1944.)

251. F. E. Satterthwaite: *Error control in matrix calculation.*

The arithmetic evaluation of matrix expressions is often rather complicated. One of the causes of this is the fact that relatively minor errors (such as rounding errors) introduced in an early step may be magnified to such an extent in succeeding steps that the final result is useless. Iterative methods to meet this difficulty have been reviewed very completely by Hotelling. In this paper a different approach is taken. Conditions on the norm of a matrix are determined so that a Doolittle process will not magnify errors to more than two or three decimal places. It is then pointed out that if an approximation to the inverse of the matrix is available, most problems can be rearranged so that the required norm conditions are met. A Doolittle process may then be used to any number of decimal places with assurance that errors will not accumulate to more than a limited number of decimal places. (Received July 8, 1944.)

252. Abraham Wald: *On cumulative sums of random variables.*

Let $\{z_i\}(i=1, 2, \cdots$ ad inf.) be a sequence of independent random variables each having the same distribution. Denote by Z_j the sum of the first j elements of the sequence. Let $a>0$ and $b<0$ be two constants and denote by n the smallest integer for which either $Z_n \geqq a$ or $Z_n \leqq b$. Neglecting the quantity by which Z_n may differ from a or b (this can be done if the mean value of $|z_i|$ is small), the probability that $Z_n \geqq c$ for $c=a$ and $c=b$ is derived, and the characteristic function of n is obtained. The probability distribution of n when z_i is normally distributed is derived. These results have application to various statistical problems and to problems in molecular physics dealing with the random walk of particles in the presence of absorbing barriers. (Received July 8, 1944.)

253. Abraham Wald and Jacob Wolfowitz: *Statistical tests based on permutations of the observations.*

It was pointed out by Fisher that statistical tests of exact size, based on permutations of the observations, can be carried out without assuming anything about the underlying distributions except their continuity. Scheffé has proved that, for an important class of hypotheses, these tests are the only ones with regions of exact size. Tests based on permutations of the observations have been constructed by Fisher, Pitman, Welch, and the present authors. In the present paper, the authors prove a theorem on the limiting normality of the distribution, in the universe of permutations, of a class of linear forms. Application of this theorem gives the limiting normality (in the universe of permutations, of course) of the correlation coefficient, and of a statistic introduced by Pitman to test the difference between two means. The limiting distribution of the analysis of variance statistic in the universe of permutations is also obtained. (Received July 8, 1944.)

TOPOLOGY

254. Samuel Eilenberg and N. E. Steenrod: *Axiomatic approach to homology.*

The homology (and cohomology) groups are studied starting from a system of axioms, fulfilled by all the homology theories usually considered. It is shown that in the case of complexes, more generally, in the case of absolute neighborhood retracts, the axioms have a unique interpretation. A similar discussion is carried out for products. (Received July 7, 1944.)

255. Mariano Garcia: *Orbit-components and component orbits.*

Let $f(X)=X$ be a homeomorphism where X is compact and metric. The *orbit-*

component of a point $x\in X$ is all points y of X such that, given neighborhoods of x and y, some orbit intersects both. The homeomorphism f is *uniformly pointwise almost periodic* if f is pointwise recurrent and the gaps between iterates returning to a neighborhood are uniformly bounded for all points of X. Among other results the author proves: (1) f is uniformly pointwise almost periodic if and only if the orbit-components are identical with the orbit-closures. (2) Let f be pointwise recurrent and let $G_1, G_2, \cdots$ be a sequence of orbits or component orbits (see Bull. Amer. Math. Soc. vol. 50 (1944) p. 260) with non-empty limit inferior k and limit superior L; then (a) if M is any closed invariant set in L which intersects k, every component of L intersects M; (b) L is the closure of a component orbit. (3) If f is pointwise recurrent, (a) every orbit-component is the closure of a component orbit; (b) f is uniformly pointwise almost periodic if each component of X is contained in the orbit-closure of any of its points. (Received July 6, 1944.)

256. W. H. Gottschalk: *A note on pointwise nonwandering mappings.*

Let X be a topological space, that is, a space satisfying the first three postulates of a Hausdorff space. Let $f(X)\subset X$ be a continuous mapping and let $h(X)=X$ be a homeomorphism. In the terminology of Birkhoff, it is said that $x\in X$ is *nonwandering* (n.) under f provided that to each neighborhood U of x there correspond infintely many positive integers n such that $U\cdot f^n(U)\neq\Lambda$; also, f is *pointwise nonwandering* (p.n.) provided that each point of X is n. under f. The following theorems are proved. (1) If f is p.n., then so also is f^k for every positive integer k. (2) If A and B are closed and connected sets such that $A+B=X$ and $A\cdot B=x\in X$, if $A\cdot h(A)\neq\Lambda$ and $B\cdot h(B)\neq\Lambda$, and if x is n., then x is fixed. (3) If X is connected and if h is p.n., then each cut point of X is periodic. These results allow a weakening of hypothesis from pointwise recurrence to p.n. in several theorems of Ayres and Whyburn on the behavior of cyclic elements under a homeomorphism. (Received July 5, 1944.)

257. W. H. Gottschalk: *Totally disconnected sets and almost periodic properties.*

Let X be a metric space with metric ρ, let $f(X)\subset X$ be a continuous mapping and let $h(X)=X$ be a homeomorphism. The mapping f is said to be *pointwise recurrent* (p.r.) provided that if $x\in X$, then to each $\epsilon>0$ there corresponds a positive integer n such that $\rho(x, f^n(x))<\epsilon$. The mapping f is said to be *uniformly pointwise almost periodic* (u.p.a.p.) provided that to each $\epsilon>0$ there corresponds a positive integer N so that if $x\in X$, then in every set of N consecutive positive integers appears an integer n such that $\rho(x, f^n(x))<\epsilon$. In the usage of Whyburn, it is said that h is *pointwise regularly almost periodic* (p.r.a.p.) provided that if $x\in X$, then to each $\epsilon>0$ there corresponds a positive integer n such that $\rho(x, f^{mn}(x))<\epsilon$ $(m=1, 2, \cdots)$; also, h is *regularly almost periodic* (r.a.p.) provided that to each $\epsilon>0$ there corresponds a positive integer n such that if $x\in X$, then $\rho(x, f^{mn}(x))<\epsilon$ $(m=1, 2, \cdots)$. The following theorems are proved. (1) If X is compact and totally disconnected and if f is p.r., then f is u.p.a.p. (2) If h is p.r.a.p., then each orbit-closure is totally disconnected. (3) If X is a compact two-dimensional topological manifold (with or without boundary) and if h is r.a.p., then h is periodic. (4) If X is compact and if h is both p.r.a.p. and u.p.a.p., then h is r.a.p. (5) If X is compact and totally disconnected and if h is p.r.a.p., then h is r.a.p. (Received July 5, 1944.)

258. S. B. Myers: *Arcs and geodesics in metric spaces.*

In a metric space M, a geodesic arc is defined to be the shortest rectifiable arc joining its end points. Menger has shown that if bounded sets are compact in M, then any two points joinable by a rectifiable arc are joinable by a geodesic. In this paper it is proved that the same conclusion holds under the weaker hypotheses of local compactness and almost completeness of M; the method involves a proof of the existence of a limiting curve in the sense of Fréchet for a family of curves of bounded length in M. Applications of these results are made to the class of *geodesic metric spaces*; that is, rectifiably connected metric spaces in which distance is identical with the greatest lower bound of arc length. This class contains all convex, complete metric spaces. The class of locally compact geodesic metric spaces contains all symmetric Finsler manifolds, in particular all Riemannian manifolds. A series of results obtained by Hopf and Rinow for Riemannian manifolds are here generalized to locally compact geodesic metric spaces; for example, in a locally compact geodesic metric space completeness is equivalent to the compactness of bounded sets. (Received June 13, 1944.)

259. A. W. Tucker: *Antipodal-point theorems proved by an elementary lemma.*

Let K be the regular simplicial n-complex determined in Euclidean $n+1$-space by $\sum|x_a|=1$, $a=1, 2, \cdots, n+1$, and let K' be any simplicial subdivision of K that has central symmetry. Then, if K' is mapped simplicially into K so that antipodal vertices go into antipodal vertices, each n-simplex of K is the image of an odd number of n-simplexes of K'. This lemma can be proved by elementary means. It implies the following antipodal-point theorems which imply and are implied by the antipodal-point theorems of Lusternik-Schnirelmann and Borsuk-Ulam (Alexandroff-Hopf, *Topologie*, pp. 487, 486). 1. If non-null vectors in Euclidean $n+1$-space are distributed continuously throughout an $n+1$-disk ($\sum x_a^2 \leq 1$), there is some pair of antipodal boundary-points whose vectors are parallel. 2. The nerve of a covering of an n-sphere ($\sum x_a^2 = 1$) by $n+1$ pairs of disjoint closed sets, no one of which contains two antipodal points, is the join of $n+1$ pairs of vertices (represented by K, above). 3. A covering of an n-disk ($\sum x_i^2 \leq 1$, $i=1, 2, \cdots, n$) by $n+1$ closed sets, no one of which contains two antipodal boundary-points, has order $n+1$. 4. If an $n-1$-sphere ($\sum x_i^2=1$) is covered by $n+1$ closed sets, no one of which contains two antipodal points, the intersection of any p ($\leq n$) of the sets contains some point whose antipode belongs to the intersection of the remaining $n-p+1$ sets. (Received July 5, 1944.)

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THE COMPACTNESS OF THE RIEMANN MANIFOLD OF AN ABSTRACT FIELD OF ALGEBRAIC FUNCTIONS

OSCAR ZARISKI

1. The existence of finite resolving systems. In an earlier paper[1] we have announced the result that the existence of a resolving system of the Riemann manifold of an abstract field of algebraic functions (in any number of variables) or—what is the same—the local uniformization theorem[2] implies the existence of *finite* resolving systems of the Riemann manifold. We have proved this result for algebraic surfaces by arithmetic considerations.[1] The proof for the general case of varieties, which at that time was in our possession,[3] and which we have promised to publish in a subsequent paper, was of similar nature, that is, it was based upon considerations involving the structure of certain infinite sequences of quotient rings. However, we have succeeded lately in finding a much simpler proof which is based on topological considerations.

Let Σ be a field of algebraic functions of several variables, over an arbitrary ground field k. By the Riemann manifold M of Σ we mean the totality of places of Σ, that is, the totality of zero-dimensional valuations v of Σ/k. If V is a projective model of Σ/k, and if H is any subset of V, we denote by $N(H)$ the subset of M consisting of those valuations v which have center in H. By a resolving system of M we mean a collection $\mathfrak{B}=\{V_\alpha\}$ of projective models (finite or infinite in number) with the property that for any v in M there exists a V_α in $\mathfrak{B}$ such that the center of v on V_α is a simple point.

The topology which we introduce in M is simply this: *we choose as a basis for the closed sets of M the sets $N(W)$, where W is any algebraic subvariety of any projective model of* Σ. We prove that if topologized in this fashion, *the set M is a compact*[4] *topological space.* From this the result announced above follows immediately. For if $\{V_\alpha\}$ is a resolving system, and if we denote by S_α the singular locus of V_α, then $N(V_\alpha-S_\alpha)$ is an open set and $\{N(V_\alpha-S_\alpha)\}$ is an open covering of M.

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[1] *A simplified proof for the resolution of singularities of an algebraic surface*, Ann. of Math. vol. 43 (1942) p. 583.

[2] See loc. cit. footnote 1.

[3] That proof was presented by us at a seminar in algebraic geometry at Johns Hopkins in 1942.

[4] We use the term compact in the same sense as it is used by S. Lefschetz in his *Algebraic topology* (Amer. Math. Soc. Colloquium Publications, vol. 27, 1942). The old term is bicompact.

Hence this covering contains a finite subcovering $\{N(V_i-S_i)\}$, $i=1, 2, \cdots, m$, and this means that $\{V_1, V_2, \cdots, V_m\}$ is a finite resolving system of M.

The proof of compactness of M given in the next section is based in part on some simple algebro-geometric considerations, and in part on a theorem of Steenrod[5] on the compactness of the limit space of an inverse system of compact T_1-spaces.

2. **The Riemann manifold as the limit space of an inverse system.** Let $\mathfrak{V}=\{V_a\}$ be the collection of *all* projective models of Σ/k. By a point of V_a we mean a zero-dimensional prime ideal in a suitable coördinate ring of V_a, or, in other terms, a point is a prime one-dimensional *homogeneous* ideal in the ring of homogeneous coördinates of the general point of V_a. This defines V_a set-theoretically as a set of points. We topologize V_a by choosing as closed sets the algebraic subarieties of V_a. It is obvious that V_a then becomes a compact topological space in which points are closed sets (whence V_a is a T_1-space; however, it is not a Hausdorff space).

If V_a and V_b are two projective models of Σ/k, we denote by π_a^b the transformation of V_b onto V_a defined by the birational correspondence between V_a and V_b. We define a partial ordering $<$ of the collection $\mathfrak{V}$ as follows: $V_a < V_b$ if whenever P_a and P_b are corresponding points of V_a and V_b under π_a^b then $Q(P_a)\subseteq Q(P_b)$. Here $Q(P)$ denotes the quotient ring of P. It is clear that if $V_a < V_b$ then π_a^b is a single-valued continuous and *closed* mapping. Moreover, if V_a and V_b are arbitrary projective models of Σ/k, and if V_c denotes the join[6] of V_a and V_b, then $V_a < V_c$ and $V_b < V_c$. Hence we have here an inverse system $\{V_a;\ \pi_a^b \mid V_a < V_b\}$ of compact T_1-spaces. Let M be the limit space of the system. By Steenrod's theorem M is compact. Every point P^* of M represents an infinite collection of points $\{P_a\}$, $P_a\in V_a$, $V_a\in\mathfrak{V}$, with the property that if $V_a < V_b$ then $Q(P_a)\subseteq Q(P_b)$. We shall denote by π_a^* the mapping $P^*\rightarrow P_a$ of M into V_a. If V_a is any projective model of Σ/k and if W is any algebraic subvariety of V_a, then $(\pi_a^*)^{-1}W$ is a closed subset of M, and the closed sets obtained in this fashion form a basis for the closed subsets of M.

The compactness of the Riemann manifold of Σ/k and the implications stated in the preceding section are immediate consequences of the following theorem.

THEOREM. *There is a* (1, 1) *correspondence between the points* P^*

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[5] N. E. Steenrod, *Universal homology groups*, Amer. J. Math. vol. 58 (1936) p. 666.

[6] See our paper *Foundations of a general theory of birational correspondences*, Trans. Amer. Math. Soc. vol. 53 (1943) p. 516.

of M and the zero-dimensional valuations v of the field Σ/k. If P^* and v are corresponding elements, and if V_a is any projective model of Σ/k, then $\pi_a^* P^*$ is the center of v on V_a.

PROOF. Let v be a zero-dimensional valuation of Σ/k and let $P_{a,v}$ be the center of v on any given projective model V_a of Σ/k. For any two projective models V_a, V_b it is then true that $P_{a,v}$ and $P_{b,v}$ are corresponding points in the birational correspondence π_a^b. Hence $P_v^* = \{P_{a,v}\}$ is a point of M. *Thus every zero-dimensional valuation v determines uniquely a point P_v^* of M.*

If v_1 and v_2 are two distinct zero-dimensional valuations, then there exists at least one projective model V_a such that $P_{a,v_1} \neq P_{a,v_2}$. *Hence if $v_1 \neq v_2$ than $P^*_{v_1} \neq P^*_{v_2}$.*

Now let P^* be an arbitrary point of M, $P^* = \{P_a\}$. We denote by $\mathfrak{B}$ the least ring containing the quotient rings $Q(P_a)$. Let V_b be a fixed projective model of Σ/k and let $P_b = \pi_b^* P^*$. *We assert that if ω is a non-unit in $Q(P_b)$ then* $1/\omega \notin \mathfrak{B}$. For assume that $1/\omega \in \mathfrak{B}$. Then $1/\omega$ will belong to the ring generated by a finite number of quotient rings $Q(P_a)$, say $Q(P_{a_1})$, $Q(P_{a_2})$, $\cdots$, $Q(P_{a_m})$. Let V_c be the join of the varieties V_b, V_{a_1}, V_{a_2}, $\cdots$, V_{a_m} and let $P_c = \pi_c^* P^*$. Since $\pi_{a_i}^* P^* = P_{a_i}$ and $\pi_b^* P^* = P_b$, we have $\pi_{a_i}^c P_c = P_{a_i}$ and $\pi_b^c P_c = P_b$, and hence $Q(P_{a_i}) \subseteq Q(P_c)$, $Q(P_b) \subseteq Q(P_c)$. Therefore $1/\omega \in Q(P_c)$. This is a contradiction since any non-unit of $Q(P_b)$ is obviously also a non-unit in $Q(P_c)$.

We have therefore shown that $\mathfrak{B}$ is a proper ring (not a field). We now show that $\mathfrak{B}$ *is a valuation ring*. For this it is sufficient to show that if ξ is any element of Σ then either $\xi \in \mathfrak{B}$ or $1/\xi \in \mathfrak{B}$. We consider again a fixed projective model V_b of Σ/k. We select a system of nonhomogeneous coördinates $x_1, x_2, \cdots, x_n$ of the general point of V_b in such a fashion that the point P_b $(=\pi_b^* P^*)$ is at finite distance with respect to these coördinates. Let V_d be the projective model whose general point has as nonhomogeneous coördinates the elements $x_1, x_2, \cdots, x_n, \xi$. If the point P_d $(=\pi_d^* P^*)$ is at finite distance with respect to these coördinates, then $\xi \in Q(P_d) \subseteq \mathfrak{B}$. If P_d is a point at infinity, we observe first of all that in the above proof of our assertion $1/\omega \notin \mathfrak{B}$ we have shown incidentally the following: if V_a and V_b are *any* two projective models of Σ/k and if $\pi_a^* P^* = P_a$ and $\pi_b^* P^* = P_b$, then P_a and P_b are *corresponding* points of the birational correspondence between V_a and V_b. For on the join V_c of V_a and V_b we have the point $P_c = \pi_c^* P^*$ and the relations $Q(P_c) \supseteq Q(P_a)$, $Q(P_c) \supseteq Q(P_b)$. These relations show that if v is any zero-dimensional valuation whose center on V_c is the point P_c, then the center of v on V_a is P_a and its center on V_b is P_b. Hence P_a and P_b are indeed corresponding points.[7] With this observation in mind, let v be a zero-dimensional valuation whose center on V_b is the point P_b and whose center on V_d is the point P_d. Since P_b is at finite distance, we have $v(x_i) \geqq 0$, $i=1, 2, \cdots, n$. Since P_d is at infinity, we must have $v(\xi) < 0$. Hence $v(1/\xi) > 0$, $v(x_i/\xi) > 0$, and this shows that if we take $1/\xi, x_1/\xi, \cdots, x_n/\xi$ as nonhomogeneous coördinates of the general point of V_d, then P_d is at finite distance. Hence $1/\xi \in Q(P_d) \subseteq \mathfrak{B}$. This completes the proof of our assertion that $\mathfrak{B}$ is a valuation ring.

Let v be the valuation defined by the valuation ring $\mathfrak{B}$. We assert that *v is zero-dimensional*. For let v be of dimension s. We can find a projective model V_b on which the center of v is an s-dimensional variety W. If $P_b = \pi_b^* P^*$, then $Q(P_b) \subseteq \mathfrak{B}$ and this implies that $P_b \in W$.[8] If $s > 0$, then we can find a non-unit ω in $Q(P_b)$ such that $\omega \neq 0$ on W, whence $1/\omega \in Q(W) \subseteq \mathfrak{B}$, a contradiction. Hence $s=0$, as asserted.

The above relation $P_b \in W$ implies now $P_b = W$. This is true for any projective model V_b, that is, the center of v on any projective model V_b is the point $P_b = \pi_b^* P^*$. This completes the proof of the theorem.

3. **A generalization. Infinite direct products of projective lines.** The idea of topologizing an algebraic variety V by choosing as closed sets the algebraic subvarieties of V can be used with good effect in order to topologize the set M^* of all homomorphic mappings of any abstract field A into another abstract field K. In this general case we are dealing essentially with a generalization of the concept of the Riemann manifold of a field of algebraic functions (see the Remark at the end of the paper). We begin with some topological preliminaries.

Let $\{R_a\}$ be a system of compact topological spaces indexed by a set $A = \{a\}$. We assume that each R_a is a T_1-space; that is, that the points of R_a are closed sets. Elements of A shall be denoted by small Latin letters, $a, b, c, \cdots$; subsets of A shall be denoted by small Greek letters, $\alpha, \beta, \gamma, \cdots$. If α is a subset of A we shall denote by R_α the direct product $P_{a \in \alpha} R_a$. If $\alpha \subset \beta$ we denote by π_α^β *the projection* of R_β onto R_α. Finally, elements of R_a and R_α shall be denoted by $x_a, y_a, z_a, \cdots$ and by $x_\alpha, y_\alpha, z_\alpha, \cdots$ respectively. If $a \in \alpha$ and if $\pi_a^\alpha x_\alpha = x_a$, then x_a shall be referred to as the a-component of x_α.

We assume that for each *finite* subset α of A a topology has been assigned to R_α and that the following three conditions are satisfied: (1) *R_a is a compact topological space*; (2) *if $\alpha \subset \beta$ then π_α^β is a closed*

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[7] See our definition of corresponding points of a birational transformation, loc. cit. footnote 6, p. 505.

[8] See loc. cit. footnote 6, Theorem 3, p. 497.

mapping (mapping = single-valued continuous transformation); (3) *if α is a set with one element a then the topology assigned to R_α is exactly the topology of R_a.* It is clear that in virtue of these two conditions R_α is a T_1-space. For if x_a is the a-component of x_α, then $(\pi_a^\alpha)^{-1}x_a$ is closed and x_α is the intersection of the closed sets $(\pi_a^\alpha)^{-1}x_a$, $a \in \alpha$.

If we consider only finite subsets α of A and if we define a partial ordering in the collection $\{R_\alpha\}$ by setting $R_\alpha \prec R_\beta$ if $\alpha \subset \beta$, then we have an inverse system $\{R_\alpha; \pi_\alpha^\beta\}$. It is clear that set-theoretically the limit space R^* of the system coincides with the direct product $R^* = P_{a \in A} R_a$. However, the topology in R^* is not necessarily the usual topology of the product space, for our topology in R^* depends not only on the topology of each factor R_a but also on the topology which has been assigned to each R_α, for α any finite subset of A.

Our space R^* is compact, by Steenrod's theorem. We are dealing here with a special case of Steenrod's theorem, and the proof of the compactness of R^* can be somewhat simplified. For this reason, and also for the convenience of the reader, we shall include here a proof of the compactness of R^*.

We have to show that if a family of closed sets in R^* has the finite intersection property (that is, if every finite subfamily has a non-empty intersection), then the intersection of the entire family is non-empty. It will be sufficient to prove this for families of basic closed sets F_α^*, $F_\alpha^* = \pi_\alpha^{-1} F_\alpha$, where F_α denotes a closed set in R_α and where π_α is the projection of R^* onto R_α. Let then $\{F_\alpha^*\}$ be a family $\mathfrak{F}$ of basic closed sets which has the finite intersection property. By Zorn's lemma the family $\{F_\alpha^*\}$ is contained in a maximal family $\{G_\alpha^*\}$ of basic closed sets which has the finite intersection property. It will be sufficient to show that $\bigcap G_\alpha^*$ is non-empty. We shall therefore assume that our original family $\{F_\alpha^*\}$ is not contained properly in another family of basic closed sets which has the finite intersection property.[9]

We first observe that *the intersection of any finite collection of basic closed sets is itself a basic closed set.* For let $\{\pi_{\alpha_i}^{-1} F_{\alpha_i}\}$ be a finite collection of basic closed sets. We put $\alpha = \bigcup \alpha_i$, $F_\alpha = \bigcap (\pi_{\alpha_i}^{\alpha})^{-1} F_{\alpha_i}$. Then it is clear that $\bigcap \pi_{\alpha_i}^{-1} F_{\alpha_i} = \pi_\alpha^{-1} F_\alpha$.

In virtue of this remark and in virtue of the maximality property of the given family $\mathfrak{F}$, it follows that every finite intersection of sets in $\mathfrak{F}$ is again in the family $\mathfrak{F}$.

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[9] The idea of passing to a maximal family is taken from the proof of Tychonoff's theorem as given in Lefschetz, *Algebraic topology*, p. 19. There is only this difference: the maximal family in Lefschetz is not a family of closed sets, while ours is. This modification of the proof succeeds because we restrict ourselves to families of basic closed sets and because in our case the mappings π_α^β are closed.

For any element a in A and for any member F_α^* in $\mathfrak{F}$ let $\pi_a F_\alpha^* = F_{\alpha,a}$. If $a \notin \alpha$ then it is clear that $F_{\alpha,a} = R_a$, for then the a-component of the points of F_α^* is not restricted. If $a \in \alpha$ and if $F_\alpha^* = \pi_\alpha^{-1} F_\alpha$, then $F_{\alpha,a} = \pi_a^\alpha F_\alpha$. In either case $F_{\alpha,a}$ is a closed set in R_a, for we have assumed that π_α^β is closed whenever $\alpha \subset \beta$. For a given a the family $\mathfrak{F}_a$ of closed sets $\{F_{\alpha,a}\}$ has the finite intersection property. Since R_a is compact, the intersection $\bigcap_\alpha F_{\alpha,a}$ is non-empty. Let x_a be a point common to all the sets in $\mathfrak{F}_a$. Then $\pi_a^{-1} x_a$ is a basic closed set (since R_a is a T_1-space) which meets every set F_α^* in $\mathfrak{F}$. Consequently $\pi_a^{-1} x_a \in \mathfrak{F}$, $x_a \in \mathfrak{F}_a$, and the intersection $\bigcap_\alpha F_{\alpha,a}$ consists only of the point x_a.

Let then $x = \{x_a\}$, where $x_a = \bigcap_\alpha F_{\alpha,a}$. We show that x is a common point of the sets F_α^* in $\mathfrak{F}$. Since $\pi_a^{-1} x_a \in \mathfrak{F}$, for any a, it follows that $\bigcap_{a \in \alpha} \pi_a^{-1} x_a \in \mathfrak{F}$, that is, $\pi_\alpha^{-1} x_\alpha \in \mathfrak{F}$, where $x_\alpha = \pi_\alpha x$. Therefore $\pi_\alpha^{-1} x_\alpha$ meets F_α^*, that is, $\pi_\alpha^{-1} F_\alpha$; hence $x_\alpha \in F_\alpha$ and $x \in \pi_\alpha^{-1} x_\alpha \subset \pi_\alpha^{-1} F_\alpha = F_\alpha^*$, q.e.d.

Now let K be a fixed abstract field and let the sets R_a be projective lines over K, so that the points of each set R_a are in (1, 1) correspondence with the elements of K together with the symbol ∞. We topologize R_a by choosing as closed sets the finite subsets of R_a. Then each R_a becomes a compact topological T_1-space.

We still have to topologize each set R_α, for α a finite subset of A. For this purpose we introduce on each line R_a a pair of homogeneous coördinates x_{a1}, x_{a2} and we define an algebraic variety V_α by the following parametric equations (in which the $X_{(\epsilon)}^{(\alpha)}$ denote the homogeneous coördinates of the general point of V_α):

$$\rho \cdot X^{(\alpha)}_{\epsilon_1\epsilon_2\cdots\epsilon_n} = x_{a_1\epsilon_1} x_{a_2\epsilon_2} \cdots x_{a_n\epsilon_n}, \tag{1}$$

where $\alpha = \{a_1, a_2, \cdots, a_n\}$ and where each ϵ_j can take the values 1 or 2. It is well known that V_α is a Segre variety, of dimension n, immersed in a projective space of dimension $2^n - 1$. The points of V_α are in (1, 1) correspondence with n-tuples of ratios $\{x_{a2}/x_{a1}\}$, $a \in \alpha$, that is, with the points of the direct product $R_\alpha = R_{a_1} \times R_{a_2} \times \cdots \times R_{a_n}$. It should be noted that here we only consider points X^α whose homogeneous coördinates are in K. *We topologize V_α by choosing as closed sets the algebraic subvarieties of V_α.* Then it is clear that each V_α becomes a compact topological T_1-space.

If $\alpha = \{a_1, a_2, \cdots, a_n\}$ and if β is a subset of α, say if $\beta = \{a_1, a_2, \cdots, a_m\}$, $m < n$, then the projection π_β^α of V_α onto V_β is given by the equations:

$$X^{(\beta)}_{\epsilon_1\epsilon_2\cdots\epsilon_m} : X^{(\beta)}_{\delta_1\delta_2\cdots\delta_m} = X^{(\alpha)}_{\epsilon_1\epsilon_2\cdots\epsilon_m\gamma_{m+1}\cdots\gamma_n} : X^{(\alpha)}_{\delta_1\delta_2\cdots\delta_m\gamma_{m+1}\cdots\gamma_n},$$

where each ϵ, δ and γ can take *independently* the values 1 or 2. Thus π_β^α is a single-valued *rational* transformation of V_α onto V_β, and therefore π_β^α is closed and $(\pi_\beta^\alpha)^{-1}$ is open. It is clear that the closed sets in the infinite direct product R^*, as defined above, are the sets defined by (finite or infinite) systems of homogeneous equations, each equation involving the variables $X^{(\alpha)}$ relative to some finite subset α of A.

4. The space of homomorphic mappings of one abstract field into another. We now further specialize our application by assuming that the set A is a field. The space R^* is then the space of all single-valued transformations $x^*: a \rightarrow x_a = x_{a1}/x_{a2}$, of the field A into the set consisting of the elements of the field K and of the symbol ∞. We shall now express in an appropriate homogeneous form the conditions that a given mapping x^* be a homomorphism. Let α be a subset of A consisting of three elements, $\alpha = \{a_1, a_2, a_3\}$. On the corresponding variety V_α let F_{a_1,a_2,a_3} be the algebraic subvariety obtained by imposing on the 6 parameters x_{i1}, x_{i2}, $i = a_1, a_2, a_3$, the following condition:

$$x_{a_11}x_{a_22}x_{a_32} + x_{a_12}x_{a_21}x_{a_32} = x_{a_12}x_{a_22}x_{a_31}. \tag{2}$$

Similarly we define another algebraic subvariety G_{a_1,a_2,a_3} of V_α by the equation

$$x_{a_11}x_{a_21}x_{a_32} = x_{a_12}x_{a_22}x_{a_31}. \tag{3}$$

Let $x_{a_j1}/x_{a_j2} = x_j$, $j = 1, 2, 3$, where x_j may be ∞. Suppose that equation (2) holds true. Then if x_1 and x_2 are both different from ∞ we find $x_3 = x_1 + x_2$. If $x_1 = \infty$ and $x_2 \neq \infty$, then $x_{a_12} = 0$, $x_{a_11} \cdot x_{a_22} \neq 0$, whence (2) yields $x_{a_32} = 0$, that is, $x_3 = \infty$. Assume now that equation (3) holds. Again we find that $x_3 = x_1x_2$, if both x_1 and x_2 are different from ∞. If $x_1 = \infty$ and $x_2 \neq 0$, then $x_{a_12} = 0$, $x_{a_11} \cdot x_{a_21} \neq 0$, and (3) yields $x_{a_32} = 0$, that is, $x_3 = \infty$. Thus the equations (2) and (3) are the homogeneous counterparts of the equations $x_3 = x_1 + x_2$ and $x_3 = x_1x_2$ respectively, and they include the conventions which are usually made for the symbol ∞. We can therefore assert that x^* represents a homomorphic mapping of A into (K, ∞) if and only if the following conditions are satisfied: for any three elements a_1, a_2, a_3 of A such that respectively $a_3 = a_1 + a_2$ or $a_3 = a_1a_2$, the projection $\pi_\alpha^{-1}x^*$ (where $\alpha = \{a_1, a_2, a_3\}$) must lie respectively on F_{a_1,a_2,a_3} or on G_{a_1,a_2,a_3}. Therefore, if we denote by M the set of all *homomorphic* mappings of A into (K, ∞), we see that

$$M = \bigcap_\alpha \overset{-1}{\pi_\alpha} F_{a_1a_2a_3} \bigcap_\beta \overset{-1}{\pi_\beta} G_{b_1b_2b_3},$$

where the index α ranges over all sets $\alpha = \{a_1, a_2, a_3\}$ such that $a_3 = a_1 + a_2$, and the index β ranges over the sets $\beta = \{b_1, b_2, b_3\}$ such that $b_3 = b_1 b_2$. We see thus that M is an intersection of basic closed subsets of R^*. Hence M is closed, and since R^* is compact M is also compact.

The case which is of special interest to us is that in which K is a subfield of A. In this case we are interested in the relative homomorphisms of A into (K, ∞), that is, in the homomorphisms x^* which leave each element of K invariant. If M^* is the set of all these relative homomorphisms, then it is clear that M^* is the intersection of M with the closed set $\bigcap_{a \in K} \pi_a^{-1} a$. Here, according to our notations, $\pi_a^{-1} a$ denotes that subset of R^* which consists of the points x^* whose a-component x_a is a itself $(a \in K)$. Hence also M^* is a compact space.

It is convenient to describe in algebro-geometric terms the relative topology induced in M^* by the topology of M. Let $x_1, x_2, \cdots, x_n$ be a finite set of elements of A. For each x_i we introduce a pair of homogeneous parameters x_{i1}, x_{i2} such that $x_{i1}/x_{i2} = x_i$. We consider the algebraic variety Z over K whose general point has as homogeneous coördinates the quantities $X_{(\epsilon)}$ defined by the parametric equations

$$\rho X_{\epsilon_1 \epsilon_2 \cdots \epsilon_n} = x_{1\epsilon_1} x_{2\epsilon_2} \cdots x_{n\epsilon_n}, \tag{4}$$

where each ϵ_j can take the values 1 or 2. If the quantities x_i are algebraically independent, then the variety Z coincides with the variety V_α defined by the equations (1), α being the subset $\{x_1, x_2, \cdots, x_n\}$ of A. But in general Z is a subvariety of V_α. If $x^* \in M^*$, then the mapping x^* of A into (K, ∞) must preserve all the algebraic relations between $x_1, x_2, \cdots, x_n$ over K, since x^* is a homomorphism. It follows that the point $\pi_\alpha x^*$ of V_α must lie on Z. Now we observe that the homomorphism x^* defines a unique valuation of A/K whose residue field is K itself and whose center on Z is the point $\pi_\alpha x^*$. Conversely, every valuation of A/K whose residue field is K defines a relative homomorphic mapping of A/K onto (K, ∞). We conclude that *if W is any algebraic subvariety of Z, then the set of all valuations of A/K having K as residue field and having center on W is a closed subset of M^*.* By taking different finite subsets $\{x_1, x_2, \cdots, x_n\}$ of A and different subvarieties W of Z we obtain a family of closed subsets of M^* which form a basis for the closed subsets of M^*.

REMARK. Suppose that A is a field of algebraic functions in any number of variables, over a given ground field k. We identify the field K with the algebraically closed field determined by k. The Riemann manifold M of A is the set of all zero-dimensional valuations v

of A. By the ground field extension $k \to K$ we can embed A in a field $A' = KA$. Every relative homomorphic mapping of A' onto (K, ∞) determines uniquely a zero-dimensional valuation of A'/K, and vice versa. Every zero-dimensional valuation of A'/K induces a unique zero-dimensional valuation of A/K, but a given zero-dimensional valuation of A/K may be extendable in more than one way to a zero-dimensional valuation of A'/K. It follows that the Riemann manifold M' of A'/K coincides with the space M^* of all relative homomorphic mappings of A' onto (K, ∞) and is therefore a compact space. The Riemann manifold M of A/K is obtainable from M' by topological identification and therefore can also be converted into a compact topological space. That is precisely what we have proved in §2.

THE JOHNS HOPKINS UNIVERSITY

THE COMPLETION OF A THEOREM OF KANTOR

GERALD B. HUFF

The nature of the problem. Let a homaloidal net be defined by its order n and its multiplicities $s_1, s_2, \cdots, s_\rho$ at set P_ρ^2 of ρ general points in the plane. The positive integers $n, s_1, s_2, \cdots, s_\rho$ satisfy the equations

$$
\begin{aligned}
s_1^2 + s_2^2 + \cdots + s_\rho^2 - n^2 &= -1, \\
s_1 + s_2 + \cdots + s_\rho - 3n &= -3.
\end{aligned}
\tag{1}
$$

A planar Cremona transformation C is set up by putting this net into projective correspondence with a net of lines in another plane. If a complete and regular linear system $\Sigma_{p,d}$ of dimension d. the generic curve of the system having the genus p, has the order x_0 and the multiplicities $x_1, x_2, \cdots, x_\rho$, then $x=\{x_0; x_1, \cdots, x_\rho\}$ is called the characteristic of $\Sigma_{p,d}$. The image of $\Sigma_{p,d}$ under C is another linear system of the same p, d and a characteristic $x'=\{x_0'; x_1', \cdots, x_\rho'\}$ at the set Q_ρ^2 of the fundamental points of C^{-1}. x' is related to x by the substitution

$$
\begin{aligned}
x_0' &= nx_0 - r_1x_1 - \cdots - r_\rho x_\rho, \\
x_1' &= s_1x_0 - \alpha_{11}x_1 - \cdots - \alpha_{1\rho}x_\rho, \\
&\cdots\cdots\cdots\cdots\cdots\cdots, \\
x_\rho' &= s_\rho x_0 - \alpha_{\rho 1}x_1 - \cdots - \alpha_{\rho\rho}x_\rho.
\end{aligned}
\tag{2}
$$

The sets of numbers $\{s_i, \alpha_{i1}, \cdots, \alpha_{i\rho}\}$ are the characteristics of the principal curves of C at P_ρ^2 and satisfy the equations

$$
\begin{aligned}
\alpha_{i1}^2 + \alpha_{i2}^2 + \cdots + \alpha_{i\rho}^2 - s_i^2 &= 1, \\
\alpha_{i1} + \alpha_{i2} + \cdots + \alpha_{i\rho} - 3s_i &= -1.
\end{aligned}
\tag{3}
$$

The linear substitution (2) has as absolute invariants the forms

$$
\begin{aligned}
(xx) &\equiv x_1^2 + x_2^2 + \cdots + x_\rho^2 - x_0^2; \\
(lx) &\equiv x_1 + x_2 + \cdots + x_\rho - 3x_0.
\end{aligned}
\tag{4}
$$

The problems considered in this paper arise from the fact that the converses of two of the above statements do not hold. There are sets of positive integers satisfying equations (1) which are not associated with any homaloidal net and there are linear substitutions (2) which

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leave (xx), (lx) invariant which are not associated with any planar Cremona transformation. A solution of (1) or a linear substitution (2) which does have the geometric meaning described above will be said to be proper.

In his prize memoir [8][1] of 1884, S. Kantor stated the following proposition: *a linear substitution* (2) *leaving* (xx) *and* (lx) *invariant and in which* n, s_i, r_j, α_{ij} *are non-negative integers is proper.* He gave two "proofs" of the assertion. In 1931, Cooolidge [4] recognized the importance of the proposition and supported it with an argument like one of Kantor's. At the time Coble [1] pointed out that the proof was not valid. In 1934, this writer [5] constructed an example ($\rho=11$) which showed that the proposition was not true; and, in 1940, by using the specific results [2] of Coble on irreducible solutions of (3), it was possible to prove [6, p. 865] that Kantor's theorem was true for $\rho<11$.

The purpose of this paper is to exhibit further necessary conditions on a proper linear substitution which will also be sufficient for all values of ρ.

1. Two lemmas.

LEMMA 1. *Let* $\{p\}=\{p_0; p_1, \cdots, p_\rho\}$ *be an integer solution of equations* (3) *such that* $p_0 \geqq 0$ *and* $p_1 \geqq p_2 \geqq \cdots \geqq p_\rho$. *Then* $2p_0-p_1-p_2-p_3 \geqq 0$.

It is easily verified that for each of $p_0=0, 1$ there is a unique solution, and that each satisfies the lemma. In the case $p_0>1$, it may be shown that $p_0>p_1$. Indeed, $p_0=p_1$ requires that $p_0=0, 1$ and it is easy to show that $p_1>p_0$ is impossible for $p_0>1$. Thus for $p_0>1$ we may write

$$p_1=p_0-b_1; \qquad p_2=p_0-b_2; \qquad p_3=p_0-b_3,$$

where $0<b_1 \leqq b_2 \leqq b_3$. Substitution in the quadratic relation of (3) yields

$$p_4^2+\cdots+p_\rho^2+b_1^2+b_2^2+b_3^2+2p_0^2=1+2(b_1+b_2+b_3)p_0.$$

But

$$3 \leqq b_1^2+b_2^2+b_3^2+p_4^2+\cdots+p_\rho^2.$$

Adding these yields

$$2(p_0^2+1) \leqq 2(b_1+b_2+b_3)p_0.$$

---

[1] Numbers in brackets refer to the bibliography at the end of the paper.

Hence,

$$p_0 < b_1 + b_2 + b_3,$$

$$3p_0 - b_1 - b_2 - b_3 < 2p_0;$$

or

$$p_1 + p_2 + p_3 < 2p_0.$$

The restriction that $p_1 \geqq p_2 \geqq \cdots \geqq p_\rho$ is not essential to the proof. Indeed, it is clear that if the inequality holds for the three largest of p_i, then it must hold for any three.

LEMMA 2. *If an integer linear substitution $L(I)$ of the form* (2) *leaves* (xx), (lx) *invariant, and if the numbers* $\{n; r_j\}$ *constitute a proper solution of* (1), *then the $L(I)$ is proper.*

Since the characteristic $\{n; r_j\}$ is proper, it is associated with a Cremona transformation which has the description

$$L(C)\qquad \begin{aligned} x_0' &= nx_0 - \sigma_1 x_1 - \cdots - \sigma_\rho x_\rho,\\ x_1' &= r_1 x_1 - \beta_{11} x_1 - \cdots - \beta_{1\rho} x_\rho,\\ &\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot,\\ x_\rho' &= r_\rho x_1 - \beta_{\rho 1} x_1 - \cdots - \beta_{\rho\rho} x_\rho. \end{aligned}$$

Consider the product[2] $L(I)L(C)$ as being in the form (2) with primed coefficients. It is a linear substitution with (xx) and (lx) as invariants and hence [3]:

(i) $\{n'; s_i'\}$ and $\{n'; r_j'\}$ satisfy (1);
(ii) $\{s_i'; \alpha_{ij}'\}$ and $\{r_j'; \alpha_{ij}'\}$ satisfy (3); and
(iii) $s_i' s_k' - \alpha_{i1}'\alpha_{k1}' - \alpha_{i2}'\alpha_{k2}' - \cdots - \alpha_{i\rho}'\alpha_{k\rho}' = 0,\ i \neq k.$

Now $n' = n^2 - r_1^2 - \cdots - r_\rho^2 = 1$, and hence, from (i), $r_1' = \cdots = r_\rho' = s_1' = \cdots = s_\rho' = 0$. Since the characteristics $\{s_i'; \alpha_{ij}'\}$ $[r_j'; \alpha_{ij}']$ satisfy (3), have $s_i' = 0$ $[r_j' = 0]$ and are integral, they must be of type $\{0; 0^{\rho-1} - 1\}$. From (iii) it is clear that if $\alpha_{ij}' = -1$, and $\alpha_{kl}' = -1$, then $j \neq l$. $L(I)L(C)$ must then be the identity or a permutation P of the letters $x_1, \cdots, x_\rho$. Then $L(I) = L^{-1}(C)$ or $L(I) = PL^{-1}(C)$. In either case, $L(I)$ is proper.

2. The proof of the theorem.

THEOREM 1. *A linear substitution L of the form* (2) *is proper if and only if*:

(a) *the coefficients are integers,*
(b) (xx) *and* (lx) *are absolute invariants, and*

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[2] The convention for order of multiplication is $L(C)$ followed by $L(I)$.

(c) *p_0' has the same sign as p_0 if $\{p\}$ is a solution of* (3) *and* $L\{p\}=\{p'\}$.

The necessity of conditions (a), (b) is well known [9]. Now a geometric $L(C)$ may be expressed as the product of linear substitutions which are permutations of $x_1, \cdots, x_\rho$, and of

$$A_{123}: \quad \begin{aligned} x_0' &= 2x_0 - x_1 - x_2 - x_3, & & \\ x_i' &= x_i + (x_0 - x_1 - x_2 - x_3), & & i = 1, 2, 3, \\ x_j' &= x_j, & & j = 4, \cdots, \rho. \end{aligned}$$

By Lemma 1 it is clear that under A_{123} any solution $\{p\}$ of (3) with $p_0 \geqq 0$ goes into a $\{p'\}$ of $p_0' \geqq 0$. Since A_{123} is an involution, it must then send a $\{p\}$ of $p_0 < 0$ into a $\{p'\}$ of $p_0' < 0$. $L(C)$ must then have the same property.

The proof of the sufficiency of the conditions depends on the following theorem:

THEOREM 2 [7]. *Let $\{\gamma\}$ be an integer solution of equations* (1) *arranged so that $\gamma_1 \geqq \gamma_2 \geqq \cdots \geqq \gamma_\rho$. Further, let $\{\gamma\}$ satisfy the finite set of inequalities $p_0\gamma_0 - p_1\gamma_1 - \cdots - p_\rho\gamma_\rho \geqq 0$, where the characteristics $\{p\}$ run over the finite set of all proper solutions of* (3) *with $p_0 < \gamma_0$ and so ordered that $p_1 \geqq p_2 \geqq \cdots \geqq p_\rho$. Then $\{\gamma\}$ is the characteristic of a homaloidal net.*

The ordering of $\{\gamma\}$ and $\{p\}$ is not necessary, but is stated for emphasis.

To demonstrate that an L of the form (2) satisfying (a), (b), (c) is proper, note first that $\{n; r_1, \cdots, r_\rho\}$ is an integer solution of (1) as a consequence of (a), (b), and (i). The invariance of the sign of p_0 assures that the inequalities of Theorem 2 hold. Thus $\{n; r_1, \cdots, r_\rho\}$ is the characteristic of a homaloidal net. By Lemma 2, the linear substitution L is proper.

The restriction of integer coefficients is indeed essential. An example has been exhibited [6, p. 863] for $\rho = 9$ which satisfies all other conditions and is not an $L(C)$ since the numbers s_i, α_{ij} are rational. The proof of the sufficiency of the conditions of Theorem 1 could be proved by the method Coolidge uses; under the conditions given here, that argument is valid.

3. Some results.

COROLLARY 1. *If a solution $\{p\}$ of equations* (3) *satisfies*

$$p_0c_0 - p_1c_1 - \cdots - p_\rho c_\rho \geqq 0$$

for a single proper $\{c\}$ *which is a solution of* (1), *then it satisfies that relation for every proper* $\{c\}$.

COROLLARY 2. *Given solutions* $\{c\}$, $\{p\}$ *of* (1) *and* (3) *such that* $p_0>0$, $p_0c_0-p_1c_1-\cdots-p_\rho c_\rho<0$, *then* $\{c\}$ *is not proper.*

COROLLARY 3. *The linear substitution group generated by* A_{123} *and the permutations of* $x_1, x_2, \cdots, x_\rho$ *is completely characterized by conditions* (a), (b), (c) of Theorem 1.

The above results follow easily from Theorem 1. A similar theorem for characteristics for which x_0 never vanishes would be useful.

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SOUTHERN METHODIST UNIVERSITY

CONJUGATE NETS IN ASYMPTOTIC PARAMETERS

JANET MACDONALD

1. Introduction. Conjugate nets on an analytic nonruled surface in ordinary space may be studied either in conjugate parameters or in asymptotic parameters. The purpose of this paper is to make some contributions to the theory of conjugate nets in asymptotic parameters.

M. L. MacQueen has studied the pencil of quadrics each of which has second-order contact with a surface at a point and third-order contact with both curves of a conjugate net at the point. This pencil of quadrics is contained in the bundle of quadrics each of which has contact of at least the third order with both curves of a conjugate net at a point. The equation of this bundle of quadrics is deduced and the bundle is studied in this paper.

W. M. Davis in his Chicago doctoral dissertation, *Contributions to the theory of conjugate nets*, defined and studied several canonical configurations, considering the conjugate net as parametric. This paper presents some of Davis's results translated into asymptotic parameters.

A summary of that portion of the theory of conjugate nets in asymptotic parameters which is used in subsequent sections is given in §2. §3 deals with the bundle of quadrics mentioned above. Certain polar relations with respect to the quadrics of this bundle are presented in §4. In §5 certain loci and envelopes which arise in the study of a pencil of conjugate nets at a point on a surface are studied. §6 presents a study of Davis's canonical configurations in asymptotic parameters.

2. Analytic basis. The purpose of this section is to summarize[1] for later use portions of the theory of conjugate nets on an analytic nonruled surface referred to its asymptotic net in ordinary space.

Let the projective homogeneous coordinates $x_1, \cdots, x_4$ of a point P_x on a nonruled surface S in ordinary space be given by the parametric vector equation

$$x = x(u, v). \tag{2.1}$$

A necessary and sufficient condition that the asymptotic net be para-

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[1] E. P. Lane, *A treatise on projective differential geometry*, Chicago, 1942, pp. 89, 105–113, 115–120, 125–129, 150–152, 180–190.

metric is that the point P_x satisfy two partial differential equations of the form

$$
\begin{aligned}
x_{uu} &= px + \theta_u x_u + \beta x_v, \\
x_{vv} &= qx + \gamma x_u + \theta_v x_v,
\end{aligned}
\qquad (\theta = \log \beta\gamma). \tag{2.2}
$$

The coefficients are functions of u, v which are not arbitrary but must satisfy certain integrability conditions which need not be given here.

By means of equations (2.2) the third derivatives of x may be expressed as linear combinations of x, x_u, x_v, and x_{uv}.

A conjugate net N_λ on an integral surface of equations (2.2) can be represented by a curvilinear differential equation of the form

$$
dv^2 - \lambda^2 du^2 = 0 \qquad (\lambda \neq 0), \tag{2.3}
$$

where λ is a function of u, v. The two curves of the net N_λ through the point P_x will be denoted by C_λ and $C_{-\lambda}$ according as du/dv has value λ or $-\lambda$.

By the usual method, power-series expansions for the local coordinates of a point X sufficiently near a fixed point P_x and on the curve C_λ through the point P_x may be obtained:

$$
\begin{aligned}
x_1 &= 1 + 2^{-1}(p + q\lambda^2)\Delta u^2 + 6^{-1}[(p_u + p\theta_u) + 3(p_v + \beta q)\lambda \\
&\quad + 3(q_u + \gamma p)\lambda^2 + (q_v + q\theta_v)\lambda^3 + 3q\lambda\lambda']\Delta u^3 + \cdots, \\
x_2 &= \Delta u + 2^{-1}(\theta_u + \gamma\lambda^2)\Delta u^2 + 6^{-1}[(p + \theta_u^2 + \theta_{uu}) + 3\kappa\lambda \\
&\quad + 3\chi\lambda^2 + (\gamma_v + \gamma\theta_v)\lambda^3 + 3\gamma\lambda\lambda']\Delta u^3 + \cdots, \\
x_3 &= \lambda\Delta u + 2^{-1}(\beta + \theta_v\lambda^2 + \lambda')\Delta u^2 + 6^{-1}[(\beta_u + \beta\theta_u) + 3\pi\lambda \\
&\quad + 3\kappa\lambda^2 + (q + \theta_v^2 + \theta_{vv})\lambda^3 + 3\theta_v\lambda\lambda' + \lambda'']\Delta u^3 + \cdots, \\
x_4 &= \lambda\Delta u^2 + 6^{-1}(\beta + 3\theta_u\lambda + 3\theta_v\lambda^2 + \gamma\lambda^3 + 3\lambda')\Delta u^3 + \cdots,
\end{aligned} \tag{2.4}
$$

where

$$
\begin{aligned}
\lambda' &= \lambda_u + \lambda\lambda_v, & \kappa &= \beta\gamma + \theta_{uv}, \\
\pi &= p + \beta_v + \beta\theta_v, & \chi &= q + \gamma_u + \gamma\theta_u.
\end{aligned}
$$

The power series expansions for $C_{-\lambda}$ are obtained from these by first replacing λ' by $\lambda_u + \lambda\lambda_v$ and then changing the sign of λ.

The equation of the associate conjugate net is known to be

$$
dv^2 + \lambda^2 du^2 = 0. \tag{2.5}
$$

Power series expansions for a point X on a curve of the associate conjugate net are obtained from equations (2.4) by changing the sign of λ^2.

The equation of the pencil of conjugate nets determined by the net (2.3) is

$$dv^2 - \lambda^2 h^2 du^2 = 0, \tag{2.6}$$

where h is a constant independent of u and v.

Two types of congruences, Γ_1 and Γ_2, will be referred to in this paper. A line l_1 at a point P_x on a surface[2] may be regarded as determined by the points $x(1, 0, 0, 0)$ and $y(0, -a, -b, 1)$ and also by the planes

$$x_3 + bx_4 = 0, \qquad x_2 + ax_4 = 0. \tag{2.7}$$

If a and b are functions of u and v, the lines l_1 constitute a congruence Γ_1. The reciprocal line l_2 is the polar line of the line l_1 with respect to the quadric of Lie and may be regarded as determined by the points $\rho(-b, 1, 0, 0)$ and $\sigma(-a, 0, 1, 0)$ and also by the planes

$$x_4 = 0, \qquad x_1 + bx_2 + ax_3 = 0. \tag{2.8}$$

As in the case of a line l_1, if a and b are functions of u and v, the lines l_2 constitute a congruence Γ_2.

The axis at a point P_x of the conjugate net N_λ is a line l_1 for which a and b are denoted throughout this paper by a_a and b_a, respectively, and so are given by the formulas

$$a_a = 2^{-1}(\theta_v + \lambda_v/\lambda + \beta/\lambda^2), \quad b_a = 2^{-1}(\theta_u - \lambda_u/\lambda + \gamma\lambda^2). \tag{2.9}$$

The ray at a point P_x of the conjugate net N_λ is a line l_2 for which a and b are hereinafter denoted by a_r and b_r, respectively, and are given by the formulas

$$a_r = 2^{-1}(\theta_v + \lambda_v/\lambda - \beta/\lambda^2), \quad b_r = 2^{-1}(\theta_u - \lambda_u/\lambda - \gamma\lambda^2). \tag{2.10}$$

The principal lines of the two curves of the conjugate net N_λ at a point P_x are the associate conjugate tangents and have the equations

$$x_4 = 0, \qquad x_3^2 + \lambda^2 x_2^2 = 0. \tag{2.11}$$

The principal join is a line l_2 for which a and b are hereinafter denoted by a_j and b_j, respectively, and are given by the formulas

$$a_j = 2^{-1}(\theta_v + \lambda_v/\lambda + 5\beta/3\lambda^2), \quad b_j = 2^{-1}(\theta_u - \lambda_u/\lambda + 5\gamma\lambda^2/3). \tag{2.12}$$

The points

$$(-b_j - ia_j\lambda, 1, i\lambda, 0), \quad (-b_j + ia_j\lambda, 1, -i\lambda, 0) \tag{2.13}$$

are the principal points.

---

[2] Ibid., pp. 150–152.

The equations of the ray-point cubic at a point of the conjugate net N_λ are

$$lx_2x_3 - \beta x_2^3 - \gamma x_3^3 = 0, \qquad x_4 = 0, \tag{2.14}$$

where l is defined by

$$l = 2x_1 + (\theta_u - \lambda_u/\lambda)x_2 + (\theta_v + \lambda_v/\lambda)x_3. \tag{2.15}$$

The equations of the ray conic determined by the conjugate net N_λ at the point P_x are

$$4\beta\gamma x_2x_3 - l^2 = 0, \qquad x_4 = 0. \tag{2.16}$$

3. **Bundle of quadrics.** This section is devoted to deducing the equation of the bundle (linear two-parameter family) of quadrics each of which has contact of at least the third order with the curves C_λ and $C_{-\lambda}$ at the point P_x and to finding all the cones in the bundle.

If the general quadric, whose equation is

$$\begin{aligned} Ax_1^2 + Bx_2^2 + Cx_3^2 + Dx_4^2 + Ex_1x_2 + Fx_1x_3 + Gx_1x_4 \\ + Hx_2x_3 + Ix_2x_4 + Jx_3x_4 = 0, \end{aligned} \tag{3.1}$$

is to pass through the point $P_x(1, 0, 0, 0)$, the coefficient A must be zero. Contact of at least the third order with the curves C_λ and $C_{-\lambda}$ at the point P_x may be imposed by demanding that equation (3.1) be identically satisfied as far as the terms in Δu^3 by the power-series expansions for the curve C_λ given in equations (2.4) and also by the corresponding expansions for the curve $C_{-\lambda}$. After this has been done and certain notational changes which need not be explicitly stated here have been made, equation (3.1) reduces to

$$\begin{aligned} A(x_2x_3 - x_1x_4 - 3^{-1}\gamma\lambda^2x_2x_4 - 3^{-1}(\beta/\lambda^2)x_3x_4) \\ + B(2b_ax_3x_4 - 2\lambda^2a_ax_2x_4 - \lambda^2x_2^2 + x_3^2) + Cx_4^2 = 0. \end{aligned} \tag{3.2}$$

Thus the following theorem is proved.

THEOREM 1. *The bundle of quadrics each of which has contact of at least the third order with the curves C_λ and $C_{-\lambda}$ at the point P_x is represented by equation* (3.2).

If $B=0$, equation (3.2) reduces to the equation of a pencil of quadrics having also second-order contact with the surface S at P_x. This pencil of quadrics has been studied by MacQueen.[3]

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[3] M. L. MacQueen, *Pencils of quadrics associated with a conjugate net*, Journal of the Tennessee Academy of Science vol. 15 (1940) p. 421.

The cones in the bundle of quadrics represented by equation (3.2) are found by setting the discriminant of the left member of this equation equal to zero. The resulting equation can be reduced to

$$A^2(A^2 + 4B^2\lambda^2) = 0. \tag{3.3}$$

Three cases arise according as $A = 0$, $B \neq 0$, or $A \neq 0$, $A^2 + 4B^2\lambda^2 = 0$, or $A = 0$, $B = 0$.

In the first case, when $A = 0$, $B \neq 0$, equation (3.2) becomes

$$B(x_3^2 - \lambda^2 x_2^2 - 2\lambda^2 a_a x_2 x_4 + 2b_a x_3 x_4) + Cx_4^2 = 0. \tag{3.4}$$

This equation represents a pencil of noncomposite cones with vertex at (1, 0, 0, 0) in case, further,

$$B(\lambda^2 a_a^2 - b_a^2) + C \neq 0. \tag{3.5}$$

If the inequality (3.5) becomes an equality, equation (3.4) represents the osculating planes of the curves C_λ and $C_{-\lambda}$ at the point P_x.

In the second case when $A \neq 0$, $A^2 + 4B^2\lambda^2 = 0$, there result two pencils of cones, the equation of one of which is

$$\begin{aligned} A\,[x_2x_3 - x_1x_4 - (3^{-1}\gamma\lambda^2 + i\lambda a_a)x_2x_4 \\ - (3^{-1}\beta/\lambda^2 - (i/\lambda)b_a)x_3x_4 - 2^{-1}i\lambda x_2^2 + 2^{-1}(i/\lambda)x_3^2] + Cx_4^2 = 0. \end{aligned} \tag{3.6}$$

The equation of the other pencil of cones is obtained from equation (3.6) by replacing i by $-i$. The cones represented by equation (3.6) touch the tangent plane $x_4 = 0$ in one of the associate conjugate tangents. Every one of the cones of this pencil has its vertex at one of the principal points. The other pencil of cones has the same relation to the other associate conjugate tangent and the other principal point.

In the third case, when $A = 0$, $B = 0$, equation (3.2) reduces to $x_4^2 = 0$. Hence the tangent plane counted twice must be considered as a cone in the bundle.

The tangent plane $x_4 = 0$ intersects the most general quadric of the bundle (3.2) in two lines whose equations are

$$x_4 = 0, \qquad Bx_3^2 + Ax_2x_3 - B\lambda^2 x_2^2 = 0. \tag{3.7}$$

The two lines will be distinct in case $A^2 + 4B^2\lambda^2 \neq 0$. The two lines of each pair separate the associate conjugate tangents harmonically. If $A = 0$, $B \neq 0$, the lines in which the tangent plane intersects the quadric are the conjugate tangents. If $A \neq 0$, $B = 0$, the lines are the asymptotic tangents.

4. **Polar relations.** The purpose of this section is to discuss certain

polar relations which exist among the configurations studied in this paper with respect to certain quadrics of the bundle (3.2).

The equations of the polar line of a line l_1 with respect to the most general quadric of the bundle (3.2) are $x_4=0$ and

$$\begin{aligned} A[x_1 + (b + 3^{-1}\gamma\lambda^2)x_2 + (a + 3^{-1}\beta/\lambda^2)x_3] \\ + 2B[\lambda^2(a_a - a)x_2 - (b_a - b)x_3] = 0. \end{aligned} \tag{4.1}$$

This equation represents a pencil of lines with its center at the point

$$\begin{aligned} [\lambda^2(a - a_a)(a + 3^{-1}\beta/\lambda^2) + (b - b_a)(b + 3^{-1}\gamma\lambda^2), \\ b_a - b, \lambda^2(a_a - a), 0]. \end{aligned} \tag{4.2}$$

Thus we prove the following theorem.

THEOREM 2. *The polar lines of a line l_1 with respect to the quadrics of the bundle* (3.2) *form a pencil of lines in the tangent plane with the center at the point* (4.2).

For the axis, equation (4.1) reduces to

$$x_1 + b_1 x_2 + a_1 x_3 = 0. \tag{4.3}$$

Inspection of this equation leads to the following theorem.

THEOREM 3. *At a point P_x on the conjugate net N_λ the principal join is the polar line of the axis with respect to any quadric of the bundle* (3.2).

MacQueen proved[4] this theorem for any quadric of the pencil of quadrics studied by him.

Elimination of the parameters A and B from the equations of the polar line of a line l_2 with respect to the most general quadric of the bundle (3.2) yields the equation

$$\begin{aligned} \lambda^2 x_2^2 + x_3^2 + [\lambda^2 a_a(a - 3^{-1}\beta/\lambda^2) + b_a(b - 3^{-1}\gamma\lambda^2)]x_4^2 \\ + 2^{-1}\lambda^2(\theta_v + \lambda_v/\lambda + 3^{-1}\beta/\lambda^2 + 2a)x_2x_4 \\ + 2^{-1}(\theta_u - \lambda_u/\lambda + 3^{-1}\gamma\lambda^2 + 2b)x_3x_4 = 0. \end{aligned} \tag{4.4}$$

Thus we prove the following theorem.

THEOREM 4. *The locus of the polar line of a line l_2 with respect to the quadrics of the bundle* (3.2) *is the quadric cone represented by equation* (4.4).

It is evident that this cone is intersected by the tangent plane in the associate conjugate tangents.

---

[4] Ibid., p. 422.

For the ray, equation (4.4) becomes

$$
\begin{aligned}
\lambda^2 x_2^2 + x_3^2 + 4^{-1}[\lambda^2(\theta_v + \lambda_v/\lambda + \beta/\lambda^2)(\theta_v + \lambda_v/\lambda - 5\beta/3\lambda^2) \\
+ (\theta_u - \lambda_u/\lambda + \gamma\lambda^2)(\theta_u - \lambda_u/\lambda - 5\gamma\lambda^2/3)]x_4^2 \\
+ \lambda^2(\theta_v + \lambda_v/\lambda - 3^{-1}\beta/\lambda^2)x_2x_4 \\
+ (\theta_u - \lambda_u/\lambda - 3^{-1}\gamma\lambda^2)x_3x_4 = 0.
\end{aligned}
\tag{4.5}
$$

Davis's canonical plane is defined[5] to be the plane determined by the axis and the associate axis. The equation of this plane in asymptotic parameters is

$$
\gamma\lambda^2 x_2 - (\beta/\lambda^2)x_3 - 2^{-1}[(\beta/\lambda^2)(\theta_u - \lambda_u/\lambda) - \gamma\lambda^2(\theta_v + \lambda_v/\lambda)]x_4 = 0.
\tag{4.6}
$$

This plane intersects the cone (4.5) in the axis and in a line l_1 which is the reciprocal polar of the associate principal join. Likewise the cone which is the locus of the polar line of the associate ray with respect to the quadrics of the bundle for the associate conjugate net intersects Davis's canonical plane in the associate axis and in the reciprocal polar of the principal join.

The polar planes of a point with respect to all the quadrics of a bundle are known[6] to pass through a point. The point through which pass the polar planes of a point P_y with respect to all the quadrics of the bundle (3.2) is found by setting equal to zero the coefficients of A, B, and C in the equation of the polar plane of P_y with respect to the most general quadric of the bundle (3.2) and solving for the ratios of $x_1, \cdots, x_4$. Two cases arise according as $y_4 \neq 0$ or $y_4 = 0$.

When $y_4 \neq 0$, the resulting coordinates are given by

$$
\begin{aligned}
x_1 = \lambda^2 y_2^2 + y_3^2 - 3^{-1}(\gamma\lambda^2 b_a + \beta a_a)y_4^2 \\
+ 2^{-1}\lambda^2(\theta_v + \lambda_v/\lambda + 3^{-1}\beta/\lambda^2)y_2y_4 \\
+ 2^{-1}(\theta_u - \lambda_u/\lambda + 3^{-1}\gamma\lambda^2)y_3y_4, \\
x_2 = y_4(y_3 + b_a y_4), \quad x_3 = \lambda^2 y_4(y_2 + a_a y_4), \quad x_4 = 0.
\end{aligned}
\tag{4.7}
$$

Hence, for a point P_y not in the tangent plane the polar planes with respect to all quadrics of the bundle (3.2) pass through a point in the tangent plane with coordinates (4.7). If the point P_y lies on the axis, the point (4.7) is indeterminate and all the polar planes pass through the principal join.

---

[5] W. M. Davis, *Contributions to the theory of conjugate nets*, doctoral dissertation, Chicago, 1932, p. 18.

[6] Virgil Snyder and C. H. Sisam, *Analytic geometry of space*, New York, 1914, p. 171.

When $y_4=0$, that is, when P_y (supposed distinct from the point P_x) lies in the tangent plane, the point through which pass the polar planes of P_y with respect to all the quadrics of the bundle (3.2) is indeterminate. The polar planes form a pencil whose axis may be represented by the equations

$$\begin{aligned} y_3x_2 + y_2x_3 + (y_1 + 3^{-1}\gamma\lambda^2y_2 + 3^{-1}(\beta/\lambda^2)y_3)x_4 &= 0, \\ \lambda^2y_2x_2 - y_3x_3 + (\lambda^3a_\alpha y_2 - b_\alpha y_3)x_4 &= 0. \end{aligned} \tag{4.8}$$

If the point P_y lies on one of the associate conjugate tangents, the line (4.8) is the other associate conjugate tangent. If the point P_y coincides with P_x, the polar plane with respect to any quadric of the bundle is the tangent plane $x_4=0$.

5. **Theory of a pencil of conjugate nets.** This section deals with certain loci and envelopes which may be defined at a point of the surface in connection with a pencil of conjugate nets. The locus of the principal points, the envelope of the principal join, the envelope of the cone (4.5), the envelope of any particular one of the quadrics of the bundle (3.2), the envelope of Davis's canonical plane, the locus of Davis's canonical point, and the envelope of Davis's canonical quadric for nets of the pencil are studied.

The coordinates of one of the principal points for a general net of the pencil (2.6) are

$$\begin{aligned} x_1 = &- 2^{-1}(\theta_u - \lambda_u/\lambda + 5\gamma\lambda^2h^2/3) \\ &- 2^{-1}i\lambda h(\theta_v + \lambda_v/\lambda + 5\beta/3\lambda^2h^2), \\ x_2 = 1,& \qquad x_3 = i\lambda h, \qquad x_4 = 0. \end{aligned} \tag{5.1}$$

Homogeneous elimination of h leads to the equations

$$x_4 = 0, \qquad lx_2x_3 - (5/3)(\beta x_2^3 + \gamma x_3^3) = 0. \tag{5.2}$$

where l is defined by equation (2.15). The same equation is obtained if the other principal point is used. Equations (5.2) represent a cubic, which will be called the *principal cubic*. This result may be stated in the following theorem.

THEOREM 5. *At a point on a surface the locus of the principal points of all the curves of a pencil* (2.6) *of conjugate nets that pass through the point is the principal cubic of the pencil, represented by equations* (5.2).

Comparison of equations (5.2) with equations (2.14) shows that the principal cubic and the ray-point cubic are in the pencil of cubics, in the tangent plane, with equations $x_4=0$ and

$$lx_2x_3 - k(\beta x_2^3 + \gamma x_3^3) = 0, \tag{5.3}$$

where k is an arbitrary parameter. The principal cubic is characterized by $k = 5/3$, and the ray-point cubic by $k = 1$.

The principal join for a general net of the pencil (2.6) is represented by the equations $x_4 = 0$ and

$$\begin{aligned} x_1 &+ 2^{-1}(\theta_u - \lambda_u/\lambda + 5\gamma\lambda^2h^2/3)x_2 \\ &+ 2^{-1}(\theta_v + \lambda_v/\lambda + 5\beta/3\lambda^2h^2)x_3 = 0. \end{aligned} \tag{5.4}$$

Application of the theory of envelopes leads to the equations

$$x_4 = 0, \qquad 100\beta\gamma x_2x_3/9 - l^2 = 0. \tag{5.5}$$

This equation represents a conic, which will be called the *principal conic* of the pencil (2.6) of conjugate nets at the point P_x. This result may be stated in the following theorem.

THEOREM 6. *At a point on a surface the envelope of the principal joins of all the nets of a pencil* (2.6) *of conjugate nets is the principal conic, represented by equations* (5.5).

Comparison of equations (5.5) with equations (2.16) shows that the principal conic and the ray conic are in the pencil of conics, in the tangent plane, with equations

$$x_4 = 0, \qquad 4k\beta\gamma x_2x_3 - l^2 = 0, \tag{5.6}$$

where k is an arbitrary parameter. The principal conic is characterized by $k = 25/9$, and the ray conic by $k = 1$.

Replacing λ by λh in equation (4.5) and applying the theory of envelopes leads to the following equation of the envelope of the cone (4.5) for the pencil of conjugate nets (2.6):

$$\begin{aligned} 5\beta^2x_4^2m^3/3 + m^2n^2 + 5\gamma^2x_4^2n^3/3 + 25\beta^2\gamma^2x_4^4mn/8 & \\ - 625\beta^4\gamma^4x_4^8/768 &= 0, \end{aligned} \tag{5.7}$$

where m and n are polynomials in x_2, x_3, and x_4 of degree four. Hence the envelope of the cone (4.5) is a surface of order eight.

Application of the theory of envelopes to the equation of any particular one of the quadrics of the bundle (3.2) leads to the following equation:

$$\begin{aligned} (4A\beta x_3x_4/9)\{x_2x_4[A\gamma + 3B(\theta_v + \lambda_v/\lambda)] + 3Bx_2^2 - 3B\gamma x_3x_4\} & \\ = \{A(x_2x_3 - x_1x_4) + B[x_3^2 - \beta x_2x_4 & \\ + (\theta_u - \lambda_u/\lambda)x_3x_4] + Cx_4^2\}^2. & \end{aligned} \tag{5.8}$$

Hence, the envelope of any particular one of the quadrics of the bundle (3.2) is a quartic surface (5.8).

For a general net of the pencil (2.6) of conjugate nets the equation of Davis's canonical plane is

$$(5.9)\quad \begin{aligned}\gamma\lambda^2h^2x_2 - (\beta/\lambda^2h^2)x_3 - 2^{-1}[(\beta/\lambda^2h^2)(\theta_u - \lambda_u/\lambda)&\\ - \gamma\lambda^2h^2(\theta_v + \lambda_v/\lambda)]x_4 = 0.&\end{aligned}$$

Inspection shows that this plane passes through the cusp-axis,

$$(5.10)\quad 2x_3 + (\theta_u - \lambda_u/\lambda)x_4 = 0, \quad 2x_2 + (\theta_v + \lambda_v/\lambda)x_4 = 0.$$

Thus we prove the following theorem.

THEOREM 7. *At a point on the surface Davis's canonical planes with respect to all the nets of the pencil* (2.6) *form a pencil of planes with the cusp-axis as axis.*

For a general net of the pencil (2.6) the coordinates of Davis's canonical point,[7] the intersection of the ray and the associate ray, are given by

$$(5.11)\quad \begin{aligned} x_1 &= (2^{-1}\beta/\lambda^2h^2)(\theta_u - \lambda_u/\lambda) - 2^{-1}\gamma\lambda^2h^2(\theta_v + \lambda_v/\lambda),\\ x_2 &= -\beta/\lambda^2h^2, \qquad x_3 = \gamma\lambda^2h^2, \qquad x_4 = 0.\end{aligned}$$

Homogeneous elimination of the parameter h yields the equations of the flex-ray:

$$(5.12)\quad x_4 = 0, \quad x_1 + 2^{-1}(\theta_u - \lambda_u/\lambda)x_2 + 2^{-1}(\theta_v + \lambda_v/\lambda)x_3 = 0.$$

Thus we prove the theorem.

THEOREM 8. *At a point on a surface the locus of Davis's canonical point with respect to a pencil* (2.6) *of conjugate nets on the surface is the flex-ray of the pencil of conjugate nets.*

Davis's canonical quadric at a point P_x of a conjugate net is defined[8] as the unique quadric surface having the following properties:

1. The quadric has second-order contact with the surface sustaining the net.
2. The axis and the ray of the net at P_x are reciprocal polars with respect to the quadric.
3. The quadric passes through the point P_y which is the harmonic conjugate of P_x with respect to the focal points of the axis.

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[7] Davis, op. cit., p. 17.

[8] Ibid., p. 11.

The equation of Davis's canonical quadric at a point P_x of a general net of the pencil (2.6) is

$$
\begin{aligned}
&x_2x_3 - x_1x_4 + \gamma\lambda^2h^2x_2x_4 + (\beta/\lambda^2h^2)x_3x_4 \\
&\quad + [-2^{-1}\theta_{uv} + 4^{-1}\gamma_v\lambda^2h^2 + 4^{-1}\beta_u/\lambda^2h^2 \\
&\quad + 2^{-1}\gamma\lambda^2h^2(\theta_v + 2\lambda_v/\lambda) \\
&\quad + (2^{-1}\beta/\lambda^2h^2)(\theta_u - 2\lambda_u/\lambda)]x_4^2 = 0.
\end{aligned}
\tag{5.13}
$$

Application of the theory of envelopes yields an equation of a quartic surface as the equation of the envelope of Davis's canonical quadrics for all the nets of a pencil (2.6) of conjugate nets.

6. **Canonical configurations.** In W. M. Davis's dissertation, *Contributions to the theory of conjugate nets*, the conjugate net is considered as parametric. The purpose of this section is to exhibit some of Davis's results in asymptotic parameters and to show some relationships that exist among the configurations studied by him. The canonical configurations referred to are those defined[9] by Davis.

The equation of Davis's canonical quadric for the net N_λ may be obtained from equation (5.13) by setting $h=1$. The following equation of his associate canonical quadric may be obtained from the same equation (5.13) by setting $h=1$ and replacing λ^2 by $-\lambda^2$:

$$
\begin{aligned}
&x_2x_3 - x_1x_4 - \gamma\lambda^2x_2x_4 - (\beta/\lambda^2)x_3x_4 \\
&\quad + [-2^{-1}\theta_{uv} - 4^{-1}\gamma_v\lambda^2 - 4^{-1}\beta_u/\lambda^2 - 2^{-1}\gamma\lambda^2(\theta_v + 2\lambda_v/\lambda) \\
&\qquad - (2^{-1}\beta/\lambda^2)(\theta_u - 2\lambda_u/\lambda)]x_4^2 = 0.
\end{aligned}
\tag{6.1}
$$

If the equations of Davis's canonical quadric and associate canonical quadric are added, and if the result is divided by 2, the equation of the quadric of Wilczynski is obtained:

$$
x_2x_3 - x_1x_4 - 2^{-1}\theta_{uv}x_4^2 = 0. \tag{6.2}
$$

Hence the quadric of Wilczynski belongs to the pencil of quadrics determined by Davis's canonical quadric and associate canonical quadric.

The coordinates of Davis's canonical point for the net N_λ are given by

$$
\begin{aligned}
&x_1 = (2^{-1}\beta/\lambda^2)(\theta_u - \lambda_u/\lambda) - 2^{-1}\gamma\lambda^2(\theta_v + \lambda_v/\lambda), \\
&x_2 = -\beta/\lambda^2, \qquad x_3 = \gamma\lambda^2, \qquad x_4 = 0.
\end{aligned}
\tag{6.3}
$$

A little computation shows that the canonical point (6.3) and the canonical plane (4.6) are pole and polar with respect to both the canonical quadric and the associate canonical quadric.

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[9] Ibid., pp. 17–25.

The pencil of lines in the canonical plane with center at P_x is defined by Davis to be the first canonical pencil of the net, and the pencil of lines in the tangent plane at P_x with center at the canonical point is defined to be the second canonical pencil of the net. The flex-ray, represented by equations (5.12), belongs to the second canonical pencil.

The intersection of the tangent plane with the canonical plane is defined by Davis to be the first canonical tangent. The equations of the first canonical tangent in asymptotic parameters are $x_4=0$ and

$$\gamma\lambda^2 x_2 - (\beta/\lambda^2)x_3 = 0. \tag{6.4}$$

The line joining the canonical point to the point P_x is defined by Davis to be the second canonical tangent. The equations of this line are $x_4=0$ and

$$\gamma\lambda^2 x_2 + (\beta/\lambda^2)x_3 = 0. \tag{6.5}$$

Davis's thesis includes a study of congruences of pairs of lines which are polar lines with respect to his canonical quadric at a point of a conjugate net of a surface, when one of the lines, l_1, passes through P_x but does not lie in the tangent plane, and the other line, l_2, lies in the tangent plane but does not pass through P_x. Only such reciprocal polar lines as belong to Davis's canonical pencils are considered.

As in §2, a line l_1 may be represented by equations

$$x_2 + ax_4 = 0, \qquad x_3 + bx_4 = 0. \tag{6.6}$$

The line l_2 which is the reciprocal polar of l_1 with respect to Davis's canonical quadric may be represented by equations $x_4=0$ and

$$x_1 + (b - \gamma\lambda^2)x_2 + (a - \beta/\lambda^2)x_3 = 0. \tag{6.7}$$

The lines l_1 and l_2 will be lines of the first and the second canonical pencils (in the sense of Davis), respectively, if

$$a = 2^{-1}(\theta_v + \lambda_v/\lambda + h\beta/\lambda^2), \quad b = 2^{-1}(\theta_u - \lambda_u/\lambda + h\gamma\lambda^2), \tag{6.8}$$

where h is a parameter.

If $h=1$, then $a=a_a$, $b=b_a$, and $a-\beta/\lambda^2=a_r$, $b-\gamma\lambda^2=b_r$. Hence, for $h=1$, the line l_1 is the axis and l_2 the ray of the conjugate net N_λ. If $h=-1$, then l_1 is the associate axis. If $h=3$, then l_2 is the associate ray. If $h=2$, then l_2 is the flex-ray. If $h=0$, then l_1 is the cusp-axis. If $h=\infty$, the lines l_1 and l_2 are Davis's first and second canonical tangents, respectively. Any pair of lines l_1 and l_2 given by the same value of h in (6.8) are reciprocal polars with respect to Davis's canonical quadric. It is seen that the associate axis and the associate ray are not corresponding lines. Likewise, the flex-ray and the cusp-axis are not corresponding lines.

Davis defines at each point of a surface three tangents analogous to the tangents of Darboux. All the quadric surfaces represented by the equation

$$x_2x_3 - x_1x_4 + \gamma\lambda^2 x_2x_4 + (\beta/\lambda^2)x_3x_4 + k_4x_4^2 = 0 \tag{6.9}$$

have second order contact with the surface at P_x and intersect the surface in a curve with a triple point at P_x. The triple point tangents of this curve are defined to be *the triple tangents* of the conjugate net at the point P_x. They can be shown to intersect the ray-point cubic (2.14) in three collinear points lying on a line in Davis's second canonical pencil, defined by Davis to be *the line of collineation* of the net N_λ. The associate line of collineation is the reciprocal polar of the associate axis with respect to Davis's canonical quadric.

In addition to the lines already mentioned, the principal join and the associate principal join are also members of Davis's second canonical pencil, the principal join being characterized by $h = 11/3$, and the associate principal join by $h = 1/3$.

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CONTRACTIONS IN NON-EUCLIDEAN SPACES

F. A. VALENTINE

The existence of an extension of the range of definition of a function $f(x)$ defined on a set S of a metric space M to a metric space M' so as to preserve a contraction of the type

$$\|f(x_1), f(x_2)\|' \leqq \|x_1, x_2\| \tag{1}$$

depends upon M and M'. The author has previously shown [3, 4][1] that for $M = M'$ the extension exists when M is: (1) the n-dimensional Euclidean space; (2) the surface of the n-dimensional Euclidean sphere; (3) the general Hilbert space. In this brief article *the extension is shown to exist when each M and M' is the n-dimensional hyperbolic space.* The method used to prove this result is applied to a metric space which includes both the hemispherical and hyperbolic cases. Hence a unification of results is also obtained.

As shown in the previous papers [3, 4] a necessary and sufficient condition for a contraction to be extensible in M and M' is the property E, which is restated as follows.

PROPERTY E. *Consider in each of the metric spaces M and M' a set of spheres, such that to each sphere $S_i \in M$, having center x_i and radius r_i, there corresponds a sphere $S_i' \in M'$, having center x_i' and radius r_i'. Furthermore suppose that*

$$\begin{aligned} r_i &= r_i', \\ \|x_i', x_j'\|' &\leqq \|x_i, x_j\| \end{aligned} \tag{2}$$

for all corresponding spheres S_i and S_i', and for all corresponding pairs (S_i, S_j) and (S_i', S_j').

The spaces M and M' are said to have the extensibility property E *if conditions* (2) *and*

$$\prod_i S_i \neq 0 \tag{3}$$

imply that

$$\prod_i S_i' \neq 0. \tag{4}$$

If the above statement holds for $M = M'$, the space M is said to have property E.

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[1] Numbers in brackets refer to references at the end of the paper.

For convenience of discussion let M be an n-dimensional metric space which can be imbedded in an $(n+1)$-dimensional Euclidean space $\mathfrak{R}$. Let R_i be the Euclidean vector emanating from the origin of $\mathfrak{R}$ to the point $x_i \in M$. It is assumed that there exists a symmetric real-valued bilinear product $R_i \cdot R_j$ such that $R_i \cdot R_i = k^2 = \text{const.}$ defines the metric space M. Suppose that for any two points x_i and x_j in M, R_i and R_j determine a Euclidean plane which intersects M in a *unique* continuous curve joining x_i and x_j. This curve is defined to be a geodesic. Furthermore suppose the distance $\|x_i, x_j\|$ in M is defined to be

$$\|x_i, x_j\| = F(R_i \cdot R_j) \geqq 0, \tag{5}$$

where $F(u)$ is either a single-valued *increasing* function of u or a single-valued *decreasing* function of u.

THEOREM 1. *If the n-dimensional metric space M has the above properties, it possesses the property* E.

PROOF. To prove this we consider the case $F(u)$ is an *increasing* function of u. When $F(u)$ is decreasing the proof is obtained by a uniform change in the direction of the inequality signs. On account of a theorem of Helly[2] type [2, 1], to prove Theorem 1 it is sufficient to establish property E for $i = 1, \cdots, n+1$.

Let $\Delta(x_1, \cdots, x_{n+1})$ be the simplex (degenerate or nondegenerate) in M determined by the points x_i $(i = 1, \cdots, n+1)$. Condition (4) implies that $\Delta(x_{i_1}, x_{i_2}) \cdot S_{i_1} \cdot S_{i_2} \neq 0$. If we have $\Delta(x_{i_1}, \cdots, x_{i_{r+1}}) \cdot \prod_{j=i_1}^{i_r} S_j \neq 0$ $(i = 1, \cdots, n+1;\ 2 \leqq r \leqq n)$, then since by (4) $\prod_{j=i_1}^{i_{r+1}} S_j \neq 0$, the theorem of Helly[3] type implies in the r-dimensional subspace that $\Delta(x_{i_1}, \cdots, x_{i_{r+1}}) \cdot \prod_{j=i_1}^{i_{r+1}} S_j \neq 0$. Hence

$$\Delta(x_1, \cdots, x_{n+1}) \cdot \prod_{i=1}^{n+1} S_i \neq 0 \tag{6}$$

is established by induction. *Suppose* that $\Delta(x_1', \cdots, x_{n+1}')$ is *not covered* by the spheres S_i'. Then choose x and x' so that

$$x \in \Delta(x_1, \cdots, x_{n+1}) \cdot \prod_{i=1}^{n+1} S_i, \quad x' \in \Delta(x_1', \cdots, x_{n+1}') - \sum_{i=1}^{n+1} S_i', \tag{7}$$

and let R and R' be the Euclidean vectors emanating from the origin of $\mathfrak{R}$ to the points x and x' respectively. Conditions (2) and (5) yield the results

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[2] The theorem states: *If each n+1 sets of a family of closed bounded, convex sets of the n-dimensional Euclidean space intersect, then there is a point common to all the sets.* For a more general topological theorem of the same type, see Alexandroff and Hopf [1, p. 297].

[3] Loc. cit.

$$R_i \cdot R_j \geqq R_i' \cdot R_j' \qquad (i, j = 1, \cdots, n+1). \tag{8}$$

Also conditions (7) and the first of conditions (2) imply that $\|x_i', x'\| > \|x_i, x\|$. Hence by (5) we have

$$R' \cdot R_i' > R \cdot R_i. \tag{9}$$

Since the line of shortest length joining x_i and x_j lies in the plane determined by R_i and R_j, the simplex $\Delta(x_1, \cdots, x_{n+1})$ is contained in the smaller solid angle α determined by $R_1, \cdots, R_{n+1}$. Hence condition (7) implies that R lies inside the solid angle α. A corresponding statement with primes holds for R'. *Hence there exist* real constants a_i and a_i' such that

$$a_i \geqq 0, \qquad a_i' \geqq 0, \qquad \sum_{i=1}^{n+1} a_i \neq 0, \qquad \sum_{i=1}^{n+1} a_i' \neq 0,$$

and such that

$$R = a_i R_i, \qquad R' = a_i' R_i' \qquad (i \text{ summed}). \tag{10}$$

Multiplying (8) by $a_i a_j'$, summing on i and j, one obtains

$$(a_i R_i) \cdot (a_j' R_j) \geqq (a_i R_i') \cdot (a_j' R_j'),$$

whence by (10)

$$R \cdot (a_j' R_j) \geqq (a_i R_i') \cdot R'. \tag{11}$$

Similarly multiplying (9) by a_i, summing on i, we get

$$R' \cdot (a_i R_i') > R \cdot R. \tag{12}$$

Conditions (11) and (12) imply that

$$R \cdot (a_j' R_j) > R \cdot R. \tag{13}$$

However multiplying (9) by a_i', we get

$$R' \cdot R' = R' \cdot (a_i' R_i') > R \cdot (a_i' R_i). \tag{14}$$

Since $R \cdot R = R' \cdot R' = k^2$, conditions (13) and (14) are contradictory. Hence the assumption that $\Delta(x_1', \cdots, x_{n+1}')$ is not covered by the spheres S_i' is *false*. Since $\Delta(x_1', \cdots, x_{n+1}') \cdot S_i' \cdot S_j' \neq 0$, and since Δ is covered by the spheres S_i', a theorem[4] of Knaster, Kuratowski and

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[4] See Alexandroff and Hopf [1, p. 377]. The theorem states: *If the closed sets A_i cover the simplex T, and if each side $a_{i_1} \cdots a_{i_r}$ of T is such that $a_{i_1} \cdots a_{i_r} \subset A_1 + \cdots + A_{i_r}$, then* $A_1 \cdot A_2 \cdot \cdots \cdot A_{n+1} \neq 0$.

Mazurkiewicz implies by induction that $\prod_{i=1}^{n+1} S_i' \neq 0$. Since condition (4) now holds for each set of $n+1$ of the spheres S_i', the theorem of Helly[5] type implies that (4) holds for all the spheres S_i'.

We now readily prove the following corollary.

COROLLARY 1. *The property* E *holds for the n-dimensional hyperbolic space.*

For the hyperbolic space M this corollary is an immediate consequence of the fact that M can be defined as the points $(x_1, x_2, \cdots, x_{n+1})$ in the $(n+1)$-dimensional Euclidean space which are on one sheet of the hyperboloid[6]

$$k^2 x_1^2 - x_2^2 - x_3^2 - \cdots - x_{n+1}^2 = k^2.$$

Here $R_i \cdot R_j$ is defined to be the bilinear form

$$R_i \cdot R_j \equiv k^2 x_{i1} x_{j1} - x_{i2} x_{j2} - \cdots - x_{in+1} x_{jn+1},$$

and $F(u) = k \cosh^{-1}(u/k^2)$. These have the properties required for the proof of Theorem 1. A similar argument holds for the open hemispherical case.

The extensibility of $f(x)$ to the whole space M so as to preserve condition (1) now follows as developed in the previous work of the author [3, pp. 105–106].

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[5] Loc. cit.

[6] Coxeter [5, pp. 209, 248].

A GENERALIZATION OF MOORE'S THEOREM ON SIMPLE TRIODS

GAIL S. YOUNG, JR.

R. L. Moore has proved that the plane does not contain uncountably many mutually exclusive simple triods.[1] The generalization to spaces of higher dimension given below appears to have escaped notice, and to be of some interest.

DEFINITION. *If n is a non-negative integer, by a T_n-set we shall mean a continuum which is the sum of an n-cell, g, and an arc, t, such that $g \cdot t$ is a point which is an end point of t and a relatively interior point of g. The point $g \cdot t$ will be called a junction point.*

Obviously, a T_1-set is a simple triod.

THEOREM. *Euclidean n-space does not contain uncountably many mutually exclusive T_{n-1}-sets.*

PROOF. Supose that the theorem is false. Then for some positive number ϵ there exists an uncountable collection, G, of mutually exclusive T_{n-1}-sets such that the junction point of each is at distance greater than ϵ from the boundary of its $(n-1)$-cell. There is a point, P, which is a point of condensation of the set of all junction points of elements of G; let U be a spherical domain of n-space with center P and radius less than $\epsilon/2$. If the point X of U is a junction point of an element T of G, let $g(X)$ denote the component that contains X of the intersection of U and the $(n-1)$-cell of T. It is an easy consequence of the Alexander duality theorem that $g(X)$ separates U. Hence the collection, G', of all sets $g(X)$ is an uncountable collection of cuttings of U, and it is clearly non-separated. By a theorem due to Whyburn,[2] G' contains an uncountable saturated subcollection, G''. But if $g(X)$ is an element of G'', U contains an arc, t, which is in the T_{n-1}-set of G that contains X and which has only X in common with $g(X)$. Since the elements of G are mutually exclusive, no element of G'' separates a point of t from $g(X)$ in U, which is a contradiction.

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[1] In fact, Moore has proved that the plane does not contain uncountably many mutually exclusive triodic continua. See *Concerning triods in the plane and the junction points of plane continua*, Proc. Nat. Acad. Sci. U.S.A. vol. 14 (1928) pp. 85–88, and *Concerning triodic continua in the plane*, Fund. Math. vol. 13 (1929) pp. 261–263.

[2] Theorem 2.2 of chap. 3 of his book, *Analytic topology*, Amer. Math. Soc. Colloquium Publications, vol. 28.

ON ROTATION GROUPS OF PLANE CONTINUOUS CURVES UNDER POINTWISE PERIODIC HOMEOMORPHISMS

DICK WICK HALL

In this paper we make use of the work of G. T. Whyburn[1] on light interior transformations and on orbit decompositions of certain spaces to obtain a theorem by means of which a certain subset of the orbits of points under a periodic transformation $T(M)=M$ may be given a linear ordering. This theorem is then used to obtain an accessibility theorem for plane continuous curves similar to one previously published by L. Whyburn.[2] We take this opportunity to express our indebtedness to G. E. Schweigert for suggesting the proof of Theorem I given below and thus eliminating the longer and less interesting proof previously obtained by the author. For any $x \in M$, the orbit of x under T means $O(x)=\sum_{i=-\infty}^{\infty} T^i(x)$.

THEOREM I. *Let M be a locally connected continuum (that is, a continuous curve) and $T(M)=M$ an arbitrary periodic homeomorphism. Then if a and b are arbitrary points of M lying in different orbits under T and if axb is any simple arc in M joining a and b, then there must exist a simple arc $a'x'b'$ in M lying in the orbit of axb under T such that a' belongs to $O(a)$, b' belongs to $O(b)$ and no two points of $a'x'b'$ lie in the same orbit under T. Furthermore, the point a' may be any arbitrary preassigned point of the orbit of a.*

Proof (Schweigert). Let M' be the hyperspace obtained by decomposing the space M into its orbits under T. Then, since the orbit decomposition is continuous,[3] it follows[4] that there exists a light interior transformation $f(M)=M'$, namely, the transformation given by and associated with the orbit decomposition. Let axb be the given arc in M. Then we may assume without loss of generality that axb has precisely the point a in common with $O(a)$ and precisely the point b in common with $O(b)$. Define $K=f(axb)$. Then K is a locally connected continuum containing $c=f(a)$ and $d=f(b)$. Let cyd be an arc in K joining c to d. Now let a' be an arbitrary point of $O(a)$. Then[5]

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[1] See G. T. Whyburn, *Analytic topology*, Amer. Math. Soc. Colloquium Publications, vol. 28, 1942, pp. 182–189 and 239–262.

[2] See L. Whyburn, *Rotation groups about a set of fixed points*, Fund. Math. vol. 28 (1937) pp. 124–130, in particular p. 127.

[3] See G. T. Whyburn, loc. cit. p. 258.

[4] See G. T. Whyburn, loc. cit. p. 130.

[5] See G. T. Whyburn, loc. cit. p. 186.

there must exist a simple arc $a'x'b'$ in M such that $f(a'x'b')=cyd$ is topological. By definition of f we see that each point of $a'x'b'$ belongs to the orbit of some point of axb, and from the one-to-oneness of this transformation it is immediate that no two points of $a'x'b'$ lie in the same orbit under this transformation. This completes the proof.

COROLLARY. *The same conclusion holds for any pointwise periodic* $T(M)=M$ *if we impose the additional restriction either that* T *have equicontinuous powers*[6] *or that the period function remain bounded on the arc* axb.

The accessibility theorem for plane continuous curves mentioned in the introductory paragraph of this paper may be stated as follows.

THEOREM A (L. WHYBURN). *If* M *is a plane continuous curve and* $T(M)=M$ *is a homeomorphism and if* C *is an element of a rotation group of* M *under* T *of order at least two, then* C *has property* S.[7]

The object of our second theorem is to obtain a result similar to Theorem A, but with the emphasis in the hypothesis placed upon the type of the transformation T rather than upon the order of the rotation group under consideration. Before stating the theorem we recall certain important subsets of M. By L we denote the closed invariant subset of M consisting of those points at which the period function has an unbounded limit superior. By K we denote the collection of all fixed points of M under T. If R is a component of $M-K$, then R is an element of a rotation group under T; this rotation group consists exactly of the orbit of R under T; and its order is the number of components which it contains. The order of a rotation group under T may, of course, be either finite or infinite.

We are now in a position to state our second theorem.

THEOREM II. *Let* M *be a plane continuous curve and* $T(M)=M$ *an arbitrary homeomorphism, while* R *denotes an element of some rotation group of* M *under* T. *Then*

(a) *If* K *is locally connected then* R *has property* S *and every point of* $F(R)=\bar{R}-R$ *is regularly accessible*[8] *from* R.

(b) *If* T *is pointwise periodic and has equicontinuous powers we get the same conclusion as in* (a).

(c) *If* T *is pointwise periodic then every point of* $F(R)$ *which is not a point of* L *is regularly accessible from* R.

Proof. Note that (a) is immediate from a theorem of G. T. Whyburn,[9] and that it also follows at once from the proof of Theorem A.

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[6] See G. T. Whuburn, loc. cit. p. 258.

[7] See G. T. Whyburn, loc. cit. p. 20.

[8] See G. T. Whyburn, loc. cit. p. 111.

We give the proofs of (b) and (c) simultaneously, making use of the corollary to Theorem I in each case. It is to be noted that our proof is similar to the original proof of Theorem A.

Denote by d a positive number exceeding the diameter of the set M, and suppose that M is embedded in the upper half of the Euclidean plane. If the theorem be false there must exist a point p in $F(R)$ which is not regularly accessible from R and (unless T has equicontinuous powers) such that p does not lie in L. This means that there exists a positive number ϵ and a sequence of points $\{p_i\}$ of R converging to p such that no two points of this sequence may be joined in R by a connected set of diameter less than 17ϵ. Let C_p be a circle of radius 8ϵ having its center at the point p. No generality is lost by the following assertion:

(1) *For every i the set $O(p_i)$ lies within C_p; no two of the points p_i may be joined by a connected subset of R lying within this circle; and if T does not have equicontinuous powers then there exists an integer N such that no point of M lying within C_p has period greater than N under T.*

For some point q' of R exterior to C_p we construct arcs p_iq' in R for every integer i and we denote the first intersection of the arc p_iq' with the circle C_p by q_i. It follows from (1) that the arcs p_iq_i are pairwise disjoint and we may assume, exactly as in the proof of Theorem A, that this sequence of arcs converges to a limiting set H which is a subcontinuum of K. Making use of the corollary to Theorem I we can insure that no arc p_iq_i meets the orbit of any point of M in more than a single point, and that no two of these arcs meet the orbit of the same point of M. This means, in particular, that no two consecutive images under T of any one of these arcs will have a point in common. We may assume that the sequence $\{q_i\}$ converges monotonically on C_p to a point q.

We place the x-axis in such a position that p lies at the point $(-4\epsilon, 2d)$ and q at the point $(4\epsilon, 2d)$. By L_i $(i=\pm 1, \pm 2, \pm 3)$ we denote the line segment joining $(i\epsilon, 0)$ to $(i\epsilon, 4d)$, and by $D_{i,j}$ the interior of the rectangle formed by L_i, L_j, $y=0$, and $y=4d$.

Using the fact that H is a subset of K and either (b) or (c)[10] we may make the following assumption without loss of generality:

(2) *If x is any point of any arc p_iq_i then $O(x)$ lies within a circle having its center at some point of H and diameter $\epsilon/2$.*

Let r_is_i be a subarc of p_iq_i with its interior in $D_{-3,3}$ and its end points on L_{-3} and L_3; $r_i' s_i'$ the arc $T(r_i s_i)$; $x_i' y_i'$ a subarc of $r_i' s_i'$ with its interior in $D_{-2,2}$ and its end points on L_{-2} and L_2; and $x_i y_i$ the arc $T^{-1}(x_i' y_i')$. The existence of the arc $x_i' y_i'$ follows from (2) and by the same token we see that $x_i y_i$ is a subarc of $r_i s_i$ having its end points in the respective regions $D_{-3,-1}$, $D_{1,3}$. Now the sequence of arcs $\{x_i' y_i'\}$ may be assumed to converge to a subcontinuum H' of H which is, of course, disjoint with every one of these arcs. This means that by taking a subsequence the following assumption will hold.

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[9] See G. T. Whyburn, *Concerning the open subsets of a plane continuous curve*, Proc. Nat. Acad. Sci. U.S.A. vol. 13 (1927) pp. 650–657, in particular Theorems 1 and 5 of this paper.

[10] See G. T. Whyburn, *Analytic topology*, loc. cit. p. 252.

(3) *For any fixed integer i and every k exceeding i the arc $x_k' y_k'$ separates $D_{-2,2}$ between $x_i' y_i'$ and $x_n' y_n'$ for every n greater than k. Thus $x_k' y_k'$ separates $D_{-2,2}$ between $x_i' y_i'$ and H' for every k exceeding i.*

For any fixed value of i we know that the closed sets $O(x_i y_i)$ and H' are disjoint, which means that there will exist a region U_i in the plane containing H' but disjoint with $O(x_i y_i)$. We may assume that U_i contains $O(x_k y_k)$ for every k exceeding i. We also know, in view of this last remark, that for i fixed either $x_i y_i$ separates $D_{-1,1}$ between $x_i' y_i'$ and every $x_k' y_k'$ for k exceeding i or $x_i' y_i'$ separates this region between $x_i y_i$ and $x_k' y_k'$ for every k exceeding i. By taking a subsequence and renumbering we can insure that the same one of these two statements holds for every value of i and thus obtain the following assertion.

(4) *If i and k be any two distinct integers then in the region $D_{-1,1}$ the four arcs $x_i y_i$, $x_i' y_i'$, $x_k y_k$, $x_k' y_k'$ must occur either in the order just specified or in the alternative order $x_i' y_i'$, $x_i y_i$, $x_k' y_k'$, $x_k y_k$.*

No generality is lost by the assumption that for every integer i the arc $x_i y_i$ has an interior point within the circle C_z, having its center at a point z on the y-axis and radius sufficiently small so that any two points of M lying within C_z may be joined by an arc of M the orbit of which lies within $D_{-1,1}$. This enables us to find a simple arc $a_i b_k$ lying in M with $O(a_i b_k)$ in $D_{-1,1}$ and having exactly the points a_i, b_k in common with $x_i y_i$ and $x_k y_k$, respectively. As the two arrangements given in (4) are symmetrical we need treat only the case of the first one; the other will follow by a simple interchange of the letters i and k. From (1) we see that the arc $a_i b_k$ must contain at least one point of the closed set K, and we denote by f_{ik} the last point of K on this arc. Then if g_k be the first point of $O(x_k y_k)$ on the arc $f_{ik} b_k$, it follows that for some integer n the point $g_k' = T^n(g_k)$ lies on the arc $x_k' y_k'$. Thus the arc $f_{ik} g_k' = T^n(f_{ik} g_k)$ is a simple arc lying in $D_{-1,1}$ and joining the point f_{ik} to a point of $x_k' y_k'$, while containing no point of $x_k y_k$. This contradiction completes the proof of Theorem II.

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EQUIVALENCE OF A PROBLEM IN MEASURE THEORY TO A PROBLEM IN THE THEORY OF VECTOR LATTICES

GEORGE W. MACKEY

In connection with some other work of the author a question arises concerning the form of the general bounded linear[1] functional on certain vector lattices.[2] It is the purpose of this note to show that this question is completely equivalent to a question in measure theory which has been discussed and partially answered by Ulam [2].

Let S be an abstract set and let $\mathfrak{F}$ be the vector lattice of all real-valued functions defined on S. For each s_0 in S the function F on $\mathfrak{F}$ such that $F(f)=f(s_0)$ for all f in $\mathfrak{F}$ is clearly a linear functional. We shall call it the point functional belonging to s_0 or simply a point functional. Obviously every point functional and hence every finite linear combination of point functionals is bounded in the sense that it carries every bounded subset of $\mathfrak{F}$ into a bounded set of real numbers. Our question is as to whether every bounded linear functional on $\mathfrak{F}$ is a finite linear combination of point functionals. We shall show that this is the case if and only if there exists no countably additive measure α which is defined for all subsets of S, which is zero at points, which takes on only the values zero and one, and which does not vanish identically.

It is well known that every bounded linear functional on a vector lattice is a difference of non-negative linear functionals[3] and it is obvious that a non-negative linear functional is bounded. It follows that we need only consider non-negative linear functionals. Passages from a measure to a non-negative linear functional defined on a class of functions and vice versa are of frequent occurrence in mathematical literature. The proof of our theorem rests basically on the fact that when the methods used in effecting these passages are applied to the case at hand one obtains a natural one-to-one correspondence between the non-negative linear functionals on $\mathfrak{F}$ and the countably ad-

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[1] By a linear functional we mean a functional which is additive and homogeneous; that is, one which preserves linear combinations.

[2] See chap. 7 of [1] for definitions of the terms from the theory of vector lattices which we shall use. Numbers in brackets refer to the references cited at the end of the paper.

[3] This is proved on p. 115 of [1] for additive functionals and it is clear that an additive non-negative functional must be linear.

ditive measures which are defined on all subsets of S and take on only a finite number of different values.

LEMMA 1. *If P is a non-negative linear functional on $\mathfrak{F}$ and if $\phi_1, \phi_2, \cdots$ are the characteristic functions of mutually disjoint subsets of S then at most a finite number of the numbers $P(\phi_1), P(\phi_2), \cdots$ are different from zero.*

PROOF. Suppose the contrary and change the notation so that $P(\phi_1), P(\phi_2), \cdots$ are all different from zero. Let E_i be the set whose characteristic function is ϕ_i, $i=1, 2, \cdots$, and let $E=E_1 \cup E_2 \cup \cdots$. For each s in E_i let $g(s)=1/P(\phi_i)$ and for each s in $S-E$ let $g(s)=0$. Then for each $n=1, 2, \cdots$, $g \geqq (\phi_1/P(\phi_1))+(\phi_2/P(\phi_2))+ \cdots +(\phi_n/P(\phi_n))$. Hence $P(g) \geqq 1+1+ \cdots +1=n$ for $n=1, 2, \cdots$ and this is impossible.

LEMMA 2. *If P is a non-negative linear functional on $\mathfrak{F}$ then $P=P_1+P_2$ where P_1 is a finite linear combination of point functionals and P_2 is a non-negative linear functional which vanishes on characteristic functions of points.*

PROOF. By Lemma 1 there are at most a finite number of points in S whose characteristic functions are not taken into zero by P. Let $s_1, s_2, \cdots, s_r$ include all of these points and let $P_1(f)=P(\phi_1)f(s_1) +P(\phi_2)f(s_2)+ \cdots +P(\phi_r)f(s_r)$ for all f in $\mathfrak{F}$ where ϕ_i is the characteristic function of s_i for $i=1, 2, \cdots, r$. Then let $P_2=P-P_1$. It is obvious that P_2 is linear and vanishes on characteristic functions of points. Finally since any non-negative member of $\mathfrak{F}$ is the sum of a function which vanishes at the s_i and a linear combination of the ϕ_i with non-negative coefficients it is easily verified that P_2 is non-negative.

LEMMA 3. *If P is a non-negative linear functional on $\mathfrak{F}$ which vanishes on all characteristic functions then P vanishes identically.*

PROOF. It is obviously sufficient to show that $P(f)=0$ whenever $f \geqq 0$. For each $i=1, 2, \cdots$ let E_i be the set of all s in S such that $i-1 \leqq f(s) < i$ and let ϕ_i be the characteristic function of E_i. For each s in S there exists one and only one $i=1, 2, \cdots$ such that $s \in E_i$. Let $g(s)=if(s)$. For each $n=1, 2, \cdots$, $g-\phi_1 g-\phi_2 g- \cdots -\phi_{n-1}g \geqq n(f-\phi_1 f-\phi_2 f- \cdots -\phi_{n-1}f)$ and it is obvious that $P(h)=0$ whenever h is a bounded function. Hence for each $n=1, 2, \cdots$, $0 \leqq P(f) \leqq P(g)/n$. Thus $P(f)=0$.

LEMMA 4. *Let $\phi_1, \phi_2, \cdots$ be the characteristic functions of mutually* *disjoint subsets of S and let ϕ be the characteristic function of their union. Then if P is a non-negative linear functional on $\mathfrak{F}$, $P(\phi)=P(\phi_1)+P(\phi_2)+\cdots$.*

PROOF. By Lemma 1 there exists n_0 such that for $n>n_0$, $P(\phi_n)=0$. Since $P(\phi-\phi_1-\cdots-\phi_{n_0})=P(\phi)-P(\phi_1)-\cdots-P(\phi_{n_0})$ we have only to show that $P(\phi-\phi_1-\cdots-\phi_{n_0})=0$. In other words we may confine ourselves to the case in which $P(\phi_n)=0$ for $n=1, 2, \cdots$. For each s in S if $\phi_n(s)=1$ for some $n=1, 2, \cdots$ let $g(s)=n$; otherwise let $g(s)=0$. Then for each $n=1, 2, \cdots$, $g-\phi_1-2\phi_2-\cdots-(n-1)\phi_{n-1}\geqq n(\phi-\phi_1-\cdots-\phi_{n-1})$. Hence $P(g)\geqq nP(\phi)$. In other words $0\leqq P(\phi)\leqq P(g)/n$ for $n=1, 2, \cdots$. Thus $P(\phi)=0$.

We may now prove our equivalence theorem.

THEOREM. *Every bounded linear functional on $\mathfrak{F}$ is a finite linear combination of point functionals if and only if there exists no countably additive measure α defined on all subsets of S which is zero at points, which takes on only the values zero and one and which does not vanish identically.*

PROOF. Suppose that a measure of the sort described does exist. Let f be an arbitrary member of $\mathfrak{F}$. Using the fact that α takes on only one nonzero value, it is easy to prove the existence of a sequence $I_1, I_2, \cdots$ of closed intervals on the real line such that for each $n=1, 2, \cdots$, $I_n\supseteq I_{n+1}$, $\alpha(S-f^{-1}(I_n))=0$, and I_n is of length $1/(2^n)$. Let λ be the unique real number contained in all of the I_n's. Then it is clear that $f(s)=\lambda$ except on a set of α measure zero. In other words for each f in $\mathfrak{F}$ there is a unique real number $P(f)$ such that $f(s)=P(f)$ for "almost all" s. It is obvious that P is a non-negative linear functional on $\mathfrak{F}$ which is not identically zero and which does not vanish on characteristic functions of points. It follows then that P is a bounded linear functional on $\mathfrak{F}$ which is not a finite linear combination of point functionals. Conversely suppose that there exists a bounded linear functional on $\mathfrak{F}$ which is not a finite linear combination of point functionals. As we have already remarked we may suppose that this functional is non-negative. Hence by Lemma 2 there exists a non-negative linear functional P on $\mathfrak{F}$ which is not identically zero and which vanishes on characteristic functions of points. For each subset E of S let $\beta(E)=P(\phi_E)$ where ϕ_E is the characteristic function of E. It follows from Lemma 4 that β is a countably additive measure function and from Lemma 3 that β is not identically zero. Furthermore it is an easy consequence of Lemma 1 that $S=S_1\cup S_2\cup\cdots\cup S_n$ where each S_i has the property that for each subset E of S either $\beta(E\cap S_i)=0$ or

$\beta(E \cap S_i) = \beta(S_i)$. For at least one $i_0 = 1, 2, \cdots, n$, $\beta(S_{i_0}) \neq 0$. Let $\alpha(E) = \beta(E \cap S_{i_0})/\beta(S_{i_0})$ for each subset E of S. Then it is easily verified that α is a measure function with all properties listed in the statement of the theorem.

Using Ulam's results on two-valued measures we deduce the following corollary.

COROLLARY. Let $C_0 = \aleph_0$ and for each $n = 1, 2, \cdots$ let $C_n = 2^{C_{n-1}}$. Let $D = C_0 + C_1 + \cdots$. Then whenever the cardinal of S is less than D every bounded linear functional on $\mathfrak{F}$ is a finite linear combination of point functionals.

REFERENCES

1. G. Birkhoff, *Lattice Theory*, Amer. Math. Soc. Colloquium Publications, vol. 25, New York, 1940.

2. S. Ulam, *Zur Masstheorie in der allgemeinen Mengenlehre*, Fund. Math. vol. 16 (1930) pp. 140–150.

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THE ROLE OF INTERNAL FAMILIES IN MEASURE THEORY

ANTHONY P. MORSE

1. **Introduction.** Theorem 4.7 below is an abstract formulation of a certain closed subset-theorem[1] recently established by Randolph and myself. It has a wider range of application than similar abstractions due to Hahn[2] and to Saks.[3]

2. **Notation and terminology.** When H is a family of sets we agree that

$$\sigma(H) = \sum_{\beta \in H} \beta, \qquad \pi(H) = \prod_{\beta \in H} \beta.$$

A family R is said to be: *finitely additive* if $\sigma(H) \in R$ whenever H is a finite nonvacuous subfamily of R; *countably additive* if $\sigma(H) \in R$ whenever H is a countable nonvacuous subfamily of R; *finitely multiplicative* if $\pi(H) \in R$ whenever H is a finite nonvacuous subfamily of R; *countably multiplicative* if $\pi(F) \in R$ whenever F is a countable nonvacuous subfamily of R; α *complemental* if R is such a family of subsets of α that $\alpha - \beta \in R$ whenever $\beta \in R$.

If R is a family of sets we also agree that: R_σ is the family of all sets of the form $\sigma(H)$ where H is a countable nonvacuous subfamily of R; R_δ is the family of all sets of the form $\pi(H)$ where H is a countable nonvacuous subfamily of R; R_γ is the family of all sets of the form $\sigma(R) - \beta$ where $\beta \in R$; R^γ is the smallest $\sigma(R)$ complemental, countably additive family which contains R; R^δ is the smallest countably multiplicative, countably additive family which contains R.

DEFINITION 2.1. R is *internal* if and only if R_δ is finitely additive and $R_\gamma \subset R^\delta$.

REMARK 2.2 If R is the family of all closed subsets of a metric space then R is internal[4] and the members of R^γ are the Borel subsets of the space.

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[1] A. P. Morse and J. F. Randolph, *The ϕ rectifiable subsets of the plane*, Trans. Amer. Math. Soc. vol. 55 (1944) pp. 236–305, Theorem 3.7 together with the remarks which follow Theorem 3.4.

[2] H. Hahn, *Über die Multiplikation total-additiver Mengenfunktionen*, Annali della R. Scuola Normale Superiore Pisa (2) vol. 2 (1933) p. 437.

[3] S. Saks, *Theory of the integral*, Warsaw, 1937, p. 85.

[4] Since an open set is an R_σ.

3. Two known results in set theory.

THEOREM 3.1. *R_δ is countably multiplicative. If R is finitely additive then so is R_δ.*

PROOF. R_δ is clearly countably multiplicative. The remainder of the theorem follows from the identity

$$\prod_{x\in A} x + \prod_{y\in B} y = \prod_{x\in A}\ \prod_{y\in B} (x+y).$$

THEOREM 3.2.[5] *If $R_\gamma \subset R^\delta$ then $R^\gamma = R^\delta$.*

PROOF. Let $\tilde{\alpha} = \sigma(R) - \alpha$. Let

$$P = \underset{\alpha}{E}\,[(\alpha \in R^\delta)(\tilde{\alpha} \in R^\delta)].$$

A simple check reveals that P is a $\sigma(R)$ complemental, countably additive subfamily of R^δ. Our assumption that R_γ is contained in R^δ assures us, on the other hand, that P contains R. Accordingly $R^\gamma \subset P \subset R^\delta$. Now R^γ, being $\sigma(R)$ complemental and countably additive, is clearly countably multiplicative also. Consequently $R^\delta \subset R^\gamma$ and the desired conclusion is at hand.

4. The role of internal families in measure theory.

DEFINITION 4.1. We say ϕ *measures* S if and only if ϕ is such a function on $E_\beta[\beta \subset S]$ to $E_t[0 \leqq t \leqq \infty]$ that:

I. $\phi(0) = 0$;

II. $\phi(A) \leqq \phi(B)$ whenever $A \subset B \subset S$;

III. If H is any countable family for which $\sigma(H) \subset S$, then

$$\phi[\sigma(H)] \leqq \sum_{\beta \in H} \phi(\beta).$$

THEOREM 4.2. *If ϕ measures S and ϕ measures T then $S = T$.*

Due to Carathéodory[6] is

DEFINITION 4.3. A set A is ϕ *measurable* if and only if ϕ measures some superset S of A in such a way that

$$\phi(T) = \phi(TA) + \phi(T - A)$$

whenever $T \subset S$.

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[5] This is a corollary of a theorem proved by W. Sierpinski in his *Les ensembles boreliens abstraits*, Annales de la Société polonaise de mathématique vol. 6 (1927) p. 51.

[6] C. Carathéodory, *Über das lineare mass von Punktmengen*, Nachr. Ges. Wiss. Göttingen (1914) p. 406.

THEOREM 4.4. *If R is a family of ϕ measurable sets, ϕ measures $\sigma(R)$, then R^δ and R^γ are families of ϕ measurable sets.*

PROOF. Let M be the family of all ϕ measurable sets. Clearly M is $\sigma(R)$ complemental and countably additive.[7] Consequently $R^\delta \subset R^\gamma \subset M$.

LEMMA 4.5. *If R_s is a finitely additive family of ϕ measurable sets, ϕ measures $\sigma(R)$, $\phi[\sigma(R)] < \infty$, $B \in R^\delta$, $\epsilon > 0$, then B contains such a member C of R_s that $\phi(B - C) < \epsilon$.*

PROOF. Let K be so defined that $\beta \in K$ if and only if corresponding to each positive number η there is such a member C of R_s that

$$C \subset \beta, \qquad \phi(\beta - C) < \eta.$$

We shall complete the proof by showing in Part III below that $B \in K$.

Part I. *If H is a countable nonvacuous subfamily of K then $\sigma(H) \in K$ and $\pi(H) \in K$.*

PROOF. Let $\eta > 0$. Let $A_1, A_2, A_3, \cdots$ be a sequence whose range is H. Let $C_1, C_2, C_3, \cdots$ be such members of R_s that

$$C_n \subset A_n, \qquad \phi(A_n - C_n) < \frac{\eta}{2^n}$$

for each positive integer n.

Now

$$\phi\left[\sigma(H) - \sum_{n=1}^{\infty} C_n\right] = \phi\left[\sum_{n=1}^{\infty} A_n - \sum_{n=1}^{\infty} C_n\right] \leqq \phi\left[\sum_{n=1}^{\infty} (A_n - C_n)\right]$$
$$\leqq \sum_{n=1}^{\infty} \phi(A_n - C_n) < \sum_{n=1}^{\infty} \frac{\eta}{2^n} = \eta.$$

Accordingly if N is a sufficiently large integer we are sure that

$$\sum_{n=1}^{N} C_n \in R_s, \qquad \sum_{n=1}^{N} C_n \subset \sigma(H), \qquad \phi\left[\sigma(H) - \sum_{n=1}^{N} C_n\right] < \eta.$$

On the other hand

$$\pi(H) = \prod_{n=1}^{\infty} A_n$$

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[7] Those measure theoretic results of which we assume a previous knowledge are in H. Hahn, *Theorie der reellen Funktionen*, vol. 1, Berlin, 1921, pp. 424–427.

and $\prod_{n=1}^{\infty} C_n$ is such a member (see 3.1) of R_s that

$$\prod_{n=1}^{\infty} C_n \subset \pi(H),$$

$$\phi\left[\pi(H) - \prod_{n=1}^{\infty} C_n\right] = \phi\left[\sum_{n=1}^{\infty} \{\pi(H) - C_n\}\right] \leqq \phi\left[\sum_{n=1}^{\infty} (A_n - C_n)\right]$$

$$\leqq \sum_{n=1}^{\infty} \phi(A_n - C_n) < \sum_{n=1}^{\infty} \frac{\eta}{2^n} = \eta.$$

Part II. $R \subset K$.

PROOF. $R \subset R_s \subset K$.

Part III. $B \in K$.

PROOF. Parts I and II assure us that K is a countably multiplicative, countably additive family which contains R. Consequently $R^s \subset K$ and the conclusion that $B \in K$ follows from our hypothesis that $B \in R^s$.

THEOREM 4.6. *If R_s is a finitely additive family of ϕ measurable sets, ϕ measures $\sigma(R)$, $B \in R^s$, $\phi(B) < \infty$, $\epsilon > 0$, then B contains such a member C of R_s that $\phi(B - C) < \epsilon$.*

PROOF. Let Φ be such a function on the subsets of $\sigma(R)$ that

$$\Phi(\alpha) = \phi(B\alpha) \quad \text{whenever} \quad \alpha \subset \sigma(R).$$

Check that Φ measures $\sigma(R)$ and that 4.5 may be applied to yield the desired conclusion.

THEOREM 4.7. *If R is an internal family of ϕ measurable sets, ϕ measures $\sigma(R)$, $B \in R^\gamma$, $\phi(B) < \infty$, $\epsilon > 0$, then B contains such a member C of R_s that $\phi(B - C) < \epsilon$.*

PROOF. Use 4.6, 2.1, and 3.2.

DEFINITION 4.8. We say ϕ is a *Borelian measure* with respect to R if and only if: R is an internal family of ϕ measurable sets; ϕ measures $\sigma(R)$; corresponding to each subset A of $\sigma(R)$ there is a set β for which

$$\beta \in R^\gamma, \qquad A \subset \beta, \qquad \phi(A) = \phi(\beta).$$

THEOREM 4.9. *If ϕ is a Borelian measure with respect to R, A is a ϕ measurable set, $\phi(A) < \infty$, $\epsilon > 0$, then A contains such a member C of R_s that $\phi(A - C) < \epsilon$.*

PROOF. Let B', B'', B''' be such sets that

$$A \subset B' \in R^\gamma, \qquad \phi(B') = \phi(A),$$

$$B' - A \subset B'' \in R^{\gamma}, \qquad \phi(B'') = \phi(B' - A),$$
$$B''' = B' - B''.$$

Clearly

$$B''' \in R^{\gamma}, \qquad B''' = B' - B'' \subset B' - (B' - A) \subset A,$$
$$\phi(A - B''') \leqq \phi(B' - B''') \leqq \phi(B'') = \phi(B') - \phi(A) = 0.$$

Application of 4.7 to the set B''' completes the proof.

THEOREM 4.10. *If R is the family of all closed subsets of a metric space S, ϕ measures S in such a way that closed sets are ϕ measurable, B is a Borel set, $\phi(B) < \infty$, $\epsilon > 0$, then B contains such a closed set C that $\phi(B - C) < \epsilon$.*

PROOF. Clearly R is an internal family for which $R = R_{\delta}$, and $\sigma(R) = S$. Application of 4.7 completes the proof. Using 4.9 we obtain

THEOREM 4.11. *If R is the family of all closed subsets of a metric space S, ϕ is a Borelian measure with respect to R, A is ϕ measurable, $\phi(A) < \infty$, $\epsilon > 0$, then A contains such a closed set C that $\phi(A - C) < \epsilon$.*

REMARK 4.12. Theorems 4.9 and 4.11 are generalizations of a result due to Hahn.[8] For corollaries and special cases of Theorems 4.7, 4.9, 4.10, and 4.11, see Saks, op. cit., Theorem 6.5 on page 68, Theorem 6.6 on page 69, the correct portions of Theorem 9.7+ on page 85, the proof of Lemma 5.1 on page 114, Lemma 15.1 on page 152.

Let us now examine, in the light of an example, the just cited Theorem 9.7+ and my own Theorem 4.7. Let S be the ordinary real numbers metrized in the customary manner. Let F be the family of all closed subsets of S, G the family of all open subsets of S. Let $R = F_{\sigma}G_{\delta}$. It is easily seen, with the aid of 3.1, that R is a finitely additive, S complemental, internal family. Furthermore $\sigma(R) = S$ and R^{γ} is precisely the family of all Borel subsets of S. Let B be the rational numbers and let ϕ so measure S that

$$\phi(\beta) = \text{the number of numbers in } \beta B$$

whenever $\beta \subset S$. Note that $\phi(B) = \phi(S) = \infty$ but that S is a countable sum of Borel sets of finite ϕ measure. However, within the Borel set B, it is impossible to find a G_{δ} set C for which $\phi(B - C) < 1$; if this could be done then C would equal B and B itself would be a G_{δ} in contradiction to the well known fact that a dense G_{δ} is a residual set with the power of the continuum. Since $R_{\delta} \subset G_{\delta}$ it is also impossible to find, within the Borel set B, an R_{δ} set C for which $\phi(B - C) < 1$.

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[8] H. Hahn, *Theorie der reellen Funktionen*, vol. 1, Berlin, 1921, p. 447, Theorem IV.

This reveals the essential nature of the hypothesis "$\phi(B) < \infty$" in 4.7 as well as the erroneous aspects of the "more generally" part of Saks' Theorem 9.7+. Nevertheless it is easy to verify the statement obtained from Theorem 4.10 by deleting the hypothesis "$\phi(B) < \infty$" and replacing it by "each bounded set has finite ϕ measure."

REMARK 4.13. Herein we give a supplementary example which serves much the same purpose as the one just discussed in 4.12. Let S be the plane metrized in the customary manner. Introduce F, G, and R as in 4.12. Let B be those points in the plane whose first coordinates are rational. Let ϕ so measure S that

$$\phi(\beta) = \text{the Carathéodory}^{9} \text{ linear measure of } \beta B$$

whenever $\beta \subset S$. Note that $\phi(B) = \phi(S) = \infty$ but that S is a countable sum of Borel sets of finite ϕ measure. Note also (cf. 4.12) that each countable subset of S has ϕ measure zero. However, within the Borel set B, it is impossible to find a G_δ set C for which $\phi(B-C) < \infty$. To see this use the fact that the projection upon the y axis of any subset α of B has a Lebesgue measure which does not exceed $\phi(\alpha)$, and then recall the reasoning employed in 4.12.

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[9] C. Carathéodory, op. cit., pp. 420 ff.

ON LINEAR EQUATIONS IN HILBERT SPACE

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Given an infinite matrix $A = \|a_{ij}\|$ where a_{ij} is complex and

$$(1) \qquad \sum_{j=1}^{\infty} |a_{ij}|^2 < +\infty, \qquad i = 1, 2, \cdots,$$

the problem of solving the system of linear equations

$$(2) \qquad y_i = \sum_{j=1}^{\infty} a_{ij}\bar{x}_j, \qquad i = 1, 2, \cdots,$$

has been studied from several points of view. For arbitrary y_i, E. Schmidt[1] has given necessary and sufficient conditions on the a_{ij}, y_i so that the system (2) have a solution $x = (x_j) \in H_2$ (Hilbert space). Schmidt shows that if a solution exists, the solution of minimum norm is unique, and gives explicit formulas for this solution. If A defines a linear transformation T on H_2 to H_2, F. Riesz[2] gives necessary and sufficient conditions that an inverse T^{-1} exist, that is, that the solution $x = T^{-1}(y)$ where T^{-1} is a linear transformation. The following problem stands between these two: Find conditions on the elements of A so that the system (2) have a solution $x \in H_2$ for each $y \in H_2$. Such conditions will permit the use of Schmidt's formulas to express the minimal solution x for each y but this of course does not imply the existence of an inverse of the matrix A. We give a solution of this problem by a method which depends on a property, which seems new, of the m-rowed minors of the matrices $A_{i_1 \cdots i_m} = \|a_{i_s j}\|_{1 \leq s \leq m; j \geq 1}$ and on Cramer's rule.

Let

$$a(i_1, \cdots, i_m; j_1, \cdots, j_m) = \det \|a_{i_s j_t}\|_{1 \leq s, t \leq m}$$

be the determinant of the columns $j_1, \cdots, j_m$ of $A_{i_1 \cdots i_m}$. If $B = \|b_{ij}\|$ satisfies (1) and $B'_{i_1 \cdots i_m}$ is the transposed of $B_{i_1 \cdots i_m}$, the determinant $\det A_{i_1 \cdots i_m} B'_{i_1 \cdots i_m} = \det \|\sum_{k=1}^{\infty} a_{i_s k} b_{i_t k}\|_{1 \leq s, t \leq m}$ is finite. Because of the continuity of a determinant as a function of its elements

$$(3) \qquad \det A_{i_1 \cdots i_m} B'_{i_1 \cdots i_m} = \lim_n \det \left\| \sum_{k=1}^{n} a_{i_s k} b_{i_t k} \right\|_{1 \leq s, t \leq m}.$$

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[1] E. Schmidt, *Über die Auflösung linearer Gleichungen mit unendlich vielen Unbekannten*, Rend. Circ. Mat. Palermo vol. 25 (1908) pp. 53–77.

[2] F. Riesz, *Les systèmes d'equations linéaires a une infinité d'inconnues*, Paris, 1913, p. 86.

There is a theorem[3] on the minors of products of square matrices which, with slight modification in its proof, yields the identity

$$\det\left\|\sum_{k=1}^{n} a_{i_s k} b_{i_t k}\right\|_{1\leqq s,t\leqq m} \tag{4}$$
$$= \sum_{[j_1\cdots j_m]}^{n} a(i_1,\cdots,i_m;j_1,\cdots,j_m)b(i_1,\cdots,i_m;j_1,\cdots,j_m), \qquad n\geqq m,$$

where the sum is extended over all combinations $j_1,\cdots,j_m$ in $1,\cdots,n$.

THEOREM 1. *If* A, B *satisfy* (1), *then*

$$\det A_{i_1\cdots i_m}B'_{i_1\cdots i_m} = \sum_{[j_1\cdots j_m]}^{\infty} a(i_1,\cdots,i_m;j_1,\cdots,j_m)b(i_1,\cdots,i_m;j_1,\cdots,j_m),$$

where the sum is extended over all combinations of positive integers $j_1,\cdots,j_m$. *The series converges absolutely and*

$$\left|\det A_{i_1\cdots i_m}B'_{i_1\cdots i_m}\right| \leqq [\det A_{i_1\cdots i_m}\overline{A}'_{i_1\cdots i_m}]^{1/2}[\det B_{i_1\cdots i_m}\overline{B}'_{i_1\cdots i_m}]^{1/2}.$$

PROOF. By Schwartz' inequality we have, from (4) and (3) with $B=\overline{A}$,

$$\sum_{[j_1\cdots j_m]}^{n} \left|a(i_1,\cdots,i_m;j_1,\cdots,j_m)b(i_1,\cdots,i_m;j_1,\cdots,j_m)\right|$$
$$\leqq \left[\sum_{[j_1\cdots j_m]}^{n}\left|a(i_1,\cdots,i_m;j_1,\cdots,j_m)\right|^2\right]^{1/2} \cdot\left[\sum_{[j_1\cdots j_m]}^{n}\left|b(i_1,\cdots,i_m;j_1,\cdots,j_m)\right|^2\right]^{1/2}$$
$$\leqq [\det A_{i_1\cdots i_m}\overline{A}'_{i_1\cdots i_m}]^{1/2}[\det B_{i_1\cdots i_m}\overline{B}'_{i_1\cdots i_m}]^{1/2}, \qquad n\geqq m.$$

The conclusion is now evident as a consequence of (3).

If we define

$$|AB'| = \limsup_{m} \sup_{i_1\cdots i_m}\left|\sum_{[j_1\cdots j_m]}^{\infty} a(i_1,\cdots,i_m;j_1,\cdots,j_m) \times b(i_1,\cdots,i_m;j_1,\cdots,j_m)\right|$$

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[3] C. C. MacDuffee, *An introduction to abstract algebra*, New York, 1940, Theorem 99.1, p. 216.

we have a Schwarz inequality for matrices:

$$|AB'| \leqq |A\bar{A}'|^{1/2} |B\bar{B}'|^{1/2}.$$

The following lemma contains the Gram condition for linear dependence.

LEMMA 1. *The following statements are equivalent:*
(a) *The rows of* $A_{1\cdots m}$ *are linearly dependent.*
(b) Det $A_{1\cdots m}\bar{A}'_{1\cdots m}=0$.
(c) *All m-rowed minors of* $A_{1\cdots m}$ *equal zero.*

PROOF. That (a) implies (c) is immediate. The equivalence of (b) and (c) follows from Theorem 1 with $A=B$. It remains to show that (c) implies (a). This is evident if $m=1$. Assuming this statement for $m-1$, it is true for m if all the $(m-1)$-rowed minors of A_{m-1} vanish. If one such minor does not vanish, say

$$\det \|a_{ij}\|_{1\leqq i,j\leqq m-1} \neq 0,$$

we denote by c_k the cofactor of a_{km} in the determinant of the first m columns of $A_{1\cdots m}$. Then $c_m \neq 0$ and

$$\sum_{k=1}^{m} c_k a_{kj} = 0, \qquad j = 1, \cdots, m-1.$$

But this sum vanishes for all other values of j because of (c). Hence (c) implies (a).

THEOREM 2. *If A satisfies* (1), *the finite system*

$$y_i = \sum_{j=1}^{\infty} a_{ij}x_j, \qquad i = 1, \cdots, m, \tag{5}$$

has a solution $x \in H_2$ *for each* $y_1, \cdots, y_m$ *if and only if*

$$\det A_{1\cdots m}\bar{A}'_{1\cdots m} \neq 0.$$

PROOF. The necessity is a consequence of Lemma 1. If the condition is satisfied, then there is a nonvanishing $a(1, \cdots, m; j_1, \cdots, j_m)$ by (c) of Lemma 1 and a solution $x=(x_j)$ where $x_j=0$ for $j\neq j_1, \cdots . j_m$ and $x_{j_1}, \cdots, x_{j_m}$ are determined by Cramer's rule.

COROLLARY. *If A satisfies* (1) *and the system* (2) *has a solution* $x\in H_2$ *for each* $y \in H_2$, *then* det $A_{1\cdots m}\bar{A}'_{1\cdots m}\neq 0$ *for all m.*

An estimate of the minimum norm of the solution of the finite system (5) may be given in terms of a series of finite minors in $A_{i_1\cdots i_m}$.

Let $J=[j_1, \cdots, j_m]$ be a combination of m distinct positive integers and let S_m be a set of J such that no two J's have a common integer while every positive integer is in some $J \in S_m$. Let

$$\alpha_{i_1 \cdots i_m} = \left[\sup_{S_m} \sum_{J \in S_m} \left| a(i_1, \cdots, i_m; j_1, \cdots, j_m \right|^2 \right]^{1/2},$$

$$a_{i_1 \cdots i_m} = \left[\sum_{[j_1 \cdots j_m]} \left| a(i_1, \cdots, i_m; j_1, \cdots, j_m \right|^2 \right]^{1/2}.$$

Since S_m is a subset of the sum of all $j_1, \cdots, j_m$ we have the following lemma.

LEMMA 2. $\alpha_{i_1 \cdots i_m} \leqq a_{i_1 \cdots i_m}$,

$$\sup_{S_m} \sum_{J \in S_m} \sum_{s=1}^{m} \left| a(1, \cdots, k-1, k+1, \cdots, m; j_1, \cdots, j_{s-1} j_{s+1}, \cdots, j_m) \right|^2$$
$$\leqq \sum_{[j_1 \cdots j_{m-1}]}^{\infty} \left| a(1, \cdots, k-1, k+1, \cdots, m; j_1, \cdots, j_{m-1}) \right|^2.$$

THEOREM 3. *If A satisfies* (1) *and the finite system* (5) *has a solution $x^m \in H_2$ for each $y^m = (y_1, \cdots, y_m, 0, 0, \cdots)$, then*

$$\inf \|x^m\| \leqq \|y^m\| \left[\sum_{k=1}^{m} \left| \frac{a_{1, \cdots, k-1, k+1, \cdots, m}}{\alpha_{1 \cdots m}} \right|^2 \right]^{1/2}.$$

PROOF. By Theorem 2, $\det A_{1 \cdots m} \bar{A}'_{1 \cdots m} \neq 0$ and so $\alpha_{1 \cdots m} \neq 0$. Let $M^m_{kj_s}$ be the cofactor of a_{kj_s} in $a(1, \cdots, m; j_1, \cdots, j_m)$. The system (5) has a solution $x^m_{j_1 \cdots j_m}$ defined by

$$x^m_{j; j_1 \cdots j_m} = \begin{cases} \sum_{k=1}^{m} y_k M^m_{kj} / a(1, \cdots, m; j_1, \cdots, j_m), & j = j_1, \cdots, j_m, \\ 0, & j \neq j_1, \cdots, j_m. \end{cases}$$

We have

$$\|x^m_{j_1 \cdots j_m}\|^2 \left| a(i_1, \cdots, i_m; j_1 \cdots j_m) \right|^2$$
$$\leqq \sum_{s=1}^{m} \left| \sum_{k=1}^{m} y_k M^m_{kj_s} \right|^2 \leqq \|y^m\|^2 \sum_{s=1}^{m} \sum_{k=1}^{m} \left| M^m_{kj_s} \right|^2$$
$$= \|y^m\|^2 \sum_{s=1}^{m} \sum_{k=1}^{m} \left| a(1, \cdots, k-1, k+1, \cdots, m; j_1, \cdots, j_{s-1}, j_{s+1}, \cdots, j_m) \right|^2.$$

Hence

$$\begin{aligned}\inf \|x^m\|^2 \alpha_{1\cdots m}^2 &\leqq \|y^m\|^2 \sum_{k=1}^{m} \sup_{S_m} \sum_{J\in S_m} \sum_{s=1}^{m} \big| a(1, \cdots, k-1, \\ &\qquad\qquad k+1, \cdots, m; j_1 \cdots j_{s-1} j_{s+1} \cdots j_m) \big|^2 \\ &\leqq \|y^m\|^2 \sum_{k=1}^{m} a_{1,\cdots,k-1,k+1,\cdots,m}^2\end{aligned}$$

by Lemma 2. The conclusion follows at once.

A sufficient condition for the solution of the system (2) for each $y\in H$ may be obtained by restricting the constants

$$\alpha_m = \left[\sum_{k=1}^{m} \left| \frac{a_{1\cdots k-1,k+1,\cdots,m}}{\alpha_{1\cdots m}} \right|^2 \right]^{1/2}, \qquad m = 1, 2, \cdots .$$

THEOREM 4. *If A satisfies* (1), *its rows are linearly independent, and* $\alpha = \lim \inf_m \alpha_m < +\infty$, *then for each $y\in H_2$ the system* (2) *has a solution $x\in H_2$ such that*

$$\|x\| \leqq \alpha \|y\|.$$

PROOF. Consider any $y\in H_2$ and any $\epsilon>0$. The sequence contains a subsequence $\alpha_{m_\mu} < \alpha+\epsilon$. From Theorem 3 it follows that for each μ there is an $x^\mu = (x_j^\mu) \in H_2$ such that

$$y_i = \sum_{j=1}^{\infty} a_{ij} x_j^\mu, \qquad i = 1, \cdots, m_\mu,$$

$$\|x^\mu\| \leqq (\alpha+\epsilon)\|y\|.$$

Applying a diagonal process to (x_j^μ) one finds a subsequence $x^{\mu_\nu} = (x_j^{\mu_\nu})$ and an $x = (x_j) \in H_2$ such that

$$\lim_\nu x_j^{\mu_\nu} = x_j, \qquad j = 1, 2, \cdots,$$

$$\|x\| \leqq (\alpha+\epsilon)\|y\|.$$

Since for all ν, $N>0$ and $1 \leqq i \leqq m_{\mu_\nu}$

$$y_i - \sum_{j=1}^{\infty} a_{ij} x_j = \sum_{j=1}^{N} a_{ij}(x_j^{\mu_\nu} - x_j) + \sum_{j=N+1}^{\infty} a_{ij} x_j^{\mu_\nu} - \sum_{j=N+1}^{\infty} a_{ij} x_j,$$

x solves the system (1).

Now consider $\epsilon_n \downarrow 0$. For each n there is an $x^n \in H_2$ which solves the system (2) and such that $\|x^n\| \leqq (\alpha+\epsilon_n)\|y\|$. Repeating the diagonal process and the above argument, one finds an $x\in H_2$ such that $\|x\| \leqq \alpha\|y\|$ and which solves the system (2).

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A NOTE ON AN INEQUALITY OF E. SCHMIDT

RICHARD BELLMAN

In a note inserted in the Sitzungsberichte der Preussicher Akademie, 1932, E. Schmidt stated without proof a relation

$$(1)\qquad \left[\int_{-1}^{1} f'(x)^2 dx\right]^{1/2} \leqq k_N N^2 \left[\int_{-1}^{1} f(x)^2 dx\right]^{1/2}$$

where $f(x)$ is a polynomial of degree N, and it was subsequently shown by Hille, Szego and Tamarkin that k_N is a bounded function of N such that $\lim_{N\to\infty} k_N = \pi^{-1}$.

In a joint paper, generalizing the classical Markoff theorem

$$\max_{-1\leqq x\leqq 1} |f'(x)| \leqq N^2 \max_{-1\leqq x\leqq 1} |f(x)|$$

to general mean values of the form $\int_{-1}^{1}|f(x)|^p dx$, $p\geqq 1$, Hille, Szego and Tamarkin [1][1] gave three proofs of the inequality (1), two products of the general case of exponent p, and another for the case $p=2$, stated by the authors to be similar to the original unpublished proof of Schmidt.

It seems worthwhile to sketch a short elementary proof of the important case $p=2$, depending only upon an elementary inequality:[2]

$$(2)\qquad n\sum_{1}^{n} a_k^2 - \left(\sum_{1}^{n} a_k\right)^2 \geqq 0$$

and a simple property of the Legendre polynomials.

The required result for Legendre polynomials is [2] $P'_{n+1}(z) - P_{n-1}(z) = (2n+1)P_n(z)$, where $P_n(z)$ is the nth Legendre polynomial. From this we obtain

$$(3)\qquad \begin{aligned} P'_{2n} &= \sum_{0}^{n-1} (4k+3)P_{2k+1}, \\ P'_{2n+1} &= \sum_{0}^{n} (4k+1)P_{2k}. \end{aligned}$$

Let the polynomial of the Nth degree be expressed in Legendre polynomials

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[1] Numbers in brackets refer to the references cited at the end of the paper.

[2] The use of this inequality, instead of a more complicated one due to Hardy, was suggested by Professor Szász. It simultaneously simplifies the proof and yields a better constant than that originally obtained by the author.

$$f_N(x) = \sum_{k=0}^{N} a_k P_k(x).$$

Then, using the well known orthogonality relations of the P_n,

$$\int_{-1}^{1} f_N(x)^2 dx = 2\sum_{0}^{N} \frac{a_k^2}{2k+1}.$$

Using (3),

$$f_N'(x) = \sum_{2k\leqq N-1} (4k+1)P_{2k}(x)\left[\sum_{n\geqq k+1} a_{2n+1}\right]$$
$$+ \sum_{2k\leqq N-1} (4k+3)P_{2k+1}(x)\left[\sum_{n\geqq k+1} a_{2n}\right].$$

Thus

$$\int_{-1}^{1} f_N'(x)^2 dx = 2\sum_{2k\leqq N-1} (4k+1)\left[\sum_{r\geqq k} a_{2r+1}\right]^2$$
$$+ 2\sum_{2k\leqq N-1} (4k+3)\left[\sum_{r\geqq k} a_{2n}\right]^2.$$

Let $a_r = ((2r+1)/2)^{1/2} b_r$. Then the first sum is

$$2\sum_{2k\leqq N-1} (4k+1)\left[\sum_{r\geqq k} a_{2r+1}\right]^2$$
$$\leqq 2\left[\sum_{2k\leqq N-1} (4k+1)\right]\left[\sum \left(\frac{4k+1}{2}\right)^{1/2} b_{2k+1}\right]^2$$
$$\leqq \frac{N(N+1)(2N+1)}{4}\left[\sum b_{2k+1}\right]^2$$
$$\leqq (N+1)^4/2\left[\sum b_{2k+1}^2\right]$$

upon using the inequality (2) upon $[\sum b_{2k+1}]^2$.

The same process yields the same estimate for the second sum. Thus

$$\int_{-1}^{1} f_N'(x)^2 dx \leqq \frac{(N+1)^4}{2}\sum_{0}^{N} b_k^2 = \frac{(N+1)^4}{2}\sum_{0}^{N}\frac{2a_k^2}{2k+1}$$
$$= \frac{(N+1)^4}{2}\int_{-1}^{1} f_N(x)^2 dx.$$

This finishes the proof of (1) and yields lim sup $k_N \leqq 1/2^{1/2}$.

It is clear from the proof that by using Laguerre and Hermite polynomials with the corresponding weight functions, one can obtain similar results for the intervals $(0, \infty)$ and $(-\infty, \infty)$.

References

1. E. Hille, G. Szego, A. Tamarkin, *On some generalizations of a theorem of A. Markoff*, Duke Math. J. vol. 3 (1937) p. 729.
2. Whittaker-Watson, *Modern analysis*, Cambridge, 1935, p. 307 (IV).

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TRANSCENDENCE OF CERTAIN CONTINUED FRACTIONS

G. CUTHBERT WEBBER

Even today very little is known about which regular continued fraction expansions represent transcendental numbers. In 1851 Liouville[1] proved the transcendence of the numbers named after him, in the expansions of which the partial quotients increase very rapidly; in 1906 Maillet[2] proved the transcendence of certain expansions in which the partial quotients may be bounded. C. L. Siegel[3] proved that if the partial quotients of the expansion form an arithmetic progression of order one, the number represented is transcendental. In this paper well known transformations of continued fractions are applied, in conjunction with the Siegel theorem, to prove the transcendence of certain classes of continued fraction expansions.

We shall use the notation

$$b_0 + \frac{a_1}{b_1 +} \frac{a_2}{b_2 +} \cdots$$

to represent a continued fraction expansion, with the partial numerators and denominators a_j and b_j respectively. The jth convergent is denoted by A_j/B_j. When the expansion is regular the following notations are also used:

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[1] J. Liouville, *Sur des classes très-étendues de quantités dont la valeur n'est ni algébrique, ni même reductible à des irrationnelles algébrique*, J. Math. Pures Appl. vol. 16 (1851).

[2] E. Maillet, *Introduction à la théorie des nombres transcendants*, Paris, 1906.

[3] C. L. Siegel, *Über einige Anwendungen diophantischer Approximationen*, Abhandlung der Preussischen Akademie der Wissenschaften, Physikalische-Mathematische Klasse, no. 1 (1929) p. 29.

$$[b_0, b_1, b_2, \cdots] = [b_j]_{j=0}^{\infty}.$$

In the expansions considered $a_j = \pm 1$ $(j \geqq 1)$, the partial quotients are integers, $b_j \geqq 1$ $(j \geqq 1)$, $b_j + a_{j+1} \geqq 2$ for an infinite set of values of j, and $b_j + a_{j+1} \geqq 1$ for all j except, possibly, a finite set. Thus the expansions are semi-regular, at least after a finite number of terms, and so they converge.[4] The expansions considered in the theorems are of Hurwitz type except when one or more partial numerators are -1 in Theorem I.

Siegel, in the paper referred to above, states that the methods applied therein suffice to prove that if $x \neq 0$ is algebraic, then there does not exist between $J_\lambda(x)$ and $J_{\lambda-1}(x)$ a rational equation with rational coefficients, if λ is rational but not the half of an odd integer ($J_\lambda(x)$ is a Bessel function of type one); also, that the same result holds when λ is half an odd integer, due to the special form of $J_\lambda(x)/J_{\lambda-1}(x)$ in terms of e^{ix} and to the Lindemann theorem: The form $\sum_{\nu=1}^{n} \xi_\nu e^{\eta_\nu}$ is transcendental in each non-trivial case, if $\xi_1, \cdots, \xi_n$ and $\eta_1, \cdots, \eta_n$ are algebraic numbers.

The theory of Bessel functions gives the result that

$$\lambda x + \frac{a_1}{(\lambda + 1)x +} \frac{a_2}{(\lambda + 2)x +} \cdots$$

is equal to $iJ_{\lambda-1}(2i/x)/J_\lambda(2i/x)$ or $J_{\lambda-1}(2/x)/J_\lambda(2/x)$, according as the a_j are all $+1$ or all -1. If x and λ are chosen such that $x = d$ and $\lambda d = r$, r and d being positive integers, $r \geqq 2$, then

$$r + \frac{a_1}{(r + d) +} \frac{a_2}{(r + 2d) +} \cdots, \tag{1}$$

where the a_j are all $+1$ or all -1, is transcendental.[5]

THEOREM I. *The number* ξ, *represented by*

$$\xi = g_0 + \frac{a_1}{b_1 +} \cdots \frac{a_{k-1}}{+ b_{k-1} +} \frac{f_1}{g_1 +} \frac{a_1}{b_1 +} \cdots \frac{a_{k-1}}{+ b_{k-1} +} \frac{f_2}{g_2 +} \cdots \tag{2}$$

is transcendental when the sequence $\{g_j\}$ *is an arithmetic progression of first order whose elements are integers which are positive except, possibly,* g_0; $b_1, \cdots, b_k$ *are positive integers,* $a_m = \pm 1$ $(m = 1, \cdots, k-1)$,

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[4] O. Perron, *Die Lehre von den Kettenbrüchen*, B. G. Teubner, Leipzig, 1929, pp. 148–154.

[5] If the a_j are all $+1$, this becomes the Siegel theorem stated in the introduction.

the f_j *are either all* $+1$ *or all* -1, *and* $g_1+a_1\geqq 1$, $b_m+a_{m+1}\geqq 1$, $b_{k-1}+f_j\geqq 1$, *for* $m=1,\cdots,k-2$ *and* $j\geqq 1$.

In the proof we may consider $g_0\geqq 2$; otherwise, the first $k+1$ or $2k+2$ terms are dropped and the remaining terms renumbered to insure convergence of the semi-regular expansion. This does not affect the algebraic character of the whole expansion. The continued fraction (2) is transformed by "contraction"[6] to the continued fraction with convergents A_{jk-1}/B_{jk-1} $(j=1, 2, 3, \cdots)$. This gives

$$\begin{aligned}\xi &= \frac{1}{B_{k-1}}\Bigg\{g_0B_{k-1}+a_1B_{k-2,1}+\frac{(-1)^{k-1}a_1a_2\cdots a_{k-1}f_1}{(g_1B_{k-1}+f_1B_{k-2}+a_1B_{k-2,1})}\\ &\qquad\qquad + \frac{(-1)^{k-1}a_1\cdots a_{k-1}f_2}{(g_2B_{k-1}+f_2B_{k-2}+a_1B_{k-2,1})+}\cdots\Bigg\}\\ &= \frac{1}{B_{k-1}}\left\{g_0B_{k-1}+a_1B_{k-2,1}+\frac{(-1)^{k-1}a_1a_2\cdots a_{k-1}f_1}{\eta}\right\},^{7}\end{aligned}$$

where η is a continued fraction of type (1), at least after a finite number of terms. The latter case may occur when either or both of a_1 and f_1 are -1. Since η is transcendental it follows that ξ is also transcendental.

It is interesting to note that if the expansion is regular, the partial quotients may be made to increase as "slowly as desired" by choosing $b_j=1$ $(j=1, 2, \cdots, k)$ and k large.

THEOREM II. *If* a, d *and* c *are positive integers, either* c *or* d *being even, and* $b=c^2a+(1/2)c^2d$, $e=c^2d$, *then*

$$\alpha = [a, b, a+d, b+e, a+2d, b+2e, \cdots]$$

is a transcendental number.

From the theory of continued fractions we obtain the following result:[8]

$$\begin{aligned}&c_0b_0+\frac{c_0c_1a_1}{c_1b_1}\,+\cdots+\,\frac{c_{k-1}c_ka_k}{c_kb_k}\,+\cdots\\ &\qquad\qquad = c_0\left[b_0+\frac{a_1}{b_1}\,+\cdots+\,\frac{a_k}{b_k}\,+\cdots\right].\end{aligned}\tag{3}$$

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[6] O. Perron, *Die Lehre von den Kettenbrüchen*, B. G. Teubner, Leipzig, 1929, pp. 197–203.

[7] By $B_{k-2,1}$ is meant the result of advancing by one the subscripts of all the a_j and b_j in B_{k-2}.

[8] O. Perron, *Die Lehre von den Kettenbrüchen*, B. G. Teubner, Leipzig, 1929, p. 196.

Application of this result with $a_k=1$, $c_k=1/c_{k-1}$, $c_0=c$, $c_{2k}=c$ $(k=1, 2, 3, \cdots)$ to α yields

$$\begin{aligned} c\alpha &\doteq c[a,\ c^2a + (1/2)c^2d,\ a + d,\ c^2a + (3/2)c^2d, \\ &\qquad\qquad a + 2d,\ c^2a + (5/2)c^2d, \cdots] \\ &= [ca,\ ca + (1/2)cd,\ ca + cd,\ ca + (3/2)cd, \\ &\qquad\qquad ca + 2cd,\ ca + (5/2)cd, \cdots]. \end{aligned}$$

Since the partial quotients in $c\alpha$ form an arithmetic progression of order one with common difference $(1/2)cd$, $c\alpha$, and in turn α, is transcendental by the Siegel theorem stated in the introduction.

In the last theorem we proved the transcendence of a class of regular continued fractions in which the alternating partial quotients form an arithmetic progression of order one. The following theorem gives two more classes of a similar type.

THEOREM III. *The numbers* β *and* γ, *where*

$$\beta = [1 + 4j,\ (3 + 4j)u^2]_{j=0}^{\infty} \text{ and } \gamma = [1,\ (3 + 4j)u^2 - 2,\ 1,\ 3 + 4j]_{j=0}^{\infty},$$

u^2 *being a positive integer, are transcendental.*

Again appealing to well established theory of continued fractions[9] we have

$$\delta = 0 + \frac{z}{1\ +}\ \frac{a_2z^2}{3\ +}\ \frac{a_3z^2}{5\ +}\ \frac{a_4z^2}{7\ +}\ \cdots = \frac{e^z - e^{-z}}{e^z + e^{-z}} \text{ or } \tan z,$$

according as the a_j are all $+1$ or all -1. Apply (3) with $c_0=1/z$, $c_{2j+1}=1$, $c_{2j+2}=1/z^2$ $(j=0, 1, 2, \cdots)$ and $z=1/u$; this yields

$$\begin{aligned} \delta &= z\left[0 + \frac{1}{1\ +}\ \frac{a_2}{(3/z^2)\ +}\ \frac{a_3}{5\ +}\ \frac{a_4}{(7/z^2)\ +}\ \frac{a_5}{9\ +}\ \cdots\right] \\ &= (1/u)\left[0 + \frac{1}{1\ +}\ \frac{a_2}{3u^2\ +}\ \frac{a_3}{5\ +}\ \frac{a_4}{7u^2\ +}\ \frac{a_5}{9\ +}\ \cdots\right] \\ &= (1/u)(1/\beta) \quad \text{or} \quad (1/u)(1/\epsilon), \end{aligned}$$

according as the a_j are all $+1$ or all -1, where ϵ is defined by the above equality.

Since u^2 is a positive integer, u is an algebraic number not equal to 0, and thus z is an algebraic number. Now by the Lindemann theorem, tan z and $(e^z - e^{-z})/(e^z + e^{-z})$ are transcendental when $z \neq 0$ is algebraic. This proves the transcendence of β and ϵ.

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[9] O. Perron, p. 353.

Convert the semi-regular continued fraction for ϵ to a regular continued fraction;[10] this gives

$$\frac{1}{\epsilon} = \frac{1}{1} - \frac{1}{3u^2} - \frac{1}{5} - \frac{1}{7u^2} - \frac{1}{9} - \cdots,$$

$$\epsilon = 0 + \frac{1}{1} + \frac{1}{(3u^2 - 2)} + \frac{1}{1} + \frac{1}{3} + \frac{1}{1}$$
$$+ \frac{1}{(7u^2 - 2)} + \frac{1}{1} + \frac{1}{7} + \cdots,$$

$$\frac{1}{\epsilon} = [1, (3 + 4j)u^2 - 2, 1, 3 + 4j]_{j=0}^{\infty} = \gamma.$$

Since ϵ is transcendental, so is γ.

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[10] O. Perron, p. 159.

A NOTE ON A THEOREM OF HARDY ON FOURIER CONSTANTS

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In Note LXVI of his *Notes on some points in the integral calculus* [1],[1] Hardy proves the following theorem.

THEOREM 1 (HARDY). *If $a_1, a_2, \cdots, a_n, \cdots$ are the Fourier constants of a function of L^p, $p \geqq 1$, then $A_1, A_2, \cdots, A_n, \cdots$ are also the Fourier constants of a function of L^p, where $A_n = n^{-1}\sum_1^n a_k$.*

Hardy restricts the class of functions considered to even functions, with mean value over a period zero, and the same restriction shall be observed here.

We wish to prove a "dual" to Theorem 1, namely:

THEOREM 2. *If $b_1, b_2, \cdots, b_n, \cdots$ are the Fourier constants of a function, $g(x)$, of L^p, $p > 1$, then $B_1, B_2, \cdots, B_n, \cdots$ are also the Fourier constants of a function, $G(x)$, of L^p, where $B_n = \sum_n^\infty b_k k^{-1}$, and if $f(x) \sim \sum a_n \cos nx$ is any function of $L^{p'}$, $F(x) \sim \sum_1^\infty A_n \cos nx$, then, $1/p + 1/p' = 1$,*

$$\int_0^\pi f(x)G(x)dx + \int_0^\pi F(x)g(x)dx = 0.$$

There is a difference between the theorems. The case $p = 1$ is omitted in Theorem 2, and necessarily, since $g(x) \sim \sum_{n=2}^\infty \cos nx/\log n$ belongs to L, but the corresponding B_n do not even exist.

Hardy's method depends upon an explicit representation of the function $F(x)$ in terms of $f(x)$. In the case of the theorem to be proven, this representation does not seem to facilitate matters. The method used will depend upon some general theorems on Fourier series, together with the original theorem of Hardy. For the case $p = 2$, the proof is immediate, since it can be shown that the convergence of b_n^2 implies the convergence of B_n^2 (this result is also due to Hardy, and was the origin of this type of theorem). In particular, this series converges for a function belonging to L^p, for $p \geqq 2$, and thus there exists a function of L^2, having the B_n as Fourier coefficients.

The method of proof depends upon the observation that, purely formally, partial summation yields

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[1] Numbers in brackets refer to the references cited at the end of the paper.

$$\sum_1^\infty a_n B_n = - \sum_1^\infty A_n b_n.$$

The justification of this will constitute the principal part of the proof. Three lemmas will be required.

LEMMA 1 [2]. *A necessary and sufficient condition that the series* $\sum_1^\infty a_n \cos nx$ *should be a Fourier series of a function of* L^p, $p>1$, *is that for every* $g(x)$ *belonging to* $L^{p'}$ $(1/p+1/p'=1)$ *with Fourier coefficients* a_n', *the series* $\sum_1^\infty a_n a_n'$ *should be finite* $(C, 1)$.

A series is finite $(C, 1)$ when the arithmetic mean of its partial sums is bounded.

LEMMA 2. *If* $H(x) \sim \sum_1^\infty c_n \cos nx$ *is a function of* L^p, $p>1$, *then*

$$\left| \sum_1^N c_n \right| = O(N^{1/p}).$$

PROOF OF LEMMA 2.

$$\sum_1^N c_n = \frac{1}{\pi} \int_0^\pi H(x) \left[\sum_1^N \cos nx \right] dx.$$

Using Hölder's inequality

$$\begin{aligned} \left| \sum_1^N c_n \right| &\leqq \frac{1}{\pi} \left[\int_0^\pi \left| H(x) \right|^p dx \right]^{1/p} \left[\int_0^\pi \left| \sum_1^N \cos nx \right|^{p'} dx \right]^{1/p'} \\ &= O\left[\int_0^\pi \left| \sum_1^N \cos nx \right|^{p'} dx \right]^{1/p'} \\ &= O\left[\int_0^\pi \left| \frac{\sin Nx}{x} \right|^{p'} dx \right]^{1/p'} = O(N^{1/p}). \end{aligned}$$

This finishes the proof of the lemma.

LEMMA 3.[3] *If* $H(x) \sim \sum_1^\infty c_n \cos nx$ *belongs to* $L^{p'}$, $p>1$, *then*

$$\left| \sum_N^\infty \frac{c_n}{n} \right| = O(N^{-1/p}).$$

PROOF OF LEMMA 3. Let $\sum_1^\infty c_\nu = s_n$, then

$$\sum_n^m \frac{c_\nu}{\nu} = m^{-1} s_m - n^{-1} s_{n-1} + \sum_n^{m-1} \frac{s_\nu}{\nu(\nu+1)}.$$

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[3] The above proof of Lemma 3 and a part of the proof of Theorem 2 are due to Professor O. Szász and are simpler and more elegant than the original.

By Lemma 2, $s_m = O(m^{1/p'})$, hence $m^{-1}s_m \to 0$, and

$$\begin{aligned}\sum_n^\infty \nu^{-1}c_\nu &= \frac{-s_{n-1}}{n} + \sum_n^\infty \frac{s_\nu}{\nu(\nu+1)} \\ &= O(n^{1/p'-1}) + O\left(\sum_n^\infty \nu^{1/p'-2}\right) \\ &= O(n^{1/p'-1}) = O(n^{-1/p}).\end{aligned}$$

Proof of Theorem 2. A_n and B_n, $f(x)$, $g(x)$, $F(x)$, $G(x)$ are defined as in Theorems 1 and 2. We divide the proof into three parts. The first part will consist of showing that $G(x)$ belongs to L^p, $p>1$, whenever $g(x)$ does. In virtue of Lemma 1, it is sufficient to show

$$\sum_1^n B_\nu a_\nu = O(1) \qquad \text{as } n \to \infty,$$

where $g(x) \sim \sum b_n \cos nx$ is a function of L^p, and $f(x) \sim \sum a_n \cos nx$ is a function of $L^{p'}$.

By partial summation

$$\sum_1^n B_\nu a_\nu = B_n \sum_1^n a_\nu - \sum_1^{n-1} A_\nu b_\nu. \tag{1}$$

Using Theorem 1, $F(x) \sim \sum A_n \cos nx$ belongs to $L^{p'}$, and thus, using the generalized Parseval theorem, $\sum_1^\infty A_\nu b_\nu$ is convergent. Furthermore, $\sum_1^n a_\nu = O(n^{1/p'})$, by Lemma 2, and $B_r = O(n^{-1/p'})$, by Lemma 3. Hence

$$B_n \sum_1^n a_\nu = O(n^{1/p'})O(n^{-1/p'}) = O(1),$$

and thus $G(x)$ belongs to L^p.

The second part consists of showing that $\sum_1^\infty B_\nu a_\nu$ converges. This is immediate by the generalized Parseval theorem (Hausdorff-Young theorem) since $G(x)$ belongs to L^p, $f(x)$ to $L^{p'}$.

The third part consists of showing that $\sum_1^\infty A_n b_n = -\sum_1^\infty a_n B_n$ and using (1) this is true if, and only if, $B_n \sum_1^\infty a_\nu = o(1)$.

We have to consider two cases.

Case 1. $f(x)$ belongs to L^p, $1<p\leqq 2$. Then, by the Hausdorff-Young theorem, $\sum |a_n|^{p'}$ converges, and thus

$$\left|\sum_1^N a_n\right| = o(N^{1/p}).$$

Case 2. $f(x)$ belongs to L^p, $p>2$, and thus $g(x)$ belongs to $L^{p'}$,

$1<p'\leqq 2$. Then, as before, $\sum|b_n|^p$ converges, and this entails

$$\left|\sum_N^\infty \frac{b_n}{n}\right| = o(N^{-1/p}).$$

Thus for all $p>1$,

$$\left|B_n\sum_1^n a_\nu\right| = \begin{Bmatrix} o(N^{1/p})O(N^{-1/p}) \\ O(N^{-1/p})o(N^{1/p}) \end{Bmatrix} = o(1).$$

This finishes the proof of Theorem 2.

It is interesting to see how these results arise naturally in dealing with Fourier integrals. The processes will be purely formal, although they could be justified by similar methods.

$$f(t) = \int_0^\infty F(x)\cos xt dx,$$

$$\int_0^t f(t)dt = \int_0^\infty F(x)\frac{\sin xt}{x}dx = t\int_0^\infty\left(\int_x^\infty \frac{F(u)}{u}du\right)\cos xt\, dx.$$

The last result is obtained by integrating by parts, assuming that the integrated term drops out. Thus

$$\frac{1}{t}\int_0^t f(t)dt = \int_0^\infty\left(\int_x^\infty \frac{F(u)}{u}du\right)\cos xt\, dx.$$

Similarly

$$\int_t^\infty \frac{f(t)}{t}dt = \int_0^\infty F(x)\left[\int_t^\infty \frac{\cos xt}{t}\right]dx$$

$$= \int_0^\infty F(x)\left[\int_{xt}^\infty \frac{\cos u}{u}du\right]dx.$$

Integrating the last expression by parts, and assuming the integrated part vanishes, there results

$$\int_t^\infty \frac{f(t)}{t}dt = \int_0^\infty\left[\frac{1}{x}\int_0^x F(u)du\right]\cos xt\, dx.$$

REFERENCES

1. G. H. Hardy, *Notes on some points in the integral calculus*, Mesesnger of Mathematics vol. 58 (1928) pp. 50–52.

2. A. Zygmund, *Trigonometrical series*, Warsaw, 1935, 4.63, Lemma, p. 103.

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SOME INTEGRAL RELATIONS

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Let $u(r)$ have a continuous derivative for positive values of r, then a solution of the integral equation

$$(1)\qquad u(r) = \int_0^{\pi/2} \cos (kr \sin a) U(r \cos a) da$$

is given by the formula

$$(2)\qquad U(s) = (2/\pi)\left[u(0)ch(ks) + s\int_0^{\pi/2} ch(ks \sin b)u'(s \cos b)db\right].$$

The mean value theorem for a two-dimensional potential round a circle, due essentially to Parseval, gives the equation

$$1 = (2/\pi)\int_0^{\pi/2} \cos (kr \sin a)ch(kr \cos a)da,$$

and so the relation to be established is

$$u(r) - u(0) = (2/\pi)\int_0^{\pi/2} \cos (kr \sin a)r \cos a\, da\int_0^{\pi/2} ch(kx)u'(y)db,$$

where

$$y = r \cos a \cos b, \quad x = r \cos a \sin b = z \sin c, \quad z = (r^2 - y^2)^{1/2}.$$

When the quantities y and c are taken as new variables of integration the integral on the right becomes

$$(2/\pi)\int_0^{r} u'(y)dy\int_0^{\pi/2} \cos (kz \cos c)ch(kz \sin c)dc = \int_0^{r} u'(y)dy$$

and so the formula is established.[1] A proof of the uniqueness of the solution is not easy to obtain by substitution of an integral of type (1) in (2) but may be derived from the relation

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[1] The case $k=0$ is reducible to Abel's integral equation and is a special case of an equation $F(x)=\int_0^{\pi/2} f(x \cos a) \sin^{2n+1} a\, da$ found by Poisson in his *Researches on heat conduction*, J. École Polytech. vol. 12 (1823) p. 299. This equation was discussed by J. Liouville, ibid. vol. 15 (1835) pp. 55–60. Professor M. A. Bosoco has kindly recalled my attention to the interesting integral transformation used for this case by O. Schlömilch, Zeitschrift für Mathematik und Physik vol. 2 (1857).

$$u(s^{1/2}) = \frac{1}{2}\int_0^s \cos k(s - t)^{1/2}U(t^{1/2})dt/[t(s - t)]^{1/2}$$

and the equivalent relation

$$\int_0^\infty e^{-zs}u(s^{1/2})ds = \frac{1}{2}\int_0^\infty e^{-zt}U(t^{1/2})t^{-1/2}dt \int_0^\infty e^{-zt} \cos (kt^{1/2})t^{-1/2}dt.$$

Thus when $u(r)$ is given for positive values of r the first integral on the right is determined uniquely and so by a simple extension of Lerch's theorem $U(r)$ is unique.

The theory of exponential integrals has been advanced by the recent tabulation[2] of the function $F(x)$ which is defined by the integral

$$F(x) = \int_0^\infty e^{-m^4/4} \cos (mx)dm \tag{3}$$

$$= \frac{1}{2}\left(\frac{1}{2}\right)^{1/2} \sum_{n=0}^\infty (- 2x^2)^n\Gamma(n/2 + 1/4)/(2n)!. \tag{4}$$

The table indicates that $F(x)=0$ for $x=2.435$, 4.778, and 6.550, approximately. This is in accordance with the result of F. Bernstein[3] that $F(x)$ has an infinite number of real zeros.

The asymptotic behavior of $F(x)$ for large complex values of x has been discussed by Burwell[4] who gives, moreover, an asymptotic formula for the large zeros. The asymptotic expansion of $F(x)$ for large positive values of x is rather complicated but it is sufficient to establish the convergence of the integral

$$\int_0^\infty e^{-zt}F(t^{1/4})t^{-1/4}dt = \frac{\pi z^{-3/4}}{2} \exp (- z^{-1/2}/4), \qquad R(z) > 0, \tag{5}$$

which may be regarded as a generating function of $F(t^{1/4})t^{-1/4}$ in the sense of Abel. In the theory of Haar and others by which the asymptotic behavior of a function for large positive values of t is derived from the behavior of the generating Laplacian integral at points z on the line bounding the region of convergence, the form of the function

$$F(t^{1/4})t^{-1/4}$$

for large positive values of t is characterized by the behavior of the function on the right of (5) at the point $z=0$. The type of singularity is in this case different from that of any tabulated generating function.

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[2] G. Grimminger, *Velocity and mass distributions resulting from the lateral diffusion of a current in a stratified medium on a rotating earth*, Journal of the Franklin Institute vol. 236 (1943) pp. 413–443, 509–520.

[3] F. Bernstein, Math. Ann. vol. 79 (1918), pp. 265–268. See also G. Pólya, Messenger of Mathematics vol. 52 (1923) pp. 185–188.

[4] W. R. Burwell, Proc. London Math. Soc. (2) vol. 22 (1924) pp. 57–72.

The relation (5) was obtained originally by substituting the infinite series for $F(x)$ and integrating term by term.[5] An alternative plan is to consider the function

$$U(x, y) = \int_0^\infty e^{-ym^4} \cos (mx)dm \tag{6}$$

which is a solution of the partial differential equation

$$U_y + U_{xxxx} = 0.$$

The integral

$$u(y, z) = \int_0^\infty e^{-zt} U(t^{1/4}, y)t^{-1/4}dt = 4\int_0^\infty e^{-zx^4} U(x, y)x^2dx \tag{7}$$

is then a solution of the partial differential equation

$$u_y = 360zu + 2160z^2u_z + 1664z^3u_{zz} + 256z^4u_{zzz}$$

and may be identified with the solution

$$u(y, z) = \frac{1}{2}\left(\frac{1}{2}\right)^{1/2} \pi y^{-1/4}z^{-3/4} \exp (- y^{-1/2}z^{-1/2}/8)$$

for U is of the form $(1/x)f(yx^{-4})$ and so $u(y, z)$ is of the form

$$u(y, z) = z^{-1/2}H(yz) = z^{-1/2}H(w),$$

say. If $H(w) = w^{-1/4}S(w^{-1/2})$ the differential equation for $S(s)$ is

$$2s\, dT/ds + T = 0 \quad \text{where} \quad T = 64S''(s) - S.$$

Hence $T = Cs^{-1/2}$ and so $S = Ae^{s/8} + Be^{-s/8} - CP(s)$, where for large values of s the function P has an asymptotic expansion of type

$$P(s) = s^{-1/2}[1 + 1\cdot 3(16/s^2) + 1\cdot 3\cdot 5\cdot 7(16/s^2)^2 + \cdots].$$

The solution $S = B \exp (-s/8)$ is of the right type when s is large and w small and the proposed value of the integral seems to be correct.

There are some related integrals involving the functions V_0, V_1, V_2, V_3 defined on p. 114 of my *Partial differential equations of mathemati-*

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[5] For a justification of this process reference may be made to G. H. Hardy, Transactions of the Cambridge Philosophical Society vol. 21 (1908) pp. 1–48 and E. C. Titchmarsh, *The theory of functions*, Oxford, 1932, p. 348. The infinite series for $F(x)$ may indeed be expressed as the difference of two series involving positive terms and each of these may be integrated term by term because the resulting series are convergent. I owe the second reference to Morgan Ward.

cal physics. These functions are solutions of the partial differential equation[6]

$$V_y = V_{xxxx}$$

and are represented by the integrals

$$V_0(x, y) = 4\int_0^\infty e^{-4ys^4}\cos sx \, ch\, sx\, ds,$$

$$V_2(x, y) = 4\int_0^\infty e^{-4ys^4}\sin sx\, sh\, sx\, ds,$$

$$V_1(x, y) = 4\int_0^\infty e^{-4ys^4}(\sin sx\, ch\, sx + \cos sx\, ch\, sx)ds,$$

$$V_3(x, y) = 4\int_0^\infty e^{-4ys^4}(\sin sx\, ch\, sx - \cos sx\, sh\, sx)ds.$$

The simplest of the integrals are

$$\int_0^\infty e^{-zt}t^{-1/4}V_0(s^{1/4}, y)dt = \pi y^{-1/4}z^{-3/4}\cos(y^{-1/2}z^{-1/2}/8),$$

$$\int_0^\infty e^{-zt}t^{-1/4}V_2(s^{1/4}, y)dt = \pi y^{-1/4}z^{-3/4}\sin(y^{-1/2}z^{-1/2}/8).$$

The asymptotic forms of $V_0(x, y)$ and $V_2(x, y)$ for large values of x can be derived from Burwell's results; they can also be regarded as determined by the behavior of the functions on the right in the neighborhood of $z=0$. In the search for functions whose generating functions have prescribed singularities it should be worth while to fully explore the functions provided by solutions of the general equation of Poisson.

Generating functions depending on the error function and the integrals of Fresnel may be derived from the functions now under discussion. Thus

$$\int_0^\infty e^{-zt}[F(0) - F(t^{1/4})]t^{-5/4}dt$$
$$= \frac{\pi z^{-1/4}}{2}\sum_{n=0}^\infty (-z^{-1/4}/4)^n/[(2n+1)(n+1)!],$$

$$\int_0^\infty e^{-zt}[V(0, y) - V(t^{1/4}, y)]t^{-5/4}dt$$
$$= \frac{\pi}{16}(y^5z^3)^{-1/4}\sum_{n=0}^\infty (-8y^{1/2}z^{1/2})^{-n}/[(4n+3)(2n+2)!],$$

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[6] This is a special case of the equation $\partial^m V/\partial x^m = \partial^n V/\partial y^n$ discussed by Poisson, J. École Polytech. vol. 12 (1823) pp. 464–474.

$$\int_0^\infty e^{-st}V_2(t^{1/4}, y)t^{-5/4}dt$$
$$= \frac{\pi y^{-3/4}z^{-1/4}}{2}\sum_{n=0}^{\infty}(-8y^{1/2}z^{1/2})^{-n}/[(4n+1)(2n+1)!].$$

For the solution of the general equation of Poisson, reference may be made to H. Block, Arkiv för Matematik, Astronomi och Fysik vol. 8 (1913), G. Mikaelsson, Dissertation, Upsala, 1920 and E. Rothe, Math. Zeit. vol. 28 (1928) pp. 48–72.

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TWO PROPERTIES OF THE FUNCTION COS x

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The function $f(x)=A \cos n(x+B)$, where A, B are any real constants and n is an integer, has the properties:

(I) $f(x)$ is real valued for all real x, of period 2π, and continuous.

(II) $f(x)$ is differentiable, and there exist constants a, b such that, for all x,

$$f'(x) = af(x+b).$$

(III) Given any constants a, a', b, b', there exist constants c, d such that, for all x,

$$af(x+a') + bf(x+b') = cf(x+d).$$

The object of this note is to show that, conversely, any function $f(x)$ which has property (I) and either (II) or (III) is necessarily of the form $f(x)=A \cos n(x+B)$. The latter result is used to derive the parallelogram law of addition of forces from certain other basic assumptions.

THEOREM 1. *Let $f(x)$ have properties* (I) *and* (II). *Then there exist constants A, B and an integer n such that $f(x)=A \cos n(x+B)$.*

PROOF. It follows from (II) that $f(x)$ is of class C^∞ and hence, from (I), can be represented by a convergent Fourier series, which, moreover, may be differentiated termwise. Thus for some complex constants k_n,

$$(1)\qquad \begin{aligned} f(x) &= \sum k_n e^{inx}, \qquad f'(x) = \sum ink_n e^{inx}, \\ f'(x) - af(x+b) &= \sum k_n(in - ae^{inb})e^{inx}. \end{aligned}$$

It follows from (II) that for every integer n,

$$(2)\qquad k_n(in - ae^{inb}) = 0.$$

If $f(x)\equiv 0$ then the theorem is trivial. Otherwise, there will exist an n for which $k_n\neq 0$. It follows that

$$(3)\qquad in = ae^{inb}.$$

Taking absolute values we have

$$(4)\qquad n = \pm a.$$

Thus there can be at most two values of n for which $k_n\neq 0$, and these

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values are negatives of one another. Thus for some integer n,

$$f(x) = k_{-n}e^{-inx} + k_ne^{inx}. \tag{5}$$

Since, by (I), $f(x)$ is real valued, it follows that k_{-n} and k_n are complex conjugates, and the proof is complete.

THEOREM 2. *Let $f(x)$ have properties* (I) *and* (III). *Then there exist constants A, B and an integer n such that $f(x) = A \cos n(x+B)$.*

PROOF. Since, by (I), $f(x)$ is continuous and of period 2π, it possesses at least a formal Fourier series,[1]

$$f(x) \sim \sum k_ne^{inx}, \qquad k_n = \frac{1}{2\pi}\int_0^{2\pi} f(t)e^{-int}dt. \tag{6}$$

By (III), there exist constants c, d such that the function

$$g(x) = f(x+1) + f(x) - cf(x+d), \tag{7}$$

continuous and of period 2π, is identically zero. (This is the only consequence of (III) that we shall use.) Hence

$$0 \sim \sum k_n(e^{in} + 1 - ce^{ind})e^{inx}. \tag{8}$$

It follows that for every n,

$$k_n(e^{in} + 1 - ce^{ind}) = 0. \tag{9}$$

If $f(x) \equiv 0$ then the theorem is trivial. Otherwise, there will exist an n for which $k_n \neq 0$. For any such n it follows from (9) that

$$e^{in} + 1 = ce^{ind}. \tag{10}$$

Taking absolute values and squaring, it follows that

$$\cos n = (c^2 - 2)/2. \tag{11}$$

Hence if m and n are any two integers for which $k_m \cdot k_n \neq 0$, it follows from (11) that $\cos m = \cos n$. Hence for some integer r,

$$m = \pm n + 2\pi r. \tag{12}$$

Since π is irrational, it follows that $r = 0$ and $m = \pm n$. Thus the formal Fourier series for $f(x)$ consists of only two terms,

$$f(x) \sim k_{-n}e^{-inx} + k_ne^{inx}. \tag{13}$$

But in this case, since the functions e^{inx} are complete with respect to continuous functions, the relation $\sim$ can be replaced by an identity,

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[1] The proof given here follows a suggestion of Paul R. Halmos. The author's original proof required the unnecessary assumption that $f(x)$ be of class C^1.

$$f(x) = k_{-n}e^{-inx} + k_n e^{inx}. \tag{14}$$

Since $f(x)$ is real valued, k_{-n} and k_n must be complex conjugates, and the theorem is proved.

We shall now apply Theorem 2 to derive the law of addition of forces.[2] For simplicity, let us consider only forces acting at a fixed point in a fixed plane in which the angular coordinate x is defined. With such a force we identify the real valued function $F(x)$ which specifies the scalar component of the force in the direction x; thus a force is represented by a real valued function of period 2π. By the sum of two forces $F_1(x)$ and $F_2(x)$ we mean the function $F_1(x)+F_2(x)$. Our assumptions are the following.

(i) *All forces are geometrically similar.* By this we mean that there exists a fixed function $f(x)$ of period 2π such that any force $F(x)$ can be written in the form

$$F(x) = A_F \cdot f(x + \alpha_F), \tag{15}$$

where A_F and α_F are constants determined by $F(x)$. We need not assume that all values of the constants A_F and α_F can occur in (15), but we shall assume that there exist at least the forces $F_1(x)=f(x)$ and $F_2(x)=f(x+1)$.

(ii) *The sum of two forces is a force.* Together with (i), this implies that the function $f(x)$ has the property that for certain constants c, d and for every x,

$$f(x + 1) + f(x) = cf(x + d). \tag{16}$$

(iii) *The function $f(x)$ is continuous, non-constant, and vanishes for at most two values in the interval* $0 \leq x < 2\pi$.

The proof of Theorem 2 shows that the function $f(x)$, continuous, real valued, of period 2π, and satisfying (16), must be of the form

$$f(x) = A \cos n(x + B), \tag{17}$$

where n is an integer. The hypotheses of (iii) ensure that n can be chosen as 1. The parallelogram law of addition of forces is an immediate consequence.

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[2] See G. Darboux, Bull. Sci. Math. vol. 9 (1875) pp. 281–288; also G. D. Birkhoff, Rice Institute Pamphlet vol. 28, no. 1 (1941) pp. 46–50.

ON SOME SPECIAL DIOPHANTINE EQUATIONS

E. ROSENTHALL

1. **Introduction.** The following lemma is fundamental in the algorithm of reciprocal arrays for the solution of multiplicative diophantine equations in certain arithmetics as developed by E. T. Bell.[1]

LEMMA 1. *All sets of integers satisfying the diophantine equation*

$$xy = zw$$

are given by

$$x = ab, \qquad y = cd, \qquad z = ad, \qquad w = cb$$

and it suffices to take b and d coprime.

Rational arithmetic and the theory of ideals in an algebraic number field provide instances of these arithmetics, and in all cases the fundamental theorem of unique decomposition into prime factors is required.

By use of this algorithm Bell has obtained the complete solution of a large class of diophantine equations, and by means of an application of Lemma 1 (and results derived from it) to equations reducible in particular algebraic number fields the present writer has obtained the complete solution of some interesting diophantine equations.

In this paper Lemma 1 is generalized to an arbitrary algebraic field, and the method of proof is then applied to a multiplicative equation from which we immediately obtain a formula exhibiting all the rational integers satisfying $x^2+ay^2=z^{2n+1}$. The procedure is to replace the algebraic indeterminates in the given multiplicative equation by the principal ideals they generate; then solving this equation by the method of arrays we obtain the solution in terms of ideals. In this solution, by a use of properties of equivalent ideals all the ideals are replaced by suitable principal ideals and the complete solution of the given equation is deduced.

2. **Notations.** We shall adhere to the following notations: small letters $a, b, \cdots$ are reserved for the rational integers, the capital letters $A, B, \cdots$ (except E) for integers of the algebraic number field $\mathfrak{F}$; the letter ϵ will be reserved for the units of this field, and all other Greek letters (with or without subscripts) will denote ideals of $\mathfrak{F}$.

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[1] E. T. Bell, *Reciprocal arrays and diophantine analysis*, Amer. J. Math. vol. 35 (1933) pp. 50–66.

Parentheses enclosing a letter denote the corresponding principal ideal; thus (X), (e), $\cdots$. The conjugates of the ideal ξ are represented by ξ', ξ'', $\cdots$, $\xi^{(n-1)}$. Two ideals, α and β, are said to be equivalent if an ideal γ exists such that the products $\alpha\gamma$, $\beta\gamma$ are both principal ideals; the equivalence of α and β is expressed by $\alpha\sim\beta$.

3. **Generalization of Lemma 1.** We shall now prove the following result.

THEOREM 1. *All integers X, Y, Z, W satisfying*

$$XY = ZW \tag{3.1}$$

are given by

$$X = ST/e, \qquad Y = UV/e, \qquad Z = SV/e, \qquad W = UT/e,$$

where S, T, U, V are arbitrary and e takes only the finite set of rational integral values each equal to the norm of a representative ideal from each class.

PROOF. By Lemma 1 all solutions of

$$(X)(Y) = (Z)(W) \tag{3.2}$$

are obtainable from

$$(X) = \alpha\beta, \qquad (Y) = \gamma\delta, \qquad (Z) = \alpha\delta, \qquad (W) = \gamma\beta,$$

and the ideals of the right-hand members must be restricted to the values which make the left-hand members principal ideals. Since the products $\alpha\beta$, $\gamma\beta$ are to be principal ideals then $\alpha\sim\gamma$. Let ξ be a representative ideal of each class in which α is a member. Then $\alpha\sim\xi$ and it follows that

$$\alpha\xi'\xi'' \cdots \xi^{(n-1)} \sim \xi\xi' \cdots \xi^{(n-1)} = (e).$$

Hence $\alpha\xi'\xi'' \cdots \xi^{(n-1)}$, and consequently $\gamma\xi'\xi'' \cdots \xi^{(n-1)}$, is equivalent to a principal ideal and must therefore itself be a principal ideal (A). Therefore,

$$(eX) = \alpha\beta\xi\xi' \cdots \xi^{(n-1)} = (A)\xi\beta$$

whence $\xi\beta$ is a principal ideal. Put $\xi\beta=(B)$ and we have $(eX)=(AB)$. Also

$$(eY) = \xi\xi' \cdots \xi^{(n-1)}\gamma\delta = (C)\xi\delta = (C)(D)$$

since $\xi\delta$ must be a principal ideal (D).

Thus all solutions of (3.2) are obtainable from

$$(eX) = (AB), \qquad (eY) = (CD), \qquad (eZ) = (AD), \qquad (eW) = (CB).$$

This implies that the set of all integers satisfying (3.1) is given by

$$eX = \epsilon_1 AB, \qquad eY = \epsilon_2 CD, \qquad eZ = \epsilon_3 AD, \qquad eW = \epsilon_4 CB,$$

where $\epsilon_1\epsilon_2=\epsilon_3\epsilon_4$. Make the reversible substitution $A=\epsilon_1^{-1}S$, $B=T$, $D=\epsilon_1\epsilon_3^{-1}V$, $C=\epsilon_4^{-1}U$ and we have the required result.

4. **The equation** $X\overline{X}=z^{2n+1}$. Hereafter the field $\mathfrak{F}$ is an arbitrary quadratic number field and the conjugate of an integer X and an ideal α are denoted by $\overline{X}$ and $\bar{\alpha}$ respectively.

The following theorem will be verified by induction.

THEOREM 2. *All solutions of*

$$\alpha\bar{\alpha} = (z^{2n+1})$$

are given by

$$\alpha = \phi_1^{2n+1}\phi_2^{2n}\bar{\phi}_2\phi_3^{2n-1}\bar{\phi}_3^{2}\cdots\phi_{n+1}^{n+1}\bar{\phi}_{n+1}^{n}, \qquad (z) = \phi_1\bar{\phi}_1\phi_2\bar{\phi}_2\cdots\phi_{n+1}\bar{\phi}_{n+1}.$$

From this we can deduce the following result.[2]

THEOREM 3. *All solutions of*

$$X\overline{X} = z^{2n+1} \tag{4.1}$$

are given by

$$\begin{aligned} E^{2n+1}X &= \pm H_1^{2n+1}H_2^{2n}\overline{H}_2\cdots H_{n+1}^{n+1}\overline{H}_{n+1}^{n} \\ E^{2}z &= H_1\overline{H}_1H_2\overline{H}_2\cdots H_{n+1}\overline{H}_{n+1}, \end{aligned} \tag{4.2}$$

where $E=e_1^{n+1}e_2^{n}\cdots e_n^{2}$; $e_1, e_2, \cdots, e_n$ *each being equal to the norm of a representative ideal from each class.*

If we put $X=x+(-a)^{1/2}y$, then (4.1) becomes $x^2+ay^2=z^{2n+1}$ and equating rational and irrational parts we obtain from (4.2) an explicit representation for all rational integers satisfying this equation. Although in general the solution appears in rational form, yet for each value of a the indeterminates x, y, z can be expressed by a finite number of polynomials in integral parameters. For, for each value of a the parameter E has only a finite set of integral values and the requirement that the right-hand members of (4.2) be divisible by

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[2] For an account of the investigations on equation $x^2+ay^2=z^n$ see L. E. Dickson, *History of the theory of numbers*, vol. 2, pp. 534–543; Th. Skolem, *Diophantische Gleichungen*, Berlin, 1938, pp. 64–68; J. V. Uspensky and M. A. Heaslet, *Elementary number theory*, 1939, pp. 389–396.

E^{2n+1}, E^2 respectively may be expressed by congruential conditions upon the coordinates of the integers H_i.

EXAMPLE. From Theorem 3 it follows that all the rational integers satisfying $x^2+47y^2=z^3$ are given by

$$E^3(x+(-47)^{1/2}y) = \pm H_1^3 H_2^2 \overline{H}_2, \qquad E^2 z = H_1\overline{H}_1 H_2 \overline{H}_2,$$

where $E=e_1^2=1$, 2^2, 3^2 and H_1, H_2 are integers of $Ra(-47)^{1/2}$. It suffices to take H_2 primitive.

In order to select all integers from the above rational forms of x, y, z, it is necessary and sufficient to impose the following congruential conditions upon the coordinates of $H_1=r+sW$, $H_2=m+nW$ where $W=(1+(-47)^{1/2})/2$:

(i) $E=1$. If n even then r, s even; otherwise no restrictions on r, s.

(ii) $E=2^2$. m odd, n even, $s\equiv 2r\equiv 0 \pmod 8$; $m-4n\equiv 2 \pmod 4$, $r\equiv s\equiv 0 \pmod 4$; $m-4n\equiv 4 \pmod 8$, $r\equiv s\equiv 0$, $2 \pmod 4$; $m-4n\equiv 8 \pmod{16}$, $r\equiv s\equiv 0$, $2 \pmod 4$; $m-4n\equiv 0$, $\pm 16 \pmod{64}$, $r\equiv s \pmod 2$; $m-4n\equiv 32 \pmod{64}$ requires no restriction on r, s.

m odd, $m+5n\equiv 2 \pmod 4$, $r\equiv s\equiv 0 \pmod 4$; $m+5n\equiv 4 \pmod 8$, $r\equiv s\equiv 0 \pmod 4$; $m+5n\equiv 8 \pmod{16}$, $r\equiv 2s \pmod 4$; $m+5n\equiv 0$, $\pm 16 \pmod{64}$, r even; $m+5n\equiv 32 \pmod{64}$ requires no restriction on r and s.

(iii) $E=3^2$. $m(m+n)\not\equiv 0 \pmod 3$, $r\equiv s\equiv 0 \pmod 9$; $m-6n\equiv \pm 3 \pmod 9$, $r\equiv s\equiv 0 \pmod 9$; $m-6n\equiv \pm 9 \pmod{27}$, $r+s\equiv 0 \pmod 9$; $m-6n\equiv 0 \pmod{27}$, but $m-114n\not\equiv 0 \pmod{3^6}$, $r+s\equiv 0 \pmod 3$; $m-114n\equiv 0 \pmod{3^6}$ requires no restriction on r and s.

$m+7n\equiv \pm 3 \pmod 9$, $r\equiv s\equiv 0 \pmod 9$; $m+7n\equiv \pm 9 \pmod{27}$, $r\equiv 3s\equiv 0 \pmod 9$; $m+7n\equiv 0 \pmod{27}$ but $m+115n\not\equiv 0 \pmod{3^6}$, $r\equiv 0 \pmod 3$; $m+115n\equiv 0 \pmod{3^6}$ requires no restriction on r and s.

If in addition to the above n is even then r and s must both be even.

The above conditions were obtained by considering in turn the cases according as $H_2\overline{H}_2$ is divisible by e_1, e_1^2, $\cdots$, e_1^6.

For the proof of Theorem 2 we use the following two lemmas.

LEMMA 2. *All solutions of* $\alpha\bar{\alpha}=\gamma\beta\bar{\beta}$ *are given by* $\alpha=\theta_1\theta_2\theta_3$, $\beta=\theta_1\bar{\theta}_2$, $\gamma=\theta_3\bar{\theta}_3$.

LEMMA 3. *All solutions of* $\alpha\bar{\alpha}=\beta\gamma$ *are given by* $\alpha=\theta_1\theta_2\theta_3\theta_4$, $\beta=\theta_1\bar{\theta}_2\theta_3\bar{\theta}_3$, $\gamma=\bar{\theta}_1\theta_2\theta_4\bar{\theta}_4$.

The proof for Lemma 2 is exactly as given in a previous paper[3] where the indeterminates were integers of a unique factorization quadratic field; Lemma 3 follows immediately by a repeated application of Lemma 1.

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[3] E. Rosenthall, *On some cubic diophantine equations*, Amer. J. Math. vol. 65 (1943) pp. 664–665.

5. **Proof of Theorem 2.** We now show independently that Theorem 2 holds for $n=1$ and then complete the proof by mathematical induction. From Lemma 2 it follows that all α, (z) satisfying $\alpha\bar{\alpha}=(z^3)$ are obtainable from

$$\alpha = \theta_1\theta_2\theta_3, \qquad (z) = \theta_1\bar{\theta}_2 = \theta_3\bar{\theta}_3,$$

and applying Lemma 3 it follows that

$$\theta_3 = \lambda_1\lambda_2\lambda_3\lambda_4, \qquad \theta_1 = \lambda_1\bar{\lambda}_2\lambda_3\bar{\lambda}_3, \qquad \bar{\theta}_2 = \bar{\lambda}_1\lambda_2\lambda_4\bar{\lambda}_4.$$

Then

$$\alpha = \phi_1^3\phi_2^2\bar{\phi}_2, \qquad (z) = \phi_1\bar{\phi}_1\phi_2\bar{\phi}_2,$$

where we have put $\lambda_1=\phi_1$, $\bar{\lambda}_2\lambda_3\lambda_4=\phi_2$ since λ_2, λ_3, λ_4 always appear in this product form.

Now consider the induction from $n=s$ to $n=s+1$. Apply Lemma 2 to $\alpha\bar{\alpha}=(z^{2s+3})=(z^{2s+1})(z)(z)$. Then

$$\alpha = \theta_1\theta_2\theta_3, \qquad (z) = \theta_1\bar{\theta}_2, \qquad (z^{2s+1}) = \theta_3\bar{\theta}_3.$$

By hypothesis, Theorem 2 holds for the last equation. Hence

$$\theta_3 = \mu_1^{2s+1}\mu_2^{2s}\bar{\mu}_2 \cdots \mu_{s+1}^{s+1}\bar{\mu}_{s+1}^{s}, \qquad (z) = \mu_1\bar{\mu}_1\mu_2\bar{\mu}_2 \cdots \mu_{s+1}\bar{\mu}_{s+1};$$

and equating the two parametric representations for (z) we obtain the following equations for all the parameters concerned,

$$\{\mu_1\mu_2 \cdots \mu_{s+1}\}\{\bar{\mu}_1\bar{\mu}_2 \cdots \bar{\mu}_{s+1}\} = \theta_1\bar{\theta}_2.$$

Applying Lemma 3,

$$\mu_1\mu_2 \cdots \mu_{s+1} = \lambda_1\lambda_2\lambda_3\lambda_4, \qquad \theta_1 = \lambda_1\bar{\lambda}_2\lambda_3\bar{\lambda}_3, \qquad \bar{\theta}_2 = \bar{\lambda}_1\lambda_2\lambda_4\bar{\lambda}_4.$$

Substituting these values of the parameters in the preceding formulas for α, (z) gives

$$\alpha = \lambda_1^2\bar{\lambda}_2^2\psi_1\bar{\psi}_1\mu_1^{2s+1}\mu_2^{2s}\bar{\mu}_2 \cdots \mu_{s+1}^{s+1}\bar{\mu}_{s+1}^{s}, \qquad (z) = \lambda_1\bar{\lambda}_1\lambda_2\bar{\lambda}_2\psi_1\bar{\psi}_1, \tag{5.1}$$

where the parameters must satisfy the multiplicative equation

$$\mu_1\mu_2 \cdots \mu_{s+1} = \lambda_1\lambda_2\psi_1, \tag{5.2}$$

where we have put $\lambda_3\lambda_4=\psi_1$.

Applying the method of arrays[4] to (5.2) we obtain

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[4] See E. T. Bell, *loc. cit.*, p. 62.

$$\begin{aligned} \mu_1 &= \xi_1\eta_1\zeta_1, & \lambda_1 &= \xi_1\xi_2 \cdots \xi_{s+1},\\ \mu_2 &= \xi_2\eta_2\zeta_2, & \lambda_2 &= \eta_1\eta_2 \cdots \eta_{s+1},\\ &\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ , & \psi_1 &= \zeta_1\zeta_2 \cdots \zeta_{s+1}.\\ &\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ , & &\\ \mu_{s+1} &= \xi_{s+1}\eta_{s+1}\zeta_{s+1}. & & \end{aligned}$$

Finally, substituting in (5.1) gives

$$\alpha = \phi_1^{2s+3}\phi_2^{2s+2}\bar{\phi}_2 \cdots \phi_{s+1}^{s+3}\bar{\phi}_{s+1}^{s}\phi_{s+2}^{s+2}\bar{\phi}_{s+2}^{s+1}, \qquad (z) = \phi_1\bar{\phi}_1\phi_2\bar{\phi}_2 \cdots \phi_{s+2}\bar{\phi}_{s+2},$$

where we have put $\xi_1=\phi_1$, $\xi_2\zeta_1=\phi_2$, $\zeta_{s+1}\eta_s\bar{\eta}_{s+1}=\phi_{s+2}$, and $\xi_i\zeta_{i-1}\eta_{i-2}=\phi_i$ for $i=3, 4, \cdots, s+1$, since the parameters always occur in these product forms. This completes the induction.

6. Proof of Theorem 3.

All solutions of

$$(X)(\bar{X}) = (z^{2n+1}) \tag{6.1}$$

are obtainable from

$$(X) = \phi_1^{2n+1}\phi_2^{2n}\bar{\phi}_2 \cdots \phi_{n+1}^{n+1}\bar{\phi}_{n+1}^{n}, \qquad (z) = \phi_1\bar{\phi}_1\phi_2\bar{\phi}_2 \cdots \phi_{n+1}\bar{\phi}_{n+1}, \tag{6.2}$$

and the ideals of the right-hand members must be restricted to those values which make the left-hand members principal ideals. From $(6.2)_1$ it follows that $\phi_1^{2n+1}\phi_2^{2n-1} \cdots \phi_{n+1}\sim(1)$. Let ξ_i $(i=1, 2, \cdots, n)$ be a representative ideal from each of the classes in which ϕ_i is a member. Then we can put $\phi_i\bar{\xi}_i=(H_i)$ and $\xi_1^{2n+1}\xi_2^{2n-1} \cdots \xi_n^3\phi_{n+1}=(H_{n+1})$. Multiplying both sides of $(6.2)_1$ by $[\{\xi_1\bar{\xi}_1\}^{n+1}\{\xi_2\bar{\xi}_2\}^{n} \cdots \{\xi_n\bar{\xi}_n\}^{2}]^{2n+1}$ and $(6.2)_2$ by $[\{\xi_1\bar{\xi}_1\}^{n+1}\{\xi_2\bar{\xi}_2\}^{n} \cdots \{\xi_n\bar{\xi}_n\}^{2}]^{2}$ and also putting $\xi_i\bar{\xi}_i=(e_i)$ we obtain

$$\begin{gathered} (E^{2n+1}X) = (H_1^{2n+1}H_2^{2n}\bar{H}_2 \cdots H_{n+1}^{n+1}\bar{H}_{n+1}^{n}),\\ (E^{2}z) = (H_1\bar{H}_1H_2\bar{H}_2 \cdots H_{n+1}\bar{H}_{n+1}) \end{gathered}$$

for all solutions of (6.1). This implies that all solutions of (4.1) are given by

$$E^{2n+1}X = \epsilon_1H_1^{2n+1}H_2^{2n}\bar{H}_2 \cdots H_{n+1}^{n+1}\bar{H}_{n+1}^{n}, \qquad E^{2}z = \epsilon_2H_1\bar{H}_1 \cdots H_{n+1}\bar{H}_{n+1}.$$

Make the reversible substitution $H_{n+1}=\bar{\epsilon}_1H_{n+1}$, and then observe that $\epsilon_2\epsilon_1\bar{\epsilon}_1$ must be unity. Hence Theorem 3 is proved.

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A NOTE ON CHAIN CONDITIONS IN NILPOTENT RINGS AND GROUPS

S. A. JENNINGS

Maximal and minimal conditions for ideals in associative rings have often been considered, but little seems to be known of these conditions in non-associative rings, or of chain conditions on the non-normal subgroups of a group. Moreover, it is usual to assume the condition for one-sided ideals in noncommutative rings, and the weaker condition for two-sided ideals rarely appears. In this note we first consider a class of groups which are "nilpotent" with respect to a set of operators Ω. These groups include ordinary nilpotent groups, and associative and non-associative nilpotent rings and algebras as special cases. Our main theorem is to the effect that, for an Ω-nilpotent group, the maximal or minimal condition for Ω-subgroups implies the corresponding condition for all subgroups. As immediate consequences of this theorem it follows that, for nilpotent rings and algebras, the maximal or minimal condition for ideals implies the corresponding condition for modules, while for nilpotent groups, the maximal or minimal condition for normal subgroups implies the corresponding condition for all subgroups.

1. **Definition of Ω-nilpotency.** Let R be a group, and let Ω be a set of (left) operators of R which includes the inner automorphisms of R. R may also admit a second set of operators Φ, in which case it will be understood in what follows that all subgroups are supposed to admit Φ. The statement that a certain subgroup is an *Ω-subgroup* means that this subgroup admits Ω as well as Φ.

If A is any Ω-subgroup of R, A is normal, and we may suppose that R/A admits the same sets of operators, Ω and Φ, as R.

We take the following as our

DEFINITION OF AN Ω-NILPOTENT GROUP.[1] *R is Ω-nilpotent if there exists a strictly decreasing chain of Ω-subgroups*

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[1] Originally I discussed the chain theorems for groups for and rings separately, and while it was clear that the results were analogous, I did not succeed in unifying the proofs for the two cases. The referee, Dr. Irving Kaplansky, pointed out, however, that this could be accomplished by the use of the general notion of Ω-nilpotency which we give here, and which was suggested by him. I would like to express my thanks to the referee for his contribution to this paper.

$$R = A_1 \supset A_2 \supset A_3 \supset \cdots \supset A_m \supset A_{m+1} = 1, \tag{1}$$

terminating with the identity, and such that if $\lambda \in \Omega$ *and* $a_i \in A_i$ *then*

$$a_i^{-r}(\lambda a_i) \in A_{i+1}, \qquad i = 1, 2, \cdots, m, \tag{2}$$

for some r, *where in general* r *may be any positive or negative integer, or zero, but if* λ *happens to be an inner automorphism then* r *must equal one.*

The minimum length of all chains satisfying (1) and (2) above will be called the *exponent* of R and any chain of Ω-subgroups satisfying these conditions and having minimum length will be called a *minimum chain.*

The following results are needed for our later induction.

LEMMA A. *If* R *is an* Ω*-nilpotent group of exponent* m, *and*

$$R = A_1 \supset A_2 \supset \cdots \supset A_m \supset A_{m+1} = 1$$

is a minimum chain, then R/A_m *is an* Ω*-nilpotent group of exponent* $m-1$.

LEMMA B. *The subgroup* A_m *of any chain satisfying* (1) *and* (2) *above is contained in the centre of* R. *Moreover, every subgroup of* A_m *is an* Ω*-subgroup of* R.

LEMMA C. *If* A *is any* Ω*-subgroup of* R, *then the maximal or minimal condition for* Ω*-subgroups of* R *implies the corresponding condition for* Ω*-subgroups in* R/A.

Lemmas A and C may be readily verified. To prove Lemma B, we note that if $a_m \in A_m$, $\lambda \in \Omega$, then $a_m^{-r}(\lambda a_m) = 1$, since $A_{m+1} = 1$, and hence $\lambda a_m = a_m^r$, which implies that every subgroup of A_m admits Ω. Again, if $x \in R$, $x^{-1}a_m x = a_m$, since for inner automorphisms $r = 1$, and hence very element a_m of A_m is in the centre of R.

2. **The chain theorem.** We are now in a position to prove the following

THEOREM. *Let* R *be an* Ω*-nilpotent group. The maximal or minimal condition for* Ω*-subgroups in* R *implies the maximal or minimal condition respectively for all subgroups in* R.

We prove that if the maximal conditions for Ω-subgroups holds in R, the maximal condition for all subgroups also holds; the corresponding result for the minimal condition may be proved in a similar way.

Suppose that R has exponent m, and satisfies the maximal condition for Ω-subgroups. If $m=1$, our result is true by Lemma B. We proceed by induction and assume that the result holds for groups with exponent at most $m-1$, and hence, by Lemma C and Lemma A, for $\bar{R}=R/A_m$. Let

$$M_1 \subseteq M_2 \subseteq M_3 \subseteq \cdots$$

be an ascending chain of subgroups of R, and let $\bar{M}_i=(M_i \cup A_m)/A_m$ be the image in $\bar{R}$ of M_i. We have clearly

$$\bar{M}_1 \subseteq \bar{M}_2 \subseteq \bar{M}_3 \subseteq \cdots,$$

an ascending chain of subgroups in $\bar{R}$. Now the maximal condition for subgroups holds in $\bar{R}$, since the condition for Ω-subgroups holds in $\bar{R}$ by Lemma C, and hence there exists an integer j such that

$$\bar{M}_i = \bar{M}_{i+1} \qquad \text{for } i \geqq j.$$

This implies that

$$M_i \cup A_m = M_{i+1} \cup A_m \qquad \text{for } i \geqq j.$$

Consider $N_i = M_i \cap A_m$; clearly N_i is a subgroup of A_m, and hence by Lemma B is an Ω-subgroup of R, and we have

$$N_1 \subseteq N_2 \subseteq N_3 \subseteq \cdots,$$

an ascending chain of Ω-subgroups of R. Hence there exists an integer k such that

$$N_i = N_{i+1} \qquad \text{for } i \geqq k.$$

Under our hypothesis, therefore, we see that, provided $i \geqq \max(j, k)$,

$$\begin{aligned} &M_i \subseteq M_{i+1}, \\ (3) \qquad &M_i \cup A_m = M_{i+1} \cup A_m, \\ &M_i \cap A_m = M_{i+1} \cap A_m. \end{aligned}$$

Now A_m belongs to the centre of R, and hence M_i is normal in $M_i \cup A_m$, and therefore normal in $M_{i+1} \cup A_m$: also M_{i+1} is normal in $M_{i+1} \cup A_m$. Therefore the five subgroups M_i, M_{i+1}, $M_i \cup A_m = M_{i+1} \cup A_m$, $M_i \cap A_m = M_{i+1} \cap A_m$, A_m are all normal subgroups of $M_{i+1} \cup A_m$, and hence form a modular lattice, and hence the conditions (3) above imply $M_i = M_{i+1}$, which proves our theorem.

3. **The chain theorem of nilpotent groups.** Consider now the case where R is a nilpotent group in the usual sense.[2] It is easily seen that, if Ω is the set of inner automorphisms, R is Ω-nilpotent. For let

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[2] For definitions and properties of nilpotent groups see, for example, H. Zassenhaus, *Gruppentheorie*, vol. 1, Leipzig, 1937, pp. 118–119.

$$R = A_1 \supset A_2 \supset \cdots \supset A_m \supset A_{m+1} = 1$$

be any strictly decreasing central series of R (for example, the lower central series); we have always, if $a_i \in A_i$, $x \in R$,

$$(a_i^{-1} x^{-1} a_i x) \in A_{i+1},$$

which is precisely condition (2) above applied to an inner automorphism, with $r=1$. Since the Ω-subgroups are precisely the normal subgroups of R, we have:

"In a nilpotent group the maximal or minimal condition for normal subgroups implies the corresponding condition for all subgroups."

4. The chain theorem for nilpotent rings.

The definition of nilpotency is not well established for general non-associative rings. We give here a definition which applies to any ring, associative or not, and which is equivalent to the familiar one in the case of associative and Lie rings. This particular definition seems natural, and the definition of an Ω-nilpotent group as given earlier is a further generalization.

We shall say that R is a generalized ring, or briefly a ***ring***, which admits Φ as a set of (left) operators if the following conditions hold:

(a) R is an abelian group which admits Φ.

(b) If $a, x \in R$, then $ax \in R$, that is, R is closed under multiplication.

(c) If $a, b, x \in R$, then $x(a+b) = xa + xb$, $(a+b)x = ax + bx$, that is, the multiplications are left and right operators of R.

It is clear that associative and Lie rings and algebras, and hypercomplex systems both associative and non-associative, are rings in the above sense. We may also suppose that Φ is not empty, since the rational integers may always be adjoined to any domain of operators. By a *module* of R we mean an additive subgroup of R which admits Φ, that is, a module is a Φ-subgroup. By a *left, right or two-sided ideal* of R we mean a module which admits the elements of R as left, right or two-sided operators respectively. Unless otherwise stated, we shall understand by ideal a two-sided ideal.

If A is any ideal of R, the residue classes modulo A form a generalized ring R/A which is homomorphic to R, and we may suppose that R and R/A have the same set of operators.

Because our rings are not necessarily associative, we take the following as our

DEFINITION OF A NILPOTENT RING. *A generalized ring R will be said to be nilpotent if it has a strictly decreasing chain of ideals*

$$R = A_1 \supset A_2 \supset A_3 \supset \cdots \supset A_m \supset A_{m+1} = 0,$$

terminating with zero, such that if $x \in R$ *and* $a_i \in A_i$ *then*

$$xa_i \in A_{i+1}, \qquad a_i x \in A_{i+1}.$$

If R is an associative or Lie ring or algebra, the above definition of nilpotency is obviously equivalent to the usual one.

It is clear that a generalized ring which is nilpotent as defined above is an Ω-nilpotent abelian group, where Ω consists of the right and left multiplications of R, and the integer r of condition (2) above always equals zero. The Ω-subgroups are precisely the ideals of R, and our earlier theorem implies that

"*In any generalized nilpotent ring (and in particular, in any nilpotent associative or Lie ring or algebra) the maximal or minimal condition for ideals implies the corresponding condition for modules.*"

THE UNIVERSITY OF BRITISH COLUMBIA

SUBDIRECT UNIONS IN UNIVERSAL ALGEBRA

GARRETT BIRKHOFF

1. Preliminary definitions. By an *algebra*, we shall mean below any collection A of elements, combined by any set of single-valued operations f_α,

$$y = f_\alpha(x_1, \cdots, x_{n(\alpha)}). \tag{1}$$

The number of distinct operations (that is, the range of the variable α) may be infinite, but for our main result (Theorem 2), we shall require every $n(\alpha)$ to be finite—that is, it will concern *algebras with finitary operations*.

The concepts of *subalgebra*, *congruence relation* on an algebra, *homomorphism* of one algebra A onto (or into) another algebra with the same operations, and of the *direct union* $A_1 \times \cdots \times A_r$ of any finite or infinite class of algebras with the same operations have been developed elsewhere.[1] More or less trivial arguments establish a many-one correspondence between the congruence relations θ_i on an algebra A and the homomorphic images $H_i = \theta_i(A)$ of the algebra (isomorphic images being identified); moreover the congruence relations on A form a lattice (the *structure lattice* of A). In this lattice, the equality relation will be denoted 0; all other congruence relations will be called *proper*.

More or less trivial arguments also show (cf. *Lattice theory*, Theorem 3.20) that the isomorphic representations of any algebra A as a *subdirect union*, or subalgebra $S \leqq H_1 \times \cdots \times H_r$ of a direct union of algebras H_i, correspond essentially one-one to the sets of congruence relations θ_i on A such that $\Lambda\theta_i = 0$. In fact, given such a set of θ_i, the correspondence

$$\theta: \quad a \rightarrow [\theta_1(a), \cdots, \theta_r(a)] = [h_1, \cdots, h_r] \tag{2}$$

exhibits the desired isomorphism of A with a subalgebra of $H_1 \times \cdots \times H_r$, where $H_i = \theta_i(A)$. Incidentally, the number of S_i can

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[1] *On the structure of abstract algebras*, Proc. Cambridge Philos. Soc. vol. 31 (1935) pp. 433–454, and in the foreword to the author's *Lattice theory*. The idea of an abstract congruence relation is also developed in chap. VI, §14, of S. MacLane's and the author's *Survey of modern algebra*. Interesting remarks in this connection may be found in J. C. C. McKinsey's and A. Tarski's *The algebra of topology*, Ann. of Math. vol. 45 (1944) esp. pp. 190–191.

be infinite. What is more important, the operations of A need not even be finitary.[2]

Equally trivial arguments extend a well known theorem of Emmy Noether on commutative rings[3] to abstract algebras in general. In order to state this extension, we first define an algebra A to be *subdirectly irreducible* if in any finite or infinite representation (2), some θ_i is itself an isomorphism. This means that the meet θ^* of all proper congruence relations on A is itself a proper congruence relation. In lattice-theoretic language, it means that the structure lattice $L(A)$ of A contains a *point* $\theta^*>0$ such that $\theta>0$ implies $\theta\geqq\theta^*$. Hence if A is subdirectly irreducible, $\theta\cap\theta'=0$ in $L(A)$ implies $\theta=0$ or $\theta'=0$; such an A we shall call *weakly irreducible*. If $L(A)$ satisfies the *descending* chain condition, and A is weakly irreducible, then it is evidently also subdirectly irreducible in the strong sense.

If $L(A)$ satisfies the *ascending* chain condition, then it is evident by induction[3] that there exists a representation of 0 as the meet $0=\theta_1\cap\cdots\cap\theta_r$ of a finite number of irreducible elements. This yields the following easy generalization of Emmy Noether's theorem on commutative rings.

THEOREM 1. *Any algebra A whose structure lattice satisfies the ascending chain condition is isomorphic with a subdirect union of a finite number of weakly irreducible algebras.*

For this result, we still do not need to assume that A has finitary operations.

2. Main theorem. Our principal result is the partial extension of Theorem 1 to algebras *without* chain condition. As will be seen in §3, this result will contain as special cases many known theorems and some new theorems.

THEOREM 2. *Any algebra A with finitary operations is isomorphic with a subdirect union of subdirectly irreducible algebras.*

PROOF. For any $a\neq b$ of A, consider the set $S(a, b)$ of all congruence relations θ on A, such that $a\not\equiv b \pmod{\theta}$. If T is any linearly ordered subset of $S(a, b)$, the *union* τ of the $\theta\in T$ is defined by the rule

$$x \equiv y \pmod{\tau} \text{ means } x \equiv y \pmod{\theta} \text{ for some } \theta \in T. \tag{3}$$

It is evident that $a\not\equiv b \pmod{\tau}$, and that *if A has finitary operations,*

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[2] This is observed in N. H. McCoy and Deane Montgomery, *A representation of generalized Boolean rings*, Duke Math. J. vol. 3 (1937) p. 46, line 12.

[3] Cf. van der Waerden, *Moderne Algebra*, first ed., vol. 2, p. 36. The unicity theorem on p. 40 does not apply to abstract algebras in general, however.

then τ *is a congruence relation.* Hence, in the structure lattice $L(A)$ of A, the union of any linearly ordered subset of $S(a, b)$ exists and is in $S(a, b)$. But this is the hypothesis of the "first form" of Zorn's Lemma.[4] The conclusion is that $S(a, b)$ contains a *maximal* element, $\theta_{a,b}$. We next consider $H_{a,b}$, the homomorphic image of A, mod $\theta_{a,b}$.

Every proper congruence relation $\theta > 0$ corresponds[5] to a $\theta' > \theta_{a,b}$; and since $\theta_{a,b}$ is maximal in $S(a, b)$, this implies $a \equiv b \pmod{\theta'}$. Hence the *meet* θ^* of the $\theta > 0$ in $H_{a,b}$, defined by

$$(4) \qquad x \equiv y \pmod{\theta^*} \text{ means } x \equiv y \pmod{\theta} \text{ for all } \theta > 0,$$

will satisfy $a \equiv b \pmod{\theta^*}$, and hence $\theta^* > 0$. Hence (cf. §1) $H_{a,b}$ is subdirectly irreducible.

Finally, the meet of *all* $\theta_{a,b}$ is 0, since we have identically $x \not\equiv y \pmod{\theta_{x,y}}$. Hence, by the theorem cited in §1, paragraph 3, A is isomorphic with a subdirect union of the (subdirectly irreducible) $H_{a,b}$, q.e.d.

3. **Applications.**

Theorem 2 has importance mainly because subdirectly irreducible algebras may be specifically described in so many cases.

LEMMA 1. *A weakly irreducible distributive lattice or Boolean algebra must consist of* 0 *and* I *alone.*

PROOF FOR DISTRIBUTIVE LATTICES. For any a, the endomorphisms $\theta_a: x \to x \cup a$ and $\theta_a': x \to x \cap a$ have the property[6] that $\theta_a \cap \theta_a' = 0$, and neither defines the equality relation unless $a = 0$ or $a = I$.

PROOF FOR BOOLEAN ALGEBRAS. Let $x \equiv y \pmod{\theta_a}$ mean $|x - y| \leqq a$ (symmetric difference notation); then $\theta_a \cap \theta_{a'} = 0$, and neither defines the equality relation unless $a = 0$ or $a' = 0$ $(a = I)$.

COROLLARY 1. *Any distributive lattice is isomorphic with a ring of sets.*[7]

COROLLARY 2. *Any Boolean algebra is isomorphic with a field of sets.*[7]

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[4] Cf. J. W. Tukey, *Convergence and uniformity in topology*, Princeton, 1940, p. 7.

[5] We omit discussing the obvious isomorphism between $L(H_{a,b})$ and the sublattice of $\theta' > \theta_{a,b}$ in $L(A)$.

[6] PROOF. If $x \cup a = y \cup a$ and $x \cap a = y \cap a$, then $x = x \cap (x \cup a) = x \cap (y \cup a) = (x \cap y) \cup (x \cap a) = (y \cap x) \cup (y \cap a) = y \cap (x \cup a) = y \cap (y \cup a) = y$—by the distributive laws. θ_a and θ_a' are endomorphisms.

[7] Corollaries 1–2 are theorems of the author and Stone, respectively. Corollary 3 below is due to McCoy and Montgomery, op. cit. footnote 2. Corollary 4 is due to G. Kothe, *Abstrakte Theorie nichtkommutative Ringe*, Math. Ann. vol. 103 (1930) p. 552; N. H. McCoy *Subrings of infinite direct sums*, Duke Math. J. vol. 4 (1938) pp. 486–494.

LEMMA 2. *A subdirectly irreducible commutative ring R without nilpotent elements is a field.*[8]

PROOF. As in §1, R will have a unique minimal ideal (that is, congruence relation) J. But for any $a \neq 0$ in J, since $aa \neq 0$, $aJ > 0$. Moreover since $(aJ)R = a(JR) \leqq aJ$, aJ is an ideal, $0 < aJ \leqq J$. Consequently $aJ = J$—whence J is a *field* (Huntington's postulates) with unit e. The set $0:e$ of all $x \in R$ such that $ex = 0$ is an ideal, and $(0:e) \cap J = 0$ by what we have just shown; hence $0:e = 0$. But for any $x \in R$, $e(x - ex) = 0$; hence $(x - ex) \in 0:e = 0$, and $x = (x - ex) + ex$, $0 = ex \in J$. We conclude that $R = J$ is a field, q.e.d.

One might infer that by Theorem 2 any commutative ring without nilpotent elements was a subdirect product of fields, but this reasoning would be invalid. It is not necessarily true that a homomorphic image of rings without nilpotent elements is itself without nilpotent elements.

On the other hand, any homomorphic image of a *p-ring* (or commutative ring in which $a^p = a$ for some prime p) is itself a p-ring, and evidently without nilpotent elements, since $a^{p^n} = a$ for all n. Furthermore, a field in which $a^p = a$ can contain only p elements, and must be $GF(p)$ (or the "field" 0).

COROLLARY 3. *Any p-ring is a subdirect union of $GF(p)$, or consists of* 0 *alone.*[7]

Again, any homomorphic image of a *regular* ring in the sense of von Neumann (or ring in which any a has a "relative inverse" u such that $aua = a$) is itself regular, and evidently without nilpotent elements if commutative (since $a^n u^{n-1} = auau \cdots ua = a \neq 0$).

COROLLARY 4. *Any commutative "regular" ring is a subdirect union of fields.*[7]

If one were interested in obtaining corollaries of Theorem 1, one might show that even a *weakly* irreducible p-ring or regular ring was a field. Again, one might show (van der Waerden, op. cit. p. 32) that, in a weakly irreducible commutative ring satisfying the chain condition, every divisor of zero is nilpotent; this would yield E. Noether's theorem that every commutative ring satisfying the chain condition was a subdirect union of a finite number of primary rings.

Similarly, one can show easily that the only weakly irreducible vector space over a field F is the one-dimensional vector space $V(F; 1)$ (or 0). It follows that any vector space is a subdirect union of one-dimensional vector spaces. Actually (due to the existence of bases) a stronger result is well known.

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[8] This lemma was suggested to the author in conversation by N. H. McCoy.

LEMMA 3. *The only weakly irreducible commutative groups G are the "generalized cyclic" groups: the additive subgroups of the rationals, and those of the rationals* mod *one.*

We omit the proof, which follows easily from the fact that a commutative group with two generators is cyclic unless it contains two disjoint subgroups (the latter hypothesis would make G weakly reducible).

COROLLARY 5. *Any commutative group is a subdirect union of generalized cyclic groups.*

The center of any weakly irreducible hypercentral (alias nilpotent) group H is generalized cyclic (the proof is trivial, granted Lemma 3); the converse also holds if H is finite.[9] Hence we have the following corollary.

COROLLARY 6. *Any hypercentral group is a subdirect union of groups with generalized cyclic centers.*

Further, any weakly irreducible commutative l-group (lattice-ordered group) is known[10] to be *simply* ordered. This yields the following corollary.

COROLLARY 7. *Any commutative l-group is a subdirect union of simply ordered* 1*-groups.*

One can easily show (although we omit the proof) that any closed element in a closure algebra (in the sense of McKinsey and Tarski[11]) determines a congruence relation, essentially through relativization with respect to the complementary open set. Then from the definition of well-connectedness one obtains the following corollary.

COROLLARY 8. *Any "closure algebra" is a subdirect union of "well-connected" closure algebras.*

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[9] Theorems of Burnside (cf. H. Zassenhaus, *Gruppentheorie*, Teubner, 1937, p. 107, Satz. 11) and P. Hall, *A contribution to the theory of groups of prime-power orders*, Proc. London Math. Soc. vol. 36 (1933) p. 51, Theorem 2.49.

[10] Cf. the author's *Lattice-ordered groups*, Ann. of Math. vol. 43 (1942) p. 319.

[11] *The algebra of topology*, Ann. of Math. vol. 45 (1944) pp. 141–191. The definition of well-connectedness is on p. 147; the concept of relativization is developed on p. 151.

SIMPLE QUASIGROUPS

R. H. BRUCK

Introduction. A. A. Albert [1, II][1] has conjectured that *there exist simple loops of every finite order except order* 4. This conjecture is established in §1 by the construction of what we call *hyperabelian loops.*[2] In §2 other simple loops are constructed, in particular, loops of order $2f+1$ with subloops of order f. The concluding section of the paper is devoted to an investigation of the relationship between non-simple quasigroups (of finite or infinite order) and the loops isotopic to them (Theorem V). Theorems I and V of the paper have elsewhere been announced without proof by the author [2, §10]. For the purposes of §§1 and 2 none of the refinements of the extension theory for loops and quasigroups will be needed [1, 2, 4] and we shall be content in this introductory section with a few remarks sufficient for the proof of Lemma 1 below.

A non-empty set Q of elements $a, b, \cdots$ is said to be a *quasigroup* if (I) *to every ordered pair* $a, b \subset Q$ *there corresponds an element* $ab = c \subset Q$ and (II) *when any two of the symbols* x, y, z *of the equation* $xy = z$ *are assigned as elements of* Q *the third is uniquely determined as an element of* Q. In particular a quasigroup is called a *loop* if it possesses a (unique) unit element. Obviously every group is a loop. The *order* of a quasigroup is, by definition, its cardinal number, finite or transfinite.

If a quasigroup Q is homomorphic to a quasigroup R we may speak of R as a *proper homomorph* of Q if (i) R *is not isomorphic to* Q, (ii) R *is not a group of order one.* Correspondingly a quasigroup is *simple* if it has no proper homomorphs. This definition of simplicity is equivalent to the usual one in the case of groups, as well as to that used for finite quasigroups by G. N. Garrison [4] and to that employed by Albert [1, II] for arbitrary loops.

If a non-simple quasigroup Q has a proper homomorph R we may designate by H_p the set of elements of Q which map into the element p of R under a given homomorphism of Q upon R. Obviously H_p and H_q have common elements if and only if $p = q$. If $pq = r$, let x, y, z designate arbitrary elements of the set H_p, H_q, H_r respectively. Then,

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[1] Numbers in brackets refer to the references cited at the end of the paper.

[2] The name hyperabelian loop was suggested by the fact that the multiplication table for such a loop F_G is built upon that of a commutative (or abelian) loop. When G is a cyclic group we call F_G hypercyclic.

whenever two of x, y, z are assigned in their respective sets, the third element may be uniquely determined from the equation

$$(1) \qquad xy = z$$

as an element of its particular set. If we keep z fixed in (1), we derive a biunique correspondence $x \rightleftarrows y$ between the elements of H_p and H_q from which it follows that *each set H_p has the same cardinal number.* The equation (1) is equivalent to

$$(2) \qquad H_p \cdot H_q = H_{pq},$$

where, as usual, the product on the left of (2) designates the set consisting of all xy with $x \subset H_p$, $y \subset H_q$.

If Q is a loop with unit element e, then R is a loop with a unit element 1. In this case $H \equiv H_1$ is seen from (2) to be a subloop of Q; in fact a *proper* subloop, since $H \neq (e)$, Q. Moreover if $x \subset H_p$ then $H_p = xH$. In the light of these remarks the truth of parts (a) and (b) of the following lemma is evident.

Lemma I. (a) *If, for every proper subloop H of a loop Q, there exist at least two non-identical cosets xH, yH with common elements then Q is simple.*

(b) *If a loop has no proper subloops it is simple.*

(c) *Let G be a loop of finite order g, with a subloop F of order f, where f, g are relatively prime. If every proper subloop of G is a subloop of F then G is simple.*

Part (c) of the lemma requires some additional proof. If G is homomorphic to a proper homomorph R then $H \equiv H_1$ is a proper subloop of G, $H \subset F$. Moreover if H has order h, $h \mid g$, since the disjoint sets H_p exhaust G. Again, F is homomorphic to some loop $S \subset R$, $H \subset F$ consists of all elements of F which map into the identity of S, and so $h \mid f$. Thus h divides $(f, g) = 1$. It follows that $h = 1$, G is isomorphic to R, in contradiction to the hypothesis that R is a proper homomorph of G. Therefore G is simple.

Whether Q is a loop or not we may designate any fixed set H_q as H and put each set H_p in the form $H_p = u \cdot H$. Here $u \subset H_t$, where t is the unique element of R such that $p = tq$. This is the point of view taken by Garrison [4] and later by the author [2].

1. **Hyperabelian loops.** In this section we shall prove the following theorem.

Theorem I. *Let F be an arbitrary loop of finite order $f \geqq 2$. Let G be a commutative loop of finite order $g \geqq 3$. Then there exists a loop*

$H=F_G$ *of order* fg *with the following properties*: (i) *Every proper subloop of* H *is a subloop of* F; (ii) H *is simple. If* $g=2$, H *can be constructed with property* (i) *but not with property* (ii).

COROLLARY. *There exist simple loops of every finite order except order* 4.

The loop F_G of the theorem we call a *hyperabelian loop.*[3]

The corollary follows from Theorem I, in view of the facts that every group of prime order is simple (by Lemma I(b)!) and that the only loops of order 4 are the (non-simple) groups.

Before proceeding to the construction of the general loop F_G it will be advantageous to consider the example given by (3) below.

(3) H:

$\cdot$	1	2	3	4	5	6	7	8
1	1	2	3	4	5	6	7	8
2	2	1	4	5	6	7	8	3
3	3	4	5	6	7	8	2	1
4	4	5	6	7	8	3	1	2
5	5	6	7	8	2	1	3	4
6	6	7	8	3	1	2	4	5
7	7	8	1	2	3	4	5	6
8	8	3	2	1	4	5	6	7

(4) G:

$\cdot$	E	P	Q	R
E	E	P	Q	R
P	P	Q	R	E
Q	Q	R	E	P
R	R	E	P	Q

Here $F=(1, 2)$ is the cyclic group of order 2, and G is the cyclic group of order 4 given by (4). Thus $f=2$, $g=4$. If $r>f$ is an element of $H=F_G$, then any loop L containing r must contain the unit element 1. But it may be verified from (3), where $(\cdot)$ designates multiplication in H, that

(A) *if* $r\cdot s=1$ *for* $r, s>f$ *then either* $s\cdot r=2$ *or* $r\cdot r=2$.

Thus if L contains $r>f$ we see from (3) that L contains all elements $t\subset H$ with $t>f$. It then follows that $L\supset F$, $L=H$. Thus, in this case, the only proper subloop of H is F itself. But $3\cdot F$ contains the elements 3, 4 and $4 \cdot F$ contains the elements 4, 5. Hence H is simple, by virtue of Lemma I(a).

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[3] See the previous footnote.

In the general case, if $n>4$ is a composite integer, then n has at least one representation of the form $n=fg$ with $f\geqq 2$, $g\geqq 3$. Let G be any arbitrary commutative loop of order g, with elements $E, P, Q, \cdots$, where E is the unit element, and construct the multiplication table for G, as for example in (4). The Cayley square which is formed by deleting the sideline and headline of the multiplication table for G we shall call the *G-square*. Let F be an arbitrary loop of order f (not necessarily commutative), and designate the elements of F by $1, 2, \cdots, f$, where 1 is the unit element of F. Finally, let the elements $1, 2, \cdots, f, f+1, \cdots, n$ designate the elements of the proposed quasigroup $H=F_G$, where 1 is the unit element. In the sideline and headline of the multiplication table for G replace E by a column and row consisting of the numbers $1, 2, \cdots, f$, similarly P by the numbers $f+1, f+2, \cdots, 2f$, and so on, so that the numbers $1, 2, \cdots, n$ are now written in order in sideline and headline.

We shall first give a very simple construction of the multiplication table for $H=F_G$ (in terms of the multiplication for G, modified as above) and then indicate possible generalizations. In the G-square replace that element E which stands in the first row and column of the G-square by the F-square. Thus, as illustrated by (3), the upper left hand corner of the partially constructed multiplication table for H gives the table for F. Now replace each of the remaining elements $P, Q, \cdots$ of the first column of the G-square by a square (not a Cayley square) of f rows and columns, in the following manner: Think of the column formed by $P, Q, \cdots$ as a rectangle consisting of f columns and $n-f$ rows; in the first column write the elements $f+1, f+2, \cdots, n$ in order, and repeat these numbers, in cyclic permutation, in each successive column. (Thus the ith column, $2\leqq i\leqq f$, consists of the elements $f+i-1, f+i, \cdots, n, f, f+1, \cdots, f+i-2$ written in that order.) Next, replace the element P, wherever it appears in the G-square, by the square which has replaced P in the first column of the G-square, and do the same for Q, and so on, but not for E.

At this stage of the construction it should be clear that 1 has been made the unit element of H and, moreover, that no row or column of the H-square contains the same number more than once. (The second fact has been ensured by use of the Cayley square of a loop G.) Again, if the element E appears below the main diagonal in place (i, j), $1<j<i$, replace E by *any Cayley square* $C_{i,j}$ *of order* f, *formed from the elements* $1, 2, \cdots, f$ (for example, by the F-square itself).

Since G is commutative, E must also appear above the main diagonal, in place (j, i); and this time we replace E by the Cayley square $C'_{i,j}$ *derived from* $C_{i,j}$ *by interchanging the numbers* 1 *and* 2. We have thus partially assured the condition (A); in fact *if* $r \cdot s = 1$ *with* $r > f$, $s > f$, *and if* r, s *differ by at least* f, *then* $s \cdot r = 2$. Finally, suppose that E appears on the main diagonal of the G-square, in place (i, i) with $i > 1$, as will happen when G has an element of order two; in this case, replace E by a Cayley square $C_{i,i}$, formed from the elements $1, 2, \cdots, f$, *such that* 2 *appears in every place on the main diagonal, of* $C_{i,i}$.[4] Thus *if* $r \cdot s = 1$ *with* $r > f$, $s > f$, *and if* r, s *differ by less than* f, *then* $r \cdot r = 2$. The multiplication table of H is now complete, and satisfies condition (A); moreover the H-square is a Cayley square, in that each of the numbers $1, 2, \cdots, n$ appears exactly once in each row and column.

In view of condition (A), any subloop S of H which contains an element $r > f$ must contain the element 2. Also, by construction, $n \cdot 2 = f + 1$ and $s \cdot 2 = s + 1$ for $f < s < n$; or (to say the least!)

(B) *the mapping* $r \to r \cdot 2$ (*for* $f < r \leqq n$) *is a permutation of the elements* $f+1, \cdots, n$ *which maps no element into itself.*

It follows that S contains all the elements $f+1, \cdots, n$ and hence every element of H. This much is true even in the excluded case $g = 2$, and hence the constructed $H = F_G$ has property (i) of Theorem I for $g = 2$.

As A. A. Albert has shown [1, II], every loop of order $n = 2f$ ($f > 1$) which contains a subloop of order f is non-simple. Thus once more we assume $g \geqq 3$. What we have to say here depends essentially upon the construction of the first f columns of the H-square; so we note that

(C) *assuming* $f < r \leqq f(g-1)$, *the first* f *elements of the* r*th row of the* H*-square are, in order,* $r, r+1, \cdots, r+f-1$.

Although we gave the construction by columns, the truth of (C) is readily verified. Now let L be a proper subloop of H, so that $L \subset F$, L has order $l \leqq f$. Then, using (C), we see that the coset $(f+1) \cdot L$ contains the element $f+1$ and, in addition, $l - 1 \geqq 1$ of the elements $f+2, \cdots, 2f$. Let r be the greatest of the elements contained in $(f+1) \cdot L$; hence $f + 1 < r \leqq 2f \leqq f(g-1)$. Then, using (C) again, we see that $r \cdot L$ contains r and other elements $t > r$, but not the element $f+1$. Thus $(f+1) \cdot L$ and $r \cdot L$ are distinct cosets with the common element r. By reference to Lemma I(a) we verify that H is a simple loop.

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[4] Any Cayley (or latin) square, formed from the elements $1, 2, \cdots, f$, can be converted into a Cayley square with 2 down the main diagonal by suitable permutations of the rows.

Before attempting to generalize the construction of the loop $H=F_G$ we should realize that properties (i) and (ii) of Theorem I contain the essential features of the construction. For (i) we used conditions (A) and (B), and for (ii), condition (C). Now the only important fact about the element 2 used in (A) and (B) is that it is an element of F but not the identity; but to replace 2 by another element would not effect the construction in an essential manner. If 1 is to be the unit element of H we must have $1\cdot i=i\cdot 1=i$ for all i of H, but, aside from this restriction, condition (C) may be relaxed as regards the order in which the first f elements of the rth row appear. Then, if the first f columns of H have been completely filled in, and if the various E's have been replaced by Cayley squares as described above, the rest of the H-square may be filled in any manner, subject only to the obvious restrictions that 1 be the unit element of H and that each of the numbers $1, 2, \cdots, n$ appear exactly once in each row and column of the H-square.

Taking the case that F is the cyclic group of order $f=3$ and that G is the four-group ($g=4$) we give below two simple hypercyclic loops $H=F_G$. That given by (5) was constructed according to the first, or simpler, method, and that given by (6), with the conditions relaxed.[5] When f and g are both fairly large, considerably more freedom of construction is of course possible, without destroying the essential features (i) and (ii) of the hyerabelian loops.

(5)

·	1	2	3	4	5	6	7	8	9	10	11	12
1	1	2	3	4	5	6	7	8	9	10	11	12
2	2	3	1	5	6	7	8	9	10	11	12	4
3	3	1	2	6	7	8	9	10	11	12	4	5
4	4	5	6	2	3	1	10	11	12	7	8	9
5	5	6	7	1	2	3	11	12	4	8	9	10
6	6	7	8	3	1	2	12	4	5	9	10	11
7	7	8	9	10	11	12	2	3	1	4	5	6
8	8	9	10	11	12	4	1	2	3	5	6	7
9	9	10	11	12	4	5	3	1	2	6	7	8
10	10	11	12	7	8	9	4	5	6	2	3	1
11	11	12	4	8	9	10	5	6	7	1	2	3
12	12	4	5	9	10	11	6	7	8	3	1	2

---

[5] Note that the differences between (5) and (6) appear in the second and third rows and in the blocks of elements down the main diagonal.

(6)

·	1	2	3	4	5	6	7	8	9	10	11	12
1	1	2	3	4	5	6	7	8	9	10	11	12
2	2	3	1	6	7	8	9	10	11	12	4	5
3	3	1	2	5	6	7	8	9	10	11	12	4
4	4	5	6	2	1	3	10	11	12	7	8	9
5	5	6	7	3	2	1	11	12	4	8	9	10
6	6	7	8	1	3	2	12	4	5	9	10	11
7	7	8	9	10	11	12	3	1	2	4	5	6
8	8	9	10	11	12	4	2	3	1	5	6	7
9	9	10	11	12	4	5	1	2	3	6	7	8
10	10	11	12	7	8	9	4	5	6	3	2	1
11	11	12	4	8	9	10	5	6	7	1	3	2
12	12	4	5	9	10	11	6	7	8	2	1	3

2. **Other simple loops.** Elsewhere [3, p. 38] the author has announced without proof the following result.

LEMMA II. *Let n be any positive integer. Then an upper bound for the order of any proper sub-quasigroup is $[n/2]$ (the greatest integer in $n/2$). This bound is attained for every n.*

A. A. Albert [1, II] has published a proof of the first statement of this lemma, and has shown moreover that if a quasigroup Q of even order $n=2m$ has a sub-quasigroup of order m then Q is non-simple. We shall now construct the most general *loop* G of odd order $n=2m+1$ which has a subloop F of order $m>1$. (In case $m=1$, G is the cyclic group of order 3.)

THEOREM II. *Let F be an arbitrary loop of finite order $m>1$, consisting of the elements $1, 2, \cdots, m$, with unit element 1. Let G be a system of order $n=2m+1$, consisting of the elements $1, 2, \cdots, n$, closed under a single-valued multiplication, and containing the loop F as a multiplicative subsystem of order m. Let the multiplication table for G be given by*

$$(7)\qquad \begin{array}{c|cc} \cdot & & \\ \hline & A & B_1 \\ & C_1 & D_1 \end{array}$$

where it is understood that the sideline and headline each contain the *numbers* $1, 2, \cdots, m, m+1, \cdots, n$ *written in natural order. If* A *denotes the Cayley square for* F *then the following conditions are necessary and sufficient in order that* G *be a loop*:

(i) B_1 *must be obtainable by deleting the last row of a Cayley square* B *on the elements* $m+1, \cdots, n$, *of which the first row consists of these elements in natural order.*

(ii) C_1 *must be obtainable by deleting the last column of a Cayley square* C *on the elements* $m+1, \cdots, n$, *of which the first column consists of those elements in natural order.*

(iii) D_1 *must be obtainable from a Cayley square* D *on the elements* $0, 1, 2, \cdots, m$, *in the following manner. Each of the elements* $1, 2, \cdots, m$ *occupies the same position in* D_1 *as in* D. *The latter is partially determined by* B *and* C, *in that if* i *(for* $m+1 \leqq i \leqq n$*) appears in the row numbered* r_i *of the last column of* C *and in the column numbered* c_i *of the last row of* B $(1 \leqq r_i, c_i \leqq m+1)$ *then* 0 *must appear at the intersection of the* r_i*th row and* c_i*th column of* D. *Moreover* i *must appear at the intersection of the* r_i*th row and* c_i*th column of* D_1.

COROLLARY. *For every integer* $m>1$, *there exists a loop* G *of order* $n=2m+1$, *with the following properties*: (a) G *contains an arbitrary loop* F *of order* m. (b) *Every proper subloop of* G *is a subloop of* F. (*By Lemma* I(c), G *is simple.*)

With a few fairly obvious changes Theorem II can be altered to give the construction of the most general *quasigroup* of order $n=2m+1$ with an arbitrary sub-quasigroup of order m.

PROOF OF THEOREM II. Since G is to be a loop with subloop F, the unit element of G must coincide with 1, the unit of F. Hence the first row of B_1 (first column of C_1) must consist of the elements $m+1, \cdots, n$ in natural order. Since A is a Cayley square on the elements $1, 2, \cdots, m$, none of these elements can appear in any row or column of B_1 or of C_1. In particular B_1 consists of m rows and $m+1$ columns formed from the $m+1$ elements $m+1, \cdots, n$. In view of the fact that G is to be a loop, there can be no repetitions in any row or column of B_1; hence each column of B_1 lacks exactly one of the elements $m+1, \cdots, n$, clearly a different one for each column. Thus the missing elements form a uniquely determined final row, which, when added to B_1, yields a Cayley square B on the elements $m+1, \cdots, n$, the first row of which contains the elements in natural order. Thus condition (i) is necessary in order that G be a loop, and similar considerations show the necessity of (ii). Now if the element i $(m+1 \leqq i \leqq n)$ is missing from the r_ith column of B_1, or appears in the corresponding column of the last row of B, then i must appear somewhere in the r_ith column of D_1. Similarly if the same i is missing from the c_ith row of C_1, it must appear in the c_ith row of D_1. Finally, since this i can appear at most once in each row and column of C_1, it must appear at the intersection of the r_ith row and c_ith column of D_1, and nowhere else in D_1. Moreover, since the elements $1, 2, \cdots, m$ appear nowhere in B and C, each must appear exactly once in each row and column of D_1. Thus if D be obtained from D_1 by replacing each $i>m$ by 0, D must be a Cayley square; it follows that (iii) is necessary in order that G be a loop.

Conversely, let B, C be Cayley squares with first row and final column respectively consisting of the elements $m+1, \cdots, n$ in natural order, let D be a Cayley square on the elements $0, 1, 2, \cdots, m$ with 0 appearing in the places prescribed by (iii), and let B_1, C_1, D_1 be derived from B, C, D as described in the theorem. Then, in the inside of the multiplication table (7), each of the elements $1, 2, \cdots, n$ will appear exactly once in each row and column, inasmuch as A is the given Cayley square for F. Thus G will be a quasigroup, and, in fact, a loop, since it will have unit element 1. Thus the conditions (i), (ii) (iii) are sufficient.

When it comes to the actual problem of constructing G we find it necessary to give further consideration to the relationships between the Cayley squares B, C and D. If B and C are given, subject to the mild restrictions of (i) and (ii), then the positions of the element 0 in D are fixed. But if D' is an arbitrary Cayley square on $0, 1, \cdots, m$ we may permute the rows and columns of D' in order to bring 0 into the prescribed places, and hence there is no difficulty about constructing D. Suppose, on the other hand, that D and B are given arbitrarily. Since D is given, the positions of 0 are fixed in D. Since the last row of B is given, the positions of the elements $m+1, \cdots, n$ in D_1 are fixed, and hence the last column of C is fixed. Thus the first and last columns of C are fixed, and we are faced with the problem of constructing a Cayley square of order $m+1 \geqq 3$ with two given columns. When $m=2$ the remaining column of C is uniquely determined, but whether the construction is always possible for $m>2$ is not obvious.

There is no difficulty in constructing a loop G with the properties of the corollary. For example, let B be the symmetric Cayley square in which successive rows are obtained by cyclic permutations of the first row, and let C be identical with B. Then D will have 0 down the main diagonal, and D_1 will have down the main diagonal the elements $n, m+1, m+2, \cdots, n-1$ in that order. If $i>m$ is any element of G not in F, the elements $i_0=i$, $i_1=i\cdot i$, $i_2=i_1\cdot i_1, \cdots, i_m=i_{m-1}\cdot i_{m-1}$ will comprise all elements of G which are not in F, and the subloop generated by i will coincide with G. This establishes the corollary. If F is commutative, the G so constructed will be commutative if and only if D is commutative; but D can be commutative [5, p. 728, or 3, pp. 35–36] if and only if its order $m+1$ is even, or if m is odd, $m=2k+1>1$. Hence we have obtained the following result.

THEOREM III. *Let k be any positive integer, and let F be an arbitrary commutative loop of order $2k+1$. Then there exists a simple commutative loop G of order $4k+3$, with the property that every proper subloop of G is a subloop of F. (In particular F is a subloop of G of maximum possible order.)*

It should be noted conversely that if the G of Theorem II is commutative then C must be the transpose of B, and hence D must be a symmetric Cayley square of order $m+1\geqq 3$ with 0 down the main diagonal. But this can only happen for $m=2k+1\geqq 3$, and so Theorem III gives a "best possible" result of its kind.

We conclude this section with a result due to H. Griffin [5, p. 731].

THEOREM IV. *Let I_n designate a commutative loop of order $n>1$, which contains no proper subloops. Then*

(i) *I_{2m} does not exist for $m\neq 1$;*

(ii) *I_2, I_3 and I_5 are cyclic groups;*

(iii) *a non-associative I_n exists for every odd integer $n>5$.*

COROLLARY. *When I_n exists, it is simple.*

For the proof of Theorem IV we refer the reader to Miss Griffin's paper,[6] which also contains many other useful constructions. The corollary follows from Lemma I(b).

3. **The extension theory for quasigroups.** In another paper [2, §10] we have defined the extension $P=(H, Q)$ of a set H by a quasigroup Q. It will be convenient to repeat the definition. If H is a set of order m and Q a quasigroup of order n (m, n being finite or transfinite) then P is constructed so as to be a quasigroup of order mn. We suppose that to every pair p, q of elements of Q there has been defined a quasigroup $H_{p,q}$ consisting of the elements of H. No connection is

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[6] There seems to be a slight slip in the third sentence of Miss Griffin's construction, which could be corrected by replacing her sentence by these: "Now interchange the elements of the last row, except for the last element, with the principal diagonal elements immediately above them, and alter the last column correspondingly to preserve commutativity. Also change the headline and sideline of the multiplication table to agree respectively with the first row and column of the table."

postulated between the ordered products of a, b in the various $H_{p,q}$, and we denote this product in $H_{p,q}$ by[7] $\phi_{p,q}(a, b)$.

We define $P=(H, Q)$ to be the set product of H and Q, namely the set of all couples (a, p) with $a \subset H$, $p \subset Q$, where $(a, p)=(b, q)$ if and only if $a=b$, $p=q$. The ordered product of $x=(a, p)$ and $y=(b, q)$ is given by $x \cdot y=z=(c, r)$ with

$$c = \phi_{p,q}(a, b), \qquad r = pq. \tag{8}$$

There is now no difficulty in verifying that P is a quasigroup. Moreover, if we designate by H_p the set of all (a, p) with $a \subset H$, then

$$H_p \cdot H_q = H_{pq}, \tag{9}$$

whence it follows that *the set* $\mathfrak{Q}$ *of all sets* H_p *forms a quasigroup isomorphic to* Q, and P *is homomorphic to* $\mathfrak{Q}$. If, conversely, a quasigroup P is homomorphic to a quasigroup S, it is proved in [2] that P is isomorphic to an extension (H, Q) with Q isomorphic to S. Finally, *an extension* (H, Q) *is a loop with unit element* $(e, 1)$ *if and only if*: (i) Q *is a loop with unit* 1; (ii) $H_{1,1}$ *is a loop with unit* e; (iii) e *is a left unit for every* $H_{1,q}$ *and a right unit for every* $H_{p,1}$. Condition (iii) clearly implies (ii). In case P is a loop, A. A. Albert calls $H_{1,1}$ a ***normal divisor***, and H may be taken to be identical with $H_{1,1}$.

It is our purpose in this section to supply a proof of the following theorem, previously announced in [2].

THEOREM V. *Let* $P=(H, Q)$ *be a quasigroup extension of a set* H *by a quasigroup* Q. *Then every loop isotopic to* P *is isomorphic to some extension* $P_0=(H_0, Q_0)$ *where* Q_0 *is a loop isotopic to* Q *and* H_0 *is a set* (*which may be taken to be a loop*) *of the same order as* H.

COROLLARY. *Every isotope of a simple loop is simple.*

PROOF. If f, g are fixed and x, y arbitrary elements of P, consider the system P_0 with multiplication

$$xoy = xR_g^{-1} \cdot yL_f^{-1}. \tag{10}$$

Here the operators L_f^{-1}, R_g^{-1} designate respectively the inverses of the one-to-one mappings

$$x \rightarrow xg = xR_g, \qquad y \rightarrow fy = yL_f \tag{11}$$

of P upon itself. It is known [1, 3] that P_0 is a loop and has unit element $f\cdot g$ (this much may be verified without difficulty from (10) and the definition of a loop), and, moreover, that every loop isotopic to P is isomorphic to a loop P_0 defined for some f and g of P. Hence our proof will consist merely in relating (10) to the fact that $P=(H, Q)$.

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[7] The only difficulties to be found in the proof of Theorem V of this section are those of notation. It has been found advantageous in this instance to replace certain notations previously used by the author [2, 3] by those recently adopted by Albert [1].

If p, q are any elements of Q we define the one-to-one mappings R_p, L_p of Q upon itself by

$$pq = pR_q = qL_p. \tag{12}$$

Similarly, if a, b are any elements of H we define one-to-one mappings $R_a^{p,q}$, $L_b^{p,q}$ of the quasigroup $H_{p,q}$ upon itself by

$$\phi_{p,q}(a, b) = aR_b^{p,q} = bL_a^{p,q}. \tag{13}$$

Assuming that

$$f = (\alpha, u), \qquad g = (\beta, v), \qquad x = (a, p), \qquad y = (b, q), \tag{14}$$

we now may show that

$$xR_g^{-1} = (aV_p^{-1}, pR_v^{-1}), \qquad yL_f^{-1} = (bU_q^{-1}, qL_u^{-1}) \tag{15}$$

where V_p, U_q are one-to-one mappings of H upon itself, defined by

$$\begin{aligned} V_p &= R_\beta^{p',v}, & p' &= pR_v^{-1}, \\ U_q &= L_\alpha^{u,q''}, & q'' &= qL_u^{-1}. \end{aligned} \tag{16}$$

In fact, where g, x are given by (14), the definition of product in $P=(H, Q)$ yields $xR_g=x\cdot g=(\phi_{p,v}(\alpha, \beta), pv)=(aR_\beta^{p,v}, pR_v)$. The inverse mapping R_g^{-1} is uniquely defined; hence if we take it to be given by (15.1) then $x=xR_gR_g^{-1}=(aR_\beta^{p,v}, pR_v)R_g^{-1}=(aR_\beta^{p,v}\cdot V_{pv}^{-1}, p)$. It follows that R_g^{-1} has been correctly defined if and only if

$$V_{pv} = R_\beta^{p,v}.$$

In this last relation we replace p by $pR_v^{-1}=p'$ and obtain (16.1). Similarly we may verify that L_f^{-1} is correctly given by (15.2) if and only if U_q is given by (16.2).

Before proceeding, let us note that V_p depends not only upon p but upon the fixed element $g=(\beta, v)$. A similar remark holds for U_q.

By use of (10), (8), (15) and (16), a direct calculation gives

$$xoy = (\overset{0}{\phi}_{p,q}(a, b), poq) \tag{17}$$

where

$$poq = pR_v^{-1}\cdot qL_u^{-1} \tag{18}$$

and

$$\phi^{0}_{p,q}(a, b) = \phi_{p',q''}(aV_p^{-1}, bU_q^{-1}). \tag{19}$$

Now (18) defines a loop Q_0, isotopic to Q, and with unit element $u \cdot v \equiv 1$. (Compare the corresponding remarks above about (10) and P.) Moreover, for each fixed pair p, q of elements of Q_0, (19) defines a quasigroup $H^0_{p,q}$ consisting of the elements of H under the multiplication $\phi^0_{p,q}(a, b)$; in fact if p, q are given then p', q'' are determined by (16), and the equation $\phi^0_{p,q}(a, b) = c$, or

$$\phi_{p',q''}(aV_p^{-1}, bU_q^{-1}) = c,$$

enables us to determine any one of a, b, c uniquely once the other two are given. It is indeed evident that *each quasigroup* $H^0_{p,q}$ *is isotopic to the corresponding quasigroup* $H_{p',q''}$. Since the $H^0_{p,q}$ are quasigroups we see from (17) that P_0 is an extension (H_0, Q_0) of the set $H_0 \equiv H$ by the loop Q_0. But P_0 is itself a loop, and hence the proof of Theorem V is complete.

In view of (10) we know that the unit element of P_0 is

$$(e, 1) \equiv f \cdot g = (\alpha, u)(\beta, v) = (\phi_{u,v}(\alpha, \beta), uv). \tag{20}$$

It would seem of interest to give a direct proof of the facts that $H^0_{1,1}$ is a loop with unit element e, and that e is a left unit for every $H^0_{1,q}$ and a right unit for every $H^0_{p,1}$. Now, from (16), $1' = uv \cdot R_v^{-1} = u$ and $1'' = uv \cdot L_u^{-1} = v$. Moreover $eV_1^{-1} = \phi_{u,v}(\alpha, \beta) \cdot [R_\beta^{u,v}]^{-1} = \alpha$ and $eU_1^{-1} = \phi_{u,v}(\alpha, \beta)[L_\alpha^{u,v}]^{-1} = \beta$. Hence, by (19), (16) and (13),

$$\phi^0_{1,q}(e, b) = \phi_{u,q''}(\alpha, bU_q^{-1}) = bU_q^{-1} \cdot L_\alpha^{u,q''} = b$$

and

$$\phi^0_{p,1}(a, e) = \phi_{p',v}(aV_p^{-1}, \beta) = aV_p^{-1} \cdot R_\beta^{p',v} = a$$

for all a, b. In particular we also have $\phi^0_{1,1}(a, e) = \phi^0_{1,1}(e, a) = a$.

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AN APPLICATION OF LATTICE THEORY TO QUASIGROUPS

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The purpose of this note is to show that O. Ore's general formulation of the Jordan-Hölder theorems in partially ordered sets[1] [3] yields the Jordan-Hölder theorems for loops [1] which have recently been obtained by A. A. Albert [2]. We shall assume that the reader is familiar with the papers [1, 2, 3] of O. Ore and of A. A. Albert.

We begin with an examination of certain congruence relations in loops. We are led to characterize the normal divisors of A. A. Albert as those subloops which commute and associate with the elements of the loop. Our proof of the fundamental quadrilateral condition of O. Ore [3] is then based on this characterization.

A *loop* $\mathfrak{G}$ is a quasigroup with an identity element e. For each non-null subset $\mathfrak{H}\subset\mathfrak{G}$ we define a relation H_ρ on $\mathfrak{G}\mathfrak{G}$ as follows:

(1) *If* $x, y \in \mathfrak{G}$, *then* $xH_\rho y$ *in case* $y \in x\mathfrak{H}$.

THEOREM 1. *The relation* H_ρ *is a congruence relation for the loop* $\mathfrak{G}$ *if and only if*

$$\text{(i)}\ x\in(xh)\mathfrak{H}, \qquad \text{(ii)}\ (x\mathfrak{H})(y\mathfrak{H})\subset(xy)\mathfrak{H}$$

for every $x, y\in\mathfrak{G}$ *and* $h\in\mathfrak{H}$.

PROOF. Let H_ρ be a congruence relation for $\mathfrak{G}$. For each $x\in\mathfrak{G}$ and $h\in\mathfrak{H}$ we have $xH_\rho(xh)$ by the definition of H_ρ. Since H_ρ is symmetric we have $(xh)H_\rho x$, $x\in(xh)\mathfrak{H}$. If also $y\in\mathfrak{G}$, $h_1\in\mathfrak{H}$, then $yH_\rho(hy_1)$, and since H_ρ preserves multiplication we obtain $(xy)H_\rho(xh)(hy_1)$, $(xh)(yh_1)\in(xy)\mathfrak{H}$. Conversely, let $\mathfrak{H}$ satisfy (i) and (ii). If $x\in\mathfrak{G}$, choose $h\in\mathfrak{H}$ and we have $x\in(xh)\mathfrak{H}\subset x\mathfrak{H}$ by (i) and (ii). Thus H_ρ is reflexive. If also $y\in\mathfrak{G}$ and $xH_\rho y$, we have $y=xh_1$, and (i) yields $x\in(xh_1)\mathfrak{H}=y\mathfrak{H}$, $yH_\rho x$. Thus H_ρ is symmetric. If also $z\in\mathfrak{G}$ and $yH_\rho z$, we have $z=yh_2=(xh_1)h_2\in x\mathfrak{H}$ by (ii). Thus H_ρ is transitive and is an equivalence relation. The relations $xH_\rho y$, $zH_\rho w$ yield $(xz)H_\rho(yw)$ by (ii) and we conclude that H_ρ is a congruence relation for $\mathfrak{G}$.

Remark 1. If R is a congruence relation for a loop $\mathfrak{G}$, then the subset $\mathfrak{R}\equiv[x;xRe]$ is a *subloop* of $\mathfrak{G}$. For clearly $e\in\mathfrak{R}$ and $\mathfrak{R}$ is closed with

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[1] Numbers enclosed in brackets denote the references given at the end of the paper.

respect to multiplication. If we have $r_1x=r_2$, then eRr_1, xRx give $xR(r_1x)$, xRr_2, r_2Re, xRe, $x\in\mathfrak{R}$. Similarly $xr_1=r_2$ implies $x\in\mathfrak{R}$ and $\mathfrak{R}$ is a subloop of $\mathfrak{G}$. With $R=H_p$, we obtain $\mathfrak{R}=\mathfrak{H}$ and thus (i) and (ii) imply that $\mathfrak{H}$ is a subloop of $\mathfrak{G}$. (For groups $\mathfrak{G}$, every congruence relation R is of the form H_p for suitable $\mathfrak{H}\subset\mathfrak{G}$.)

Remark 2. We note the following consequences of (i) and (ii):

$$x(y\mathfrak{H}) \subseteq (xy)\mathfrak{H}, \tag{2}$$

$$(x\mathfrak{H})y \subseteq (xy)\mathfrak{H}, \tag{3}$$

$$x(\mathfrak{H}y) \subseteq (xy)\mathfrak{H}, \tag{4}$$

$$\mathfrak{H}x \subseteq x\mathfrak{H}. \tag{5}$$

For *finite* $\mathfrak{H}$ each of these inequalities[3] is an *equality*. To see this for (2), note that for fixed $x, y\in\mathfrak{G}$ we have a single-valued mapping $h\to h^*$ defined by $x(yh)=(xy)h^*$. This mapping is one-to-one since $x(yh_1)=x(yh_2)$ implies $h_1=h_2$. For a finite $\mathfrak{H}$, we conclude that the set of images $[h^*]=\mathfrak{H}$, and consequently that $(xy)\mathfrak{H}=x(y\mathfrak{H})$. The same reasoning applies to the inequalities (3), (4), and (5). It is now easy to see that if $\mathfrak{H}$ is a *finite* subloop of $\mathfrak{G}$, then H_p is a congruence relation for $\mathfrak{G}$ if and only if the set $\mathfrak{H}$ commutes and associates with the elements of $\mathfrak{G}$ in the sense that

$$x\mathfrak{H} = \mathfrak{H}x, \tag{6}$$

$$(xy)\mathfrak{H} = x(y\mathfrak{H}), \qquad (x\mathfrak{H})y = x(\mathfrak{H}y), \qquad (\mathfrak{H}x)y = \mathfrak{H}(xy) \tag{7}$$

for every $x, y\in\mathfrak{G}$.

THEOREM 2. *A subloop $\mathfrak{H}$ of a loop $\mathfrak{G}$ satisfies* (6) *and* (7) *if and only if $\mathfrak{H}$ is a normal divisor of $\mathfrak{G}$ in the sense of A. A. Albert.*

PROOF. Consider a subloop $\mathfrak{H}$ of a loop $\mathfrak{G}$ which satisfies (6) and (7). The permutation group $\mathfrak{H}_p$ generated by the right multiplication of $\mathfrak{G}$ gives rise to the cosets $x\mathfrak{H}_p$ of Albert. We first show that the cosets $x\mathfrak{H}_p=x\mathfrak{H}$. Clearly $x\mathfrak{H}_p\supset x\mathfrak{H}$. The elements of $\mathfrak{H}_p$ are finite products of permutations of the form R_h and R_h^{-1} for $h\in\mathfrak{H}$. If $y=xR_h^{-1}$, then $yh=x$, $y=y(hh_1)=(yh)h_2=xh_2$, $y=xR_{h_2}$. It follows that each element of $x\mathfrak{H}_p$ may be written in the form $xh\in x\mathfrak{H}$. We next note that

$$(x\mathfrak{H})(y\mathfrak{H}) = (xy)\mathfrak{H} \tag{8}$$

is an immediate consequence of (6) and (7). To see that the cosets $x\mathfrak{H}$ form a loop we now consider the equations

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[3] This result is due to G. N. Garrison. See §4 of his fundamental paper, *Quasi-groups*, Ann. of Math. vol. 41 (1940) pp. 474–487.

$$(9) \qquad (x\mathfrak{H})(w\mathfrak{H}) = y\mathfrak{H}, \qquad (z\mathfrak{H})(x\mathfrak{H}) = y\mathfrak{H}.$$

Solutions of (9) are provided by the solutions of $xw=y$ and $zx=y$ for w and z. The uniqueness of the solutions $w\mathfrak{H}$ and $z\mathfrak{H}$ of (9) is readily verified. We have proved that $\mathfrak{G}/\mathfrak{H}$ is a loop and hence that $\mathfrak{H}$ is a normal divisor of $\mathfrak{G}$ in the sense of A. A. Albert.

To prove the converse it suffices to use the representation $\mathfrak{H}=e\Gamma$ furnished by Albert's Theorem 3 [1]. Using this theorem, the verifications of (6) and (7) are quite simple and we shall omit them.

Remark 3. The equations (6) and (7) may be replaced by

$$(10) \qquad (x\mathfrak{H})y = x(y\mathfrak{H}),$$

$$(11) \qquad (xy)\mathfrak{H} = x(y\mathfrak{H})$$

for every $x, y \in \mathfrak{G}$. To see this, set $x=e$ in (10) to obtain (6).

THEOREM 3. *The intersection $\mathfrak{H}$ of a system $(\mathfrak{H}_\alpha;\ \alpha\in\Omega)$ of normal divisors of a loop $\mathfrak{G}$ is a normal divisor of $\mathfrak{G}$.*

PROOF. Clearly $\mathfrak{H}$ is a subloop of $\mathfrak{G}$. We shall prove $\mathfrak{H}x\subset x\mathfrak{H}$ for every $x\in\mathfrak{G}$. If $h\in\mathfrak{H}$, $\alpha\in\Omega$, we have $hx=xh_\alpha$ for some $h_\alpha\in\mathfrak{H}_\alpha$. The left cancellation law then shows that the set $[h_\alpha;\ \alpha\in\Omega]$ is singular and its sole member is an element of $\mathfrak{H}$. The remainder of the proof consists in a repetition of this argument which proves (6) and (7).

COROLLARY. *The set of all normal divisors of a loop $\mathfrak{G}$, when ordered by set inclusion, forms a complete lattice.*

PROOF. It suffices to remark that $\mathfrak{G}$ is a normal divisor of $\mathfrak{G}$. We shall use $\cup$ and $\cap$ to denote the lattice operations of this lattice.

THEOREM 4. *If $\mathfrak{H}$ and $\mathfrak{K}$ are normal divisors of a loop $\mathfrak{G}$, then $\mathfrak{H}\mathfrak{K}$ is a normal divisor of $\mathfrak{G}$ and $\mathfrak{H}\mathfrak{K}=\mathfrak{H}\cup\mathfrak{K}$.*

PROOF. Clearly $e\in\mathfrak{H}\mathfrak{K}$ and $\mathfrak{H}\mathfrak{K}$ is closed with respect to multiplication. Now if $x(hk)=h_1k_1$, we use (7) to get $(xh)k_2=h_1k_1$. Two applications of (i) and use of (6) and (7) then give $x=h_3k_3$. We compute $x(\mathfrak{H}\mathfrak{K})=(x\mathfrak{H})\mathfrak{K}=(\mathfrak{H}x)\mathfrak{K}=\mathfrak{H}(x\mathfrak{K})=\mathfrak{H}(\mathfrak{K}x)=(\mathfrak{H}\mathfrak{K})x$. Thus $\mathfrak{H}\mathfrak{K}$ is a subloop of $\mathfrak{G}$ satisfying (6). We omit the simple verification of (7). Since every normal divisor of $\mathfrak{G}$ which contains $\mathfrak{H}$ and $\mathfrak{K}$ must contain $\mathfrak{H}\mathfrak{K}$, we conclude that $\mathfrak{H}\mathfrak{K}=\mathfrak{H}\cup\mathfrak{K}$.

COROLLARY. *The lattice of normal divisors of a loop $\mathfrak{G}$ is modular.*

THEOREM 5 (QUADRILATERAL CONDITION). *If $\mathfrak{H}$ and $\mathfrak{K}$ are maximal normal divisors of a loop $\mathfrak{G}$, then $\mathfrak{H}\cap\mathfrak{K}$ is a maximal normal divisor of the loop $\mathfrak{H}$, and $\mathfrak{G}/\mathfrak{K}\cong\mathfrak{H}/\mathfrak{H}\cap\mathfrak{K}$.*

PROOF. Suppose on the contrary that $\mathfrak{L}$ is a normal divisor of the loop $\mathfrak{H}$ such that $\mathfrak{H} > \mathfrak{L} > \mathfrak{H} \cap \mathfrak{K}$. We shall prove that $\mathfrak{L}\mathfrak{K}$ is a normal divisor of $\mathfrak{G}$. We note that $\mathfrak{G} = \mathfrak{H}\mathfrak{K}$ and to verify (10) we consider the equation

$$[(hk)(h_1k_1)](h_2k_2) = (hk)[(h_2k_2)(h_3k_3)]$$

which we may write, using (8), (i), and (7) for $\mathfrak{K}$, as

$$(hh_1)h_2 = [h(h_2h_3)]k_4. \tag{12}$$

Now if $h_1 \in \mathfrak{L}$, we may use (10) for $\mathfrak{L}$ to get $h(h_2l) = [h(h_2h_3)]k_4$. We now use (7) for $\mathfrak{K}$, cancel h, use (7) for $\mathfrak{K}$ again and cancel h_2 to obtain $l = h_3k_5$. It follows that $k_5 \in \mathfrak{H}$, and hence that $k_5 \in \mathfrak{H} \cap \mathfrak{K} \subset \mathfrak{L}$. Consequently $h_3 \in \mathfrak{L}$ and we have proved $[x(\mathfrak{L}\mathfrak{K})]y \subset x[y(\mathfrak{L}\mathfrak{K})]$ for every x, $y \in \mathfrak{G}$. On the other hand, if $h_3 \in \mathfrak{L}$, we may use (10) for $\mathfrak{L}$ to get $(hh_1)h_2 = [(hl)h_2]k_4$. We now use (6) and (7) for $\mathfrak{K}$ and cancel h_2, use (6) and (7) for $\mathfrak{K}$ and cancel h to obtain $h_1 = lk_5$ from which $h_1 \in \mathfrak{L}$ follows as before. This completes the verification of (10). We apply the same reasoning to verify (11). We write

$$[(hk)(h_1k_1)](h_2k_2) = (hk)[(h_1k_1)(h_3k_3)]$$

as $(hh_1)h_2 = [h(h_1h_3)]k_4$. Then if $h_2 \in \mathfrak{L}$, we have $h(h_1l) = h[(h_1h_3)k_5]$ and we easily obtain $h_3 \in \mathfrak{L}$. Also, if $h_3 \in \mathfrak{L}$, we have $(hh_1)h_2 = (hh_1)(lk_5)$, $h_2 = lk_5$, $h_2 \in \mathfrak{L}$. Thus (11) holds for $\mathfrak{L}\mathfrak{K}$. To show that $\mathfrak{L}\mathfrak{K}$ is a subloop of $\mathfrak{G}$ it now suffices to note that $e \in \mathfrak{L}\mathfrak{K}$, $(\mathfrak{L}\mathfrak{K})(\mathfrak{L}\mathfrak{K}) = \mathfrak{L}\mathfrak{K}$, and to prove that if $(hk)(l_1k_1) = l_2k_2$, then $h \in \mathfrak{L}$. But we may write this equation as $hl_1 = l_2k_4$ and get $h = h(l_1l_3) = (hl_1)l_4 = l_5k_5$. As before, $k_5 \in \mathfrak{L}$, $h \in \mathfrak{L}$. We have proved that $\mathfrak{L}\mathfrak{K}$ is a normal divisor of $\mathfrak{G}$. Note also that $\mathfrak{G} > \mathfrak{L}\mathfrak{K} > \mathfrak{K}$, since if $h \in \mathfrak{H}$ and h does not belong to $\mathfrak{L}$, then $hk = lk_1$ implies $h = lk_2$, $k_2 \in \mathfrak{L}$, $h \in \mathfrak{L}$, a contradiction, and $\mathfrak{L} > \mathfrak{H} \cap \mathfrak{K}$. Our assumption of the existence of a normal divisor $\mathfrak{L}$ of $\mathfrak{H}$ satisfying $\mathfrak{H} > \mathfrak{L} > \mathfrak{H} \cap \mathfrak{K}$ has led to a contradiction of the assumption that $\mathfrak{K}$ is a maximal normal divisor of $\mathfrak{G}$. We conclude that $\mathfrak{H} \cap \mathfrak{K}$ is a maximal normal divisor of $\mathfrak{H}$.

To establish the isomorphism we map the cosets $x\mathfrak{K}$ of $\mathfrak{G}/\mathfrak{K}$ onto the cosets $x(\mathfrak{H} \cap \mathfrak{K})$ of $\mathfrak{H}/\mathfrak{H} \cap \mathfrak{K}$. This mapping exhausts the set $\mathfrak{H}/\mathfrak{H} \cap \mathfrak{K}$ and is one-to-one. For, if $x(\mathfrak{H} \cap \mathfrak{K}) = y(\mathfrak{H} \cap \mathfrak{K})$, then $xk = yu$ gives $x = (yu)k_1 = yk_2$, $uk_3 = k_2$, $u \in \mathfrak{K}$, $x\mathfrak{K} \subset y\mathfrak{K}$, and symmetry yields $x\mathfrak{K} = y\mathfrak{K}$. That this mapping preserves multiplication in the two loops $\mathfrak{G}/\mathfrak{H}$ and $\mathfrak{H}/\mathfrak{H} \cap \mathfrak{K}$ is evident. The proof is complete.

Theorem 5 is the main tool which is needed to apply Ore's result [3] to the partially ordered set of subloops of a loop $\mathfrak{G}$ which occur in some composition series of $\mathfrak{G}$.

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UNITED STATES NAVAL ACADEMY, POSTGRADUATE SCHOOL

TWO ELEMENT GENERATION OF A SEPARABLE ALGEBRA

A. A. ALBERT

The *minimum rank* of an algebra A over a field F is defined to be the least number $r=r(A)$ of elements $x_1, \cdots, x_r$ such that A is the set of all polynomials in $x_1, \cdots, x_r$ with coefficients in F. In what follows we shall assume that A is an *associative algebra of finite order* over an *infinite* field F.

It is well known that $r(A)=1$ if A is a separable field over F and that $r(A)=2$ if A is a total matric[1] algebra over F. Over fourteen years ago I obtained but did not publish the result that $r(A)=2$ if A is a central division algebra over F. The purpose of this note is to provide a brief proof of the generalization which states that if A is any *separable* algebra over F then $r(A)=1$ or 2 according as A is or is not commutative.

We observe first that a commutative separable[2] algebra Z is a direct sum of separable fields and that there exists a scalar extension K over F such that Z_K has a basis $e_1, \cdots, e_n$ over F for pairwise orthogonal idempotents e_i. If $u_1, \cdots, u_n$ is a basis of Z over F and $x=a_1u_1+\cdots+a_nu_n$ the powers x^i have the form

$$x^i = \sum_{j=1}^{n} b_{ij}u_j \qquad (i = 1, \cdots, n),$$

where the determinant

$$d(a_1, \cdots, a_n) = | b_{ij} |$$

is a polynomial in the parameters $a_1, \cdots, a_n$. If $c_1, \cdots, c_n$ are any

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[1] See page 95 of my *Modern higher algebra*.

[2] The definition of a separable algebra given below reduces to a direct sum of fields in the commutative case. When F is nonmodular the concept of semisimple algebra and separable algebra coincide.

distinct elements of K the quantity $x_0 = c_1e_1 + \cdots + c_ne_n$ is known[1] to generate the diagonal algebra Z_K, that is $Z_K = F[x_0]$ has a basis $x_0, x_0^2, \cdots, x_0^n$. If we express the quantities $e_1, \cdots, e_n$ linearly in terms of $u_1, \cdots, u_n$ we see that x_0 is a value of x for values a_{i0} of the a_i in K. The linear independence of $x_0, x_0^2, \cdots, x_0^n$ implies that $d(a_{10}, \cdots, a_{n0}) \neq 0$. Then $d(a_1, \cdots, a_n)$ is not identically zero and thus there exists a quantity x in Z such that Z has a basis $x, x^2, \cdots, x^n$ over F, $Z = F(x)$, $r(Z) = 1$.

An algebra A is called a separable[2] algebra if A is a direct sum of simple components A_k such that the center of every A_k is a separable field over F. If x and y are in A we define

$$F[x, y]$$

to mean the set of all polynomials

$$\sum_{i=1,\cdots,m}^{j=1,\cdots,q} a_{ij}x^iy^j \qquad (a_{ij} \text{ in } F).$$

Only a finite number of the power products x^iy^j are linearly independent and each $F[x, y]$ is a linear subspace of A, m and q may be be selected so that $F[x, y]$ has order mq over F.

A separable algebra has a unity quantity e and if $A = F[x]$ then A has a basis $x^0 = e, x, \cdots, x^{n-1}$ over F, $A = F[x, e]$. Also A is commutative. If A is not commutative and $A = F[x, y]$ then $e = x[f(x, y)]y$ and thus x and y must be nonsingular. Note then that A has a basis of power products x^iy^j where $i = 0, \cdots, m-1$ and $j = 0, \cdots, q-1$. We use these results in the proof of our principal

THEOREM. *If A is a separable algebra which is not commutative then $r(A) = 2$, $A = F[x, y]$ for nonsingular elements x and y such that $F[x]$ is separable.*

We first study the case where A is the direct product of a total matric algebra M of degree g and a division algebra D of degrees s over a separable center C over F. It is well known[3] that D contains a maximal separable subfield $W = C[x_0]$ of degree s over C and that $W = F[x_0]$. The algebra $Q = (e_{11}, \cdots, e_{gg})$, whose basis consists of a set of primitive idempotents of M whose sum is its unity element e, has the property that $Z \neq Q \times W$ is separable and commutative, and so $Z = F[x]$. If K is a scalar splitting field over C of D the algebra Z_K contains $n = gs$ primitive pairwise orthogonal idempotents whose sum is the unity element e of the total matric algebra A_K of degree n over K. Also $Z=C[x]$, $Z_K=K[x]$ and it is known[1] that there exists a quantity y_0 in A_K such that

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[3] See Theorem 4.18 of my *Structure of algebras*, Amer. Math. Soc. Colloquium Publications, vol. 24, New York, 1939.

$$A_K = K[x, y_0],$$

that is, the power products $x^i y_0^j$ taken for $i, j=1, \cdots, n$ are linearly independent in K. If $p=n^2$ and $u_1, \cdots, u_p$ are a basis of A over C we may write $y=a_1u_1+ \cdots +a_pu_p$ and express the powers $x^i y^j$ in the form

$$z_k = x^i y^j = \sum_{h=1} b_{kh} u_h \qquad (k = i + jn - n;\ i, j = 1, \cdots, n),$$

for b_{kh} in F. The determinant $d(a_1, \cdots, a_p)=|b_{kh}|$ is a polynomial in $a_1, \cdots, a_p$ with coefficients in C which is not identically zero since it is not zero for values $a_{10}, \cdots, a_{j0}$ which define y_0. It follows that $A=C[x, y]$. But $C[x]=F[x]$ so that $A=F[x, y]$.

We finally consider a separable algebra A which is the direct sum of simple components $A_1, \cdots, A_t$. By the proofs above every component $A_k=F[x_k, y_k]$, where y_k is the unity quantity e_k of A_k when A_k is commutative, $Z_k=F[x_k]$ is separable. The algebra Z which is the direct sum of $Z_1, \cdots, Z_t$ is a commutative separable algebra and so $Z=F[x]$. Let $y=y_1+ \cdots +y_t$. Since $F[x]$ contains every x_k the linear space $F[x, y]$ contains $x_k^i y^j=x_k^i y_k^j$. For $x_k^i=x_k^i e_k$ and $e_k y^j=(e_k y)^j=y_k^j$. It follows that $F[x, y]$ contains every A_k and that $F[x, y]=A$.

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LIMITS FOR THE CHARACTERISTIC ROOTS OF A MATRIX[1]

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Let A be a square matrix of order n with complex numbers as elements. The equation $|\lambda I-A|=0$ is called the characteristic equation of the matrix A, and the roots λ_i, the characteristic roots of the matrix A. Although it is not possible to make any definite statements regarding the nature of the characteristic roots for the general matrix, several authors have given upper limits to the roots. In 1900, Bendixson [1][2] obtained upper limits for the real and imaginary parts of the characteristic roots of a real matrix. In 1902, Hirsch [5] extended these results to matrices with complex numbers as elements. A limit was also given by Bromwich [2] in 1904. These limits were further refined by Browne [3] in 1930, and by Parker [7] in 1937.

In 1918, Toeplitz [8], using the results of Bendixson and Hirsch, studied the algebraic form $(Ax|x)\equiv\sum_{r,s}a_{rs}x_s\bar{x}_r$ corresponding to the matrix A, where by hypothesis the algebraic form $(x|x)=\sum_{r=1}^{n}x_r\bar{x}_r$ has the value unity. By using the fact that A may be decomposed uniquely in the form $A=B+iC$, where B and C are Hermitian, he showed that the totality of values which the algebraic form assumes lie within a rectangle with sides parallel to the real and imaginary axes. He further showed that this field of values is bounded by a convex curve. Hausdorff [4] showed that the field of values is connected, bounded, closed, and convex. In 1932, Murnaghan [6], using the fact that $(Ax|x)$ has values invariant under unitary transformations of A, showed that for normal matrices (that is, matrices which can be transformed unitarily into diagonal form) the field of values is a convex polygon. For non-normal matrices, he showed that in the general case the characteristic roots of the matrix are the foci of the curve bounding the field of values. Wintner had previously remarked that for $n=2$ the curve is an ellipse. The first three theorems below give radii of circles within which not merely the characteristic roots but also the entire field of values lies.

Let A' and $\bar{A}$ denote the transpose and conjugate, respectively, of the matrix A, and write

$$B = (A+\bar{A}')/2, \qquad C = (A-\bar{A}')/2i.$$

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[1] Partially extracted from a dissertation, University of California, 1944.

[2] Numbers in brackets refer to the Bibliography at the end of the paper.

It is evident that B and C are Hermitian. The theorem given by Parker may be stated as follows:

PARKER'S THEOREM. *If A is any square matrix, and if $2Sr$, $2S'r$, $2S''r$ are the sums of the absolute values of the elements in the rth row and the absolute values of the elements in the rth column of A, B, and C, respectively, and if S, S', S'' are the greatest of the Sr, $S'r$, $S''r$, respectively, then for any characteristic root, $\lambda=\alpha+i\beta$, of A, we have*

$$|\lambda| \leqq S, \qquad |\alpha| \leqq S', \qquad |\beta| \leqq S''.$$

If $\lambda=\alpha+i\beta$ is a characteristic root of a matrix $A=(a_{rs})$ of order n, there exists a set of numbers $(x_1, x_2, \cdots, x_n)$ such that

$$\sum_{r=1}^{n} x_r\bar{x}_r = 1,$$

which satisfy the relations

$$\lambda x_r = \sum_s a_{rs}x_s \qquad (r = 1, 2, \cdots, n). \tag{1}$$

If we multiply the rth equation in (1) by $\bar{x}_r$, and sum as to r, we obtain

$$\lambda = \sum_{r,s} a_{rs}\bar{x}_r x_s. \tag{2}$$

The numbers $\sum a_{rs}\bar{x}_r x_s$, where $\sum x_r\bar{x}_r$ has the value unity, comprise the field of values of A. Hence (2) shows that the characteristic roots lie within the field of values. In the following, λ will be considered to be any value in this field, the conclusions holding a fortiori where λ is a characteristic root.

Taking conjugates of both sides of (2), and interchanging subscripts, we have

$$\bar{\lambda} = \sum_{r,s} \bar{a}_{sr}\bar{x}_r x_s. \tag{3}$$

From (2) and (3), by addition and subtraction, it follows that

$$\alpha = \sum_{r,s} b_{rs}\bar{x}_r x_s, \qquad \beta = \sum_{r,s} c_{rs}\bar{x}_r x_s. \tag{4}$$

From the relations (2) and (4), upper limits for $|\lambda|$, $|\alpha|$, and $|\beta|$ may be obtained. Since these relations are identical in form, it is sufficient to carry through the computations for one of them. Furthermore, the theorems will be stated briefly since the context will be evident.

Several results will now be established. The best upper limit may be given by one in some cases, by another in others.

THEOREM 1. *Let* $R_r=\sum_s|a_{rs}|$, $T_s=\sum_r|a_{rs}|$, $R=\max(R_r)$, $T=\max(T_s)$. *Then*

$$|\lambda| \leqq (RT)^{1/2}.$$

Let $\xi_r=|x_r|$, so that

$$\sum_r \xi_r^2 = 1.$$

If we take the absolute value of (2), we have

$$|\lambda| \leqq \sum_{r,s} |a_{rs}|\,\xi_r\xi_s. \tag{5}$$

Rewriting $|a_{rs}|\xi_r\xi_s$ as $|a_{rs}|^{1/2}\xi_r\cdot|a_{rs}|^{1/2}\xi_s$, and applying Schwarz's inequality, it follows that

$$\begin{aligned} |\lambda| &\leqq \Big(\sum_{r,s}|a_{rs}|\,\xi_r^2\Big)^{1/2}\Big(\sum_{r,s}|a_{rs}|\,\xi_s^2\Big)^{1/2} \\ &= \Big(\sum_r R_r\xi_r^2\Big)^{1/2}\Big(\sum_s T_s\xi_s^2\Big)^{1/2} \\ &\leqq \Big(R\sum_r \xi_r^2\Big)^{1/2}\Big(T\sum_s \xi_s^2\Big)^{1/2} \\ &= (RT)^{1/2}. \end{aligned}$$

THEOREM 2. *If* λ *lies in the field of values of* A, *then,*

$$|\lambda| \leqq \Big(\sum_{r,s}|a_{rs}|^2\Big)^{1/2}.$$

Rewriting $|a_{rs}|\xi_r\xi_s$ in (5) as $|a_{rs}|\cdot\xi_r\xi_s$, and applying Schwarz's inequality, we have

$$\begin{aligned} |\lambda| &\leqq \Big(\sum_{r,s}|a_{rs}|^2\Big)^{1/2}\Big(\sum_{r,s}\xi_r^2\xi_s^2\Big)^{1/2} \\ &= \Big(\sum_{r,s}|a_{rs}|^2\Big)^{1/2}\Big(\Big(\sum_r \xi_r^2\Big)^2\Big)^{1/2} \\ &= \Big(\sum_{r,s}|a_{rs}|^2\Big)^{1/2}. \end{aligned}$$

THEOREM 3. *If λ lies in the field of values of A, then $|\lambda| \leqq M$, where M is defined by* (7) *below.*

From (2),

$$
\begin{aligned}
|\lambda| &= \Big|\sum_{r,s} a_{rs}\bar{x}_r x_s\Big| \\
&\leqq \frac{1}{n}\sum_r |a_{rr}| + \frac{1}{n}\sum_s\sum_{r<s}(|a_{rr}| - |a_{ss}|)(\xi_r^2 - \xi_s^2) \\
&\quad + 2\sum_s\sum_{r<s}|a_{rs}|\,\xi_r\xi_s \\
&= \frac{1}{n}\sum_r |a_{rr}| + \sum_s\sum_{r<s}\frac{(n-1)^{1/2}}{n}(|a_{rr}| - |a_{ss}|)\cdot\frac{\xi_r^2 - \xi_s^2}{(n-1)^{1/2}} \\
&\quad + 2\sum_s\sum_{r<s}\left(\frac{n-1}{2n}\right)^{1/2}|a_{rs}|\cdot\left(\frac{2n}{n-1}\right)^{1/2}\xi_r\xi_s \\
&\leqq \frac{1}{n}\sum_r |a_{rr}| + \left\{\frac{n-1}{n^2}\sum_s\sum_{r<s}(|a_{rr}| - |a_{ss}|)^2\right. \\
&\quad \left. + \frac{n-1}{2n}\sum_s\sum_{r<s}(|a_{rs}| + |a_{sr}|)^2\right\}^{1/2}
\end{aligned}
\tag{6}
$$

by Schwarz's inequality, and the fact that $\sum_r \xi_r^2 = 1$. That is, $|\lambda| \leqq M$, where

$$
\begin{aligned}
M = \frac{1}{n}\Bigg[\sum_r |a_{rr}| + \Bigg\{(n-1)\sum_s\sum_{r<s}(|a_{rr}| - |a_{ss}|)^2 \\
+ \frac{n(n-1)}{2}\sum_s\sum_{r<s}(|a_{rs}| + |a_{sr}|)^2\Bigg\}^{1/2}\Bigg].
\end{aligned}
\tag{7}
$$

For $n = 2$, M is the maximum value of the right-hand side of (6), and hence the minimum for this type of upper limit. For $n > 2$, the form of the maximum for the right-hand side of (6) is too complicated for practical use.

THEOREM 4. *Let* $U_r = \sum_s |a_{rs}|^2$, $V_s = \sum_r |a_{rs}|^2$. *Then* $|\lambda| \leqq (\sum_r (U_r V_r)^{1/2})^{1/2}$.

From (1),

$$\lambda x_r = \sum_s a_{rs} x_s.$$

Multiplying each by λ, and replacing λx_r from (1), we have

$$\lambda^2 x_1 = a_{11}(a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n) + \cdots + a_{1n}(a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n),$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots,$$

$$\lambda^2 x_n = a_{n1}(a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n) + \cdots + a_{nn}(a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n).$$

Multiplying the rth equation by $\bar{x}_r$, adding the n equations, and taking the absolute value of both sides, we get

$$\begin{aligned} \lambda^2 \leqq {} & (|a_{11}|\xi_1 + |a_{21}|\xi_2 + \cdots + |a_{n1}|\xi_n)(|a_{11}|\xi_1 + |a_{12}|\xi_2 \\ & + \cdots + |a_{1n}|\xi_n) + \cdots + (|a_{1n}|\xi_1 + |a_{2n}|\xi_2 + \cdots \\ & + |a_{nn}|\xi_n)(|a_{n1}|\xi_1 + |a_{n2}|\xi_2 + \cdots + |a_{nn}|\xi_n) \\ \leqq {} & \sum_r (U_rV_r)^{1/2}, \end{aligned} \tag{8}$$

since, by Schwarz's inequality, $\sum_r c_r\xi_r \leqq (\sum_r c_r^2)^{1/2}(\sum_r \xi_r^2)^{1/2} = (\sum_r c_r^2)^{1/2}$.

THEOREM 5. *For $n=2$, $|\lambda| \leqq (RT)^{1/2}$, where $RT = \max (R_rT_r)$.*

From (8),

$$\begin{aligned} \lambda^2 \leqq {} & (|a_{11}|\xi_1 + |a_{12}|\xi_2)(|a_{11}|\xi_1 + |a_{21}|\xi_2) \\ & + (|a_{12}|\xi_1 + |a_{22}|\xi_2)(|a_{21}|\xi_1 + |a_{22}|\xi_2). \end{aligned} \tag{9}$$

Here, for example, $R_2T_2 \leqq R_1T_1$ implies $|a_{22}| \leqq |a_{11}|$. Then since $\xi_1\xi_2 \leqq 1/2$, (9) gives,

$$\lambda^2 \leqq R_1T_1.$$

The probability of the truth of this theorem was suggested to me by W. V. Parker. Although no contradictions have been found, no method of proof has been discovered for $n > 2$.

The effectiveness of the above theorems can sometimes be improved by using the fact that if each diagonal element of a matrix A is increased by a, the numbers λ comprising the field of values of the matrix will be increased by a.

Let L denote the best upper limit obtained for the absolute value of λ.

THEOREM 6. *Let a represent the absolute value of the determinant of A. Then*

$$a \leqq L^n.$$

This follows directly from the fact that $a = |\lambda_1\lambda_2 \cdots \lambda_n|$.

It is interesting to note that this result is frequently better than the well known upper bound given by Hadamard: "If the elements of the matrix (a_{rs}) with $r, s \leqq n$ satisfy the condition $|a_{rs}| \leqq B$, then the following inequality holds:

$$|\det [a_{rs}]| \leqq B^n n^{n/2}."$$

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UNITED STATES MILITARY ACADEMY

HADAMARD'S THREE CIRCLES THEOREM

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Hadamard's theorem is concerned with the relation between the maximum absolute values of an analytic function on three concentric circles.[1] If we put

$$M(r) = \max_{|z|=r} |f(z)|,$$

then the theorem states that log $M(r)$ is a convex function of log r for $r'<r<r''$, if $f(z)$ is regular for $r'<|z|<r''$. This is an immediate consequence of the fact that if $|f(z)| \leqq A|z|^\lambda$ on two circles about the origin, then it is also true between the circles; and this in turn is seen by applying the principle of maximum to $f(z)/z^\lambda$. The bound is attainable within the ring only for $f(z)=\alpha z^\lambda$ with $|\alpha|=A$. Notice that this function is single-valued only if λ is an integer, so that Hadamard's bound is not in general sharp for single-valued functions. (It is the sharp bound for the class of many-valued functions, any branch of which is regular in the ring, and for which $|f(z)|$ is single-valued.)

We shall consider only single-valued functions. The problem of finding the sharp bound in Hadamard's theorem is formulated as Problem A below. (It is no essential restriction to suppose that the radius of the outer circle is 1, and that the given bound on this circle is 1.) Problems B and C raise the same question for more special classes of functions.

PROBLEM A. *Suppose* $0<q<Q<1$ *and* $p>0$. *Consider the class of functions satisfying the following conditions*: $f(z)$ *is regular for* $q \leqq |z| \leqq 1$,

$$|f(z)| \leqq 1 \quad for \quad |z| = 1, \qquad |f(z)| \leqq p \quad for \quad |z| = q.$$

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[1] The theorem was stated (without proof) in Hadamard's note, *Sur les fonctions entières*, Bull. Soc. Math. France vol. 24 (1896) pp. 186–187. His proof was apparently first published in 1912; it may be found in footnote 2, p. 94, of *Selecta: Jubilé Scientifique de M. Jacques Hadamard*, Paris, 1935. In the meantime, proofs (of a less simple nature) had been given by O. Blumenthal and by G. Faber. See Blumenthal, *Über ganze transzendente Funktionen*, Jber. Deutschen Math. Verein. vol. 16 (1907) pp. 97–109, and *Sur le mode de croissance des fonctions entières*, Bull. Soc. Math. France vol. 35 (1907) pp. 213–232; Faber, *Über das Anwachsen analytischer Funktionen*, Math. Ann. vol. 63 (1907) pp. 549–551.

Let P be the largest value of $|f(Q)|$ for any function of the given class. How much is P, and for what functions is it attained?

PROBLEM B. *The same as Problem* A, *with the additional hypothesis that the coefficients of the Laurent series for $f(z)$ are positive or zero.*

PROBLEM C. *The same as Problem* A, *with the additional hypotheses that $p<1$ and that $f(z)$ is regular also for $|z|<q$ (hence for $|z|\leqq 1$).*

REMARKS. From any function $f(z)$ with $|f(Q)|=P$ we can obtain a function $H(z)$ with $H(Q)=P$, by putting $H(z)=Pf(z)/f(Q)$. The name *extremal function* will be applied only to an admissible function with $H(Q)=P$.

We may determine a real λ such that $q^{\lambda}=p$. If λ is an integer, then $H(z)=z^{\lambda}$ is the extremal function for all three problems; but if λ is not an integer, then z^{λ} is not an admissible function. We may restrict ourselves to the latter case, and shall use n to denote the integer such that $n-1<\lambda<n$.

If we indicate the dependence of the extremal function on p by using the notation $H(z, p)$, then it is clear that for the first two problems we have

$$H(z, pq^{k}) = z^{k}H(z, p)$$

for any integer k, since a power of z times an admissible function is admissible; but for Problem C no such relation is to be expected. Consequently, there is no loss of generality in supposing $q<p<1$ (that is, $n=1$) when studying Problems A and B.

Summary of results.[3] We state here some of the principal results that are known concerning the three problems. For each of the problems, the extremal function $H(z)$ exists and is unique, and is real for real z. It is univalent if $q<p<1$. In Problems A and B, $H(z)$ is independent of Q; and the same is true in Problem C at least if $q<p<1$. We tabulate some additional results in the three cases for comparison.

PROBLEM A.

$zH'(z)/H(z)$ is an elliptic function of $\log z$.

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[3] Problem A was first solved by O. Teichmüller, *Eine Verschärfung des Dreikreisesatzes*, Deutsche Mathematik vol. 4 (1939) pp. 16–22. But we shall follow here the solution given by the author in *Analytic functions in circular rings*, Duke Math. J. vol. 10 (1943) pp. 341–354. Problem B was solved by F. Carlson, *Sur le module maximum d'une fonction analytique uniforme*, Arkiv för Mathematik, Astronomi, och Fysik vol. 26A (1938). Problem C is studied by M. H. Heins in a paper, *On a problem of Walsh concerning the Hadamard three circles theorem*, Trans. Amer. Math. Soc. vol. 55 (1944) pp. 349–372, which I had the privilege of reading before publication.

$|H(z)| = 1$ for $|z| = 1$.
$|H(z)| = p$ for $|z| = q$.
If $q < p < 1$, $w = H(z)$ maps $q < |z| < 1$ on $|w| < 1$ omitting an arc of $|w| = p$.

PROBLEM B.
$H(z)$ is an average of z^{n-1} and z^n.
$|H(z)| < 1$ for $|z| = 1$ except at $z = 1$.
$|H(z)| < p$ for $|z| = q$ except at $z = q$.
If $q < p < 1$, $w = H(z)$ is a contraction with 1 as fixed point.

PROBLEM C.
$H(z)$ is a rational function of the nth degree.
$|H(z)| = 1$ for $|z| = 1$.
$|H(z)| < p$ for $|z| = q$ except at n points.
If $q < p < 1$, $w = H(z)$ maps the unit circle onto itself.

Study of Problem A. We suppose $q < p < 1$. It may be seen from general mapping theorems that there exists a function $H(z)$ which maps the ring $q < |z| < 1$ on $|w| < 1$ omitting an arc of $|w| = p$. We may suppose that p is the midpoint of the omitted arc. The function $H(z)$ is regular on the boundaries of the ring, and we have

$$|H(z)| = 1 \quad \text{for} \quad |z| = 1, \qquad |H(z)| = p \quad \text{for} \quad |z| = q.$$

If $f(z)$ is any admissible function, then $|f(z)/H(z)| \leqq 1$ on both boundaries of the ring. We could apply the principle of maximum to conclude that $|f(Q)/H(Q)| \leqq 1$, were it not for the fact that $H(z)$ has a zero in the ring, so that $f(z)/H(z)$ has a pole. The fundamental lemma of the author's paper provides an extension of this principle which enables the conclusion to be drawn nevertheless. The lemma states that *if a function is regular in a circular ring except for one simple pole, and does not exceed* 1 *in absolute value on the boundaries, then it is less than* 1 *on the radius opposite the pole.* Applying this lemma, we verify that $H(z)$ is the desired extremal function.

By applying Schwarz's reflection principle, we can continue $H(z)$ to the whole plane excluding 0 and ∞. The reflections on the outer and inner boundaries give the relations

$$H(1/z) = 1/H(z), \qquad H(q^2/z) = p^2/H(z),$$

if we use the fact that $H(z)$ is real for real z. From these it follows that

$$H(q^2z) = p^2H(z),$$

and hence

$$q^2 zH'(q^2 z)/H(q^2 z) = zH'(z)/H(z).$$

Thus $zH'(z)/H(z)$ is an elliptic function of $\log z$ with the periods $2 \log q$ and $2\pi i$.

It is not difficult to obtain the explicit formula

$$H(z) = z\theta(qz/p)/\theta(pz/q)$$

where

$$\theta(z) = \sum_{k=-\infty}^{\infty} q^{k^2} z^k.$$

This enables easy calculation of the extremal function.

Study of Problem B. Here

$$f(z) = \sum_{k=-\infty}^{\infty} c_k z^k$$

with $c_k \geqq 0$, hence $M(r) = f(r)$. Evidently

$$M''(r) \geqq 0,$$

the equality holding only if $M(r) = c_0 + c_1 r$. If $q < p < 1$, we can determine positive c_0 and c_1 so that $M(1) = 1$, $M(q) = p$; that is, so that

$$c_0 + c_1 = 1, \qquad c_0 + c_1 q = p.$$

With this determination of c_0 and c_1, the extremal function is

$$H(z) = c_0 + c_1 z.$$

This result of Carlson, which concerns a special class of functions, has an interesting application to the more general class previously considered. In fact, if we no longer suppose that $f(z)$ has positive coefficients, we have nevertheless that the average of $|f(z)|^2$ on $|z| = r$ is

$$\sum_{k=-\infty}^{\infty} |c_k|^2 r^{2k},$$

which is a power series in r^2 with positive coefficients, so that Carlson's result may be applied. If we suppose given that the average of $|f(z)|^2$ on $|z| = 1$ does not exceed 1, and that the average on $|z| = q$ does not exceed p^2 $(q < p < 1)$, then the function $H(z)$ having the largest quadratic mean on $|z| = Q$ is of the form

$$H(z) = c_0 + c_1 z,$$

where c_0 and c_1 are subject to the conditions

$$|c_0|^2+|c_1|^2=1, \qquad |c_0|^2+|c_1|^2q^2=p^2.$$

Some lemmas. We consider now some results that are used in the study of Problem C. The results concern functions constant in absolute value on a circle, and interpolation by bounded functions.

In the first place, an equality such as $|F(z)|=1$ either holds identically on $|z|=1$ or at but a finite number of points, provided $F(z)$ is regular on $|z|=1$. For on the circle, $|F(z)|=1$ is equivalent to

$$F(z)\overline{F}(1/z)=1.$$

Since the left side is regular on the circle, the result follows.

Suppose now that $F(z)$ is regular for $|z|\leqq 1$ and that $|F(z)|=1$ for $|z|=1$. Let $a_1, a_2, \cdots, a_n$ be the zeros of $F(z)$ in $|z|<1$. Then we find that

$$F(z)=\alpha\prod_{k=1}^{n}\frac{z-a_k}{1-\bar{a}_kz} \qquad (|\alpha|=1)$$

by applying the principle of maximum and minimum to $F(z)$ divided by the product on the right. The zeros and poles of $F(z)$ are inverse with respect to the given circle $|z|=1$. A similar result holds for any other circle. If $|F(z)|$ were constant on two circles about the origin, $F(z)$ being regular within the larger circle, then the zeros and poles would have to be inverse with respect to both circles, that is, the zeros at 0 and the poles at ∞, and hence $F(z)=\alpha z^n$.

Concerning interpolation by bounded functions, we need the following theorem. *Let $z_1, z_2, \cdots, z_n, \zeta$ be $n+1$ distinct points in $|z|<1$, and let $w_1, w_2, \cdots, w_n, \omega$ be any $n+1$ points in $|w|\leqq 1$. Consider the class of functions $F(z)$ regular for $|z|<1$ and with $|F(z)|\leqq 1$ there. The number of such functions satisfying the interpolating conditions*

$$F(z_k)=w_k \qquad (k=1,2,\cdots,n)$$

may be 0, 1, or ∞. If there is just one such function, then it is rational of less than the nth degree, and satisfies $|F(z)|=1$ for $|z|=1$. If there are infinitely many such functions, then the possible values of $F(\zeta)$ fill a closed circle; the additional condition

$$F(\zeta)=\omega$$

will determine the function uniquely if and only if ω is on the boundary of that circle.

The proof is by induction. Consider first the case $n=0$. The number of functions is infinite. The possible values of $F(\zeta)$ fill the circle $|w| \leqq 1$. The condition $F(\zeta) = \omega$ determines $F(z)$ uniquely, if and only if $|\omega| = 1$.

Suppose now that $n > 0$. In case $|w_n| = 1$, if there is any solution it is $F(z) = w_n$, which is a rational function of the zeroth degree and satisfies $|F(z)| = 1$ for $|z| = 1$. In case $|w_n| < 1$, we establish a one-to-one correspondence between the given functions and those satisfying certain interpolating conditions at the points $z_1, \cdots, z_{n-1}$ by means of the equation

$$G(z) = \frac{F(z) - w_n}{1 - \bar{w}_n F(z)} : \frac{z - z_n}{1 - \bar{z}_n z} \cdot$$

The various desired conclusions about $F(z)$ follow easily from the corresponding conclusions about $G(z)$.

Study of Problem C. Here the case $q < p < 1$ has a very simple solution (but the general case cannot be reduced to it). For if the condition $|f(z)| \leqq p$ for $|z| = q$ is replaced by the weaker condition $|f(q)| \leqq p$, then it follows (using Schwarz's lemma) that the maximum possible value of $|f(Q)|$ is attained by the linear function $H(z)$ which maps the unit circle onto itself, with ± 1 as fixed points, and $H(q) = p$. But this function clearly satisfies $|H(z)| \leqq p$ for $|z| = q$ (under the hypothesis $q < p < 1$), and hence is the required extremal function.

For the general case, Heins reaches his results by a rather indirect method. It will be convenient to modify the statement of Problem C by supposing that $f(z)$ is regular for $|z| < 1$ rather than for $|z| \leqq 1$. The condition $|f(z)| \leqq 1$ for $|z| = 1$ may be replaced by $|f(z)| < 1$ for $|z| < 1$. With this modification, the existence of an extremal function $H(z)$ is clear from the theory of normal families. It will be shown later that this function is unique, regular for $|z| = 1$, and that $|H(z)| = 1$ for $|z| = 1$.

As a first step, we show that if $H(z)$ is regular for $|z| = 1$, then $|H(z)| = 1$ there. Otherwise, $|H(z)| = 1$ at only a finite number of points on $|z| = 1$, and hence we can find a small arc AB of $|z| = 1$, near 1, where $|H(z)| < 1$. Choose $K > 1$ so that $K|H(z)| < 1$ on this arc. Now it is easy to construct a function $g(z)$, regular for $|z| \leqq 1$ except at A and B, with constant absolute values on every circular arc joining A and B, these values varying from K on the given arc AB to 1 on the complementary arc of the unit circle. Since $|H(z)g(z)| \leqq 1$ for $|z| = 1$, except at A and B, and $|H(z)g(z)| < 1$ near these points, it follows that $|H(z)g(z)| < 1$ for $|z| < 1$. The function $H(z)g(z)$ may fail to be an admissible $f(z)$ by being too large on $|z| = q$. However,

$g(z)$ is larger at Q than on $|z|=q$, so that if we divide $H(z)g(z)$ by the maximum of $g(z)$ on $|z|=q$, we have a function $f(z)$ which is admissible and for which $|f(Q)|>|H(Q)|$, which is impossible.

Next we see that $|H(z)|=p$ has only a finite number of roots on $|z|=q$. For otherwise it would be an identity, and $H(z)$ would be rational. But then $H(z)$ would also be regular for $|z|=1$, and hence $|H(z)|=1$ there. From the fact that $|H(z)|$ is constant on two circles, we should conclude that $H(z)=z^n$, and hence $p=q^n$, which is the case we have excluded.

We shall now show that in the hypothesis that $|f(z)|\leqq p$ for $|z|=q$, only a finite number of points on $|z|=q$ have any weight. Let $H(z)$ be an extremal function, and let $z_1, z_2, \cdots, z_l$ be the points on $|z|=q$ where $|H(z)|=p$. Then if $|F(z)|<1$ for $|z|<1$, and $|F(z_k)|\leqq p$ $(k=1, 2, \cdots, l)$, we can conclude that $|F(Q)|\leqq P$ (that is, we obtain the same bound for $|F(Q)|$ as if we had supposed that $|F(z)|\leqq p$ for $|z|=q$). For if there were such a function with $|F(Q)|>P$, then we could also find a function with $|F(z_k)|<p$ and $F(Q)>P$. But then we see that

$$f(z) = (1-\epsilon)H(z) + \epsilon F(z)$$

is admissible (if ϵ is sufficiently small), and that $f(Q)>P$.

The next step is to see that $H(z)$ can be determined by interpolation. We shall show that if $|F(z)|<1$ for $|z|<1$, if $F(z_k)=H(z_k)$ $(k=1, 2, \cdots, l)$, and if $F(Q)=H(Q)$, then $F(z)=H(z)$ identically. Consider first the interpolating problem defined by $F(z_k)=H(z_k)$. The possible values of $F(Q)$ are restricted to a circle including P. If P were not on the boundary, then $F(Q)>P$ would be possible. Thus P is on the boundary, and the additional condition $F(Q)=P$ serves to determine $F(z)$ uniquely.

Thus $H(z)$ is a rational function of at most the lth degree, with $|H(z)|=1$ for $|z|=1$. From this the uniqueness of $H(z)$ follows at once. For if both $H_1(z)$ and $H_2(z)$ were extremal, then so also would be their average. This average must also satisfy the condition $|H(z)|=1$ for $|z|=1$, which is possible only if $H_1(z)=H_2(z)$ on $|z|=1$ and hence identically. Furthermore, if $H(z)$ is extremal, so also is $\bar{H}(z)$; hence $\bar{H}(z)=H(z)$, or $H(z)$ is real for real z.

Now consider the degree of $H(z)$. In the first place, $|H(z)|=p$ has l different roots on $|z|=q$, and these are all of even order, since $|H(z)|\leqq p$ on $|z|=q$. In other words, the equation

$$H(z)H(q^2/z) = p^2$$

has at least $2l$ roots, so that $H(z)$ cannot be of less than the lth degree. Thus $H(z)$ is exactly of the lth degree, and the roots of the displayed equation all lie on $|z|=q$, and are double roots.

As in the introduction, let n be the integer such that $n-1<\lambda<n$. We shall give a sketch of the proof that $l=n$.

In the first place, it is not difficult to show that P is a continuous function of q, p, Q; and using this fact, it may be shown that (for $|z|<1$) $H(z)$ depends continuously on these parameters.

Next we notice that l is a function of n only. For as long as we exclude the case in which λ is an integer, l is the number of double roots of $H(z)H(q^2/z)-p^2$ on $|z|=q$. Since this function is regular and depends continuously on the parameters for $q^2<|z|<1$, and has roots only on $|z|=q$, it cannot gain or lose a root.

Recalling that the degree of $H(z)$ is equal to the number of zeros in $|z|<1$, it is easy to see that the degree is a lower semi-continuous function of the parameters. Since the degree is n when $\lambda=n$, it cannot be less than n when λ is slightly less than n. Combined with the preceding result, this shows that $l\geqq n$.

Finally, by an ingenious method which we cannot consider here, Heins finds (for any given n) some cases in which it can be shown that the degree of $H(z)$ does not exceed n. This completes the proof that $H(z)$ is of exactly the nth degree when $n-1<\lambda<n$, and that $|H(z)|=p$ has exactly n different roots on $|z|=q$.

UNIVERSITY OF CALIFORNIA

THE SUMMER MEETING IN WELLESLEY

The fiftieth Summer Meeting of the Society and the twenty-sixth Colloquium were held at Wellesley College, Wellesley, Massachusetts, on Sunday and Monday, August 13–14, 1944. The Mathematical Association of America met on Saturday, August 12, and the Institute of Mathematical Statistics met on Saturday and Sunday, August 12–13. About three hundred persons attended the meetings among whom were the following two hundred thirty-two members of the Society:

C. R. Adams, Louise Adams, R. B. Adams, R. P. Agnew, E. J. Akutowicz, Warren Ambrose, T. W. Anderson, R. F. Arens, H. E. Arnold, K. J. Arnold, L. A. Aroian, S. P. Avann, D. H. Ballou, Ruth Bari, J. L. Barnes, J. J. Barron, H. P. Beard, Stefan Bergman, Lipman Bers, Garrett Birkhoff, G. D. Birkhoff, M. L. Boas, R. P. Boas, H. W. Bode, A. H. Bowker, J. G. Bowker, J. V. Breakwell, R. H. Breusch, H. W. Brinkmann, E. T. Browne, R. E. Bruce, R. C. Buck, Leonarda Burke, W. D. Cairns, R. H. Cameron, B. H. Camp, M. E. Carlen, W. B. Carver, W. F. Cheney, S. S. Chern, M. D. Clement, I. S. Cohen, Nancy Cole, J. B. Coleman, Esther Comegys, T. F. Cope, L. P. Copeland, Richard Courant, A. P. Cowgill, J. H. Curtiss, M. D. Darkow, M. M. Day, F. F. Decker, C. E. Dimick, Arnold Dresden, William H. Durfee P. S. Dwyer, J. E. Eaton, Samuel Eilenberg, Paul Erdös, H. S. Everett, W. H. Fagerstrom, E. E. Fedder, Will Feller, R. M. Foster, J. S. Frame, C. W. Franklin, Philip Franklin, R. E. Fullerton, D. G. Fulton, A. S. Galbraith, C. A. Garabedian, H. A. Garabedian, R. E. Gaskell, H. M. Gehman, Abe Gelbart, B. H. Gere, Michael Goldberg, R. O. Goodman, W. H. Gottschalk, Lewis Greenwald, V. G. Grove, E. J. Gumbel, P. R. Halmos, M. G. Harrison, G. E. Hay, K. E. Hazard, O. C. Hazlett, G. A. Hedlund, A. E. Heins, M. R. Hestenes, T. H. Hildebrandt, Einar Hille, D. L. Holl, T. R. Hollcroft, Harold Hotelling, S. E. Hotelling, E. M. Hull, M. G. Humphreys, C. C. Hurd, Witold Hurewicz, B. M. Ingersoll, R. N. Johanson, A. W. Jones, B. W. Jones, F. B. Jones, H. A. Jordan, G. K. Kalisch, Wilfred Kaplan, M. E. Kellar, J. L. Kelley, L. M. Kelley, D. E. Kibbey, S. H. Kimball, J. R. Kline, M. E. Ladue, Rafael Laguardia, A. L. Lanckton, R. E. Lane, J. A. Larrivee, Joseph Lehner, Norman Levinson, Jaime Lifshitz, Charles Loewner, L. H. Loomis, E. R. Lorch, W. S. Loud, N. H. McCoy, E. A. McDougle, P. H. McGrath, E. J. McShane, C. C. MacDuffee, Saunders MacLane, Ingo Maddaus, M. G. Malti, R. J. Marcou, Morris Marden, W. T. Martin, P. R. Masani, A. E. Meder, B. E. Meserve, D. S. Miller, R. v. Mises, E. B. Mode, C. N. Moore, R. K. Morley, G. D. Mostow, C. A. Nelson, Otto Neugebauer, P. B. Norman, L. R. Norwood, C. O. Oakley, R. E. O'Connor, Rufus Oldenburger, P. S. Olmstead, Glynn Owens, Gordon Pall, R. S. Pate, R. S. Phillips, Everett Pitcher, Harry Pollard, A. L. Putnam, J. F. Randolph, C. H. Rawlins, Mina Rees, C. F. Rehberg, Eric Reissner, H. J. Riblet, Harris Rice, R. G. D. Richardson, C. E. Rickart, E. K. Ritter, H. A. Robinson, S. L. Robinson, R. E. Root, P. C. Rosenbloom, L. L. Runge, H. G. Russell, Raphael Salem, F. E. Satterthwaite, Henry Scheffé, O. F. G. Schilling, C. H. W. Sedgewick, R. W. Sedgewick, Abraham Seidenberg, H. S. Sharp, Seymour Sherman, W. A. Shewhart, Max Shiffman, D. M. Smiley, M. F. Smiley, Andrew Sobczyk, A. H. Sprague, E. R. Stabler, M. E. Stark, E. P. Starke, H. W. Steinhaus, R. W. Stokes, R. R. Stoll, M. H. Stone, D. J. Struik, J. S. Stubbe, M. M. Sullivan, Otto Szász, J. D. Tamarkin, G. B. Thomas, M. M. Torrey,

J. I. Tracey, W. J. Trijitzinsky, A. W. Tucker, J. W. Tukey, H. S. Vandiver, Andrew Vazsonyi, J. H. Van Vleck, Abraham Wald, G. L. Walker, R. M. Walter, S. E. Warschawski, A. H. Wheeler, M. E. White, W. F. Whitmore, Hassler Whitney, D. V. Widder, V. A. Widder, E. W. Wilson, Jacob Wolfowitz, H. A. Wood, M. M. Young, S. D. Zeldin, J. A. Zilber, M. F. Zucker, Antoni Zygmund.

The Colloquium Lectures on the subject *Selected topics in the theory of semi-groups* were delivered by Professor Einar Hille of Yale University on Sunday and Monday mornings and afternoons. President M. H. Stone, Professor G. D. Birkhoff, Professor C. C. MacDuffee and Professor J. D. Tamarkin presided at the lectures.

On Sunday at 3:15 P.M., Professor C. C. MacDuffee gave an address entitled *On the composition of algebraic forms of higher degree.* Professor T. H. Hildebrandt presided.

Presiding officers at the sessions for contributed papers were: Probability and Statistics, Sunday morning, Professor B. H. Camp; Analysis, Sunday morning, Professor R. P. Agnew; General Session, Monday morning, Professor M. R. Hestenes; Algebra and Applied Mathematics, Monday afternoon, Professor Gordon Pall and Dean R. G. D. Richardson; Geometry and Topology, Monday afternoon, Professor A. W. Tucker and Professor P. R. Halmos.

Headquarters for the meeting were in Tower Court. Rooms in the dormitory were made available for those attending the meeting.

On Saturday evening, Wellesley College received the mathematicians and their guests in the Recreation Building.

Tea was served Saturday and Monday afternoons in the Great Hall of Tower Court.

A photograph was taken on Sunday at 1:15 P.M.

On Sunday evening there was a dinner for members of the Society, Association, and Institute and their guests. Professor D. V. Widder was toastmaster. Miss Mildred McAfee, President of Wellesley College and Captain, United States Naval Reserve, welcomed the guests. Professor M. H. Stone spoke on American mathematics in the present war. Professor J. S. Frame presented a resolution of thanks to President McAfee, the local committee and all who assisted them for their excellent arrangements and cordial hospitality. Following the dinner, Professor David Barnett of Wellesley College gave a piano recital.

The Council met at 9:20 P.M. on August 13, 1944, in Great Hall, Tower Court.

The Secretary announced the election of the following nineteen persons to membership in the Society (the address given is that of the member at the time of his application):

Dr. Louise Belai, Our Lady of Cincinnati College;

Mr. John Valentine Breakwell, Tufts College;
Mr. Richard Harland Brown, United States Navy Department;
Mr. Paul D'Arco, DePaul University;
Mr. Edward William Hazleton, Illinois Institute of Technology;
Mr. Russell Frederick Heckman, Ingwersen Manufacturing Company, Inc., Denver, Colo.;
Major William Hodgkinson, Jr., Army Air Forces, Washington, D. C.;
Mr. James Mann Hurt, University of Texas;
Mr. Phillip Sanford Jones, University of Michigan;
Miss Margaret Elizabeth Kellar, American Mathematical Society, New York, N. Y.;
Professor Rafael Laguardia, Instituto de Matematicas de la Facultad de Ingeniería, Montevideo, Uruguay;
Mr. Paul Ellis Meadows, Cranbrook School, Bloomfield Hills, Mich.;
Miss Alice E. Andrews Priestley, Lafayette College, Easton, Pa.;
Ida Roettinger (Mrs. Wilfred Kaplan), University of Michigan;
Professor Helen Adams Rouce, Langston University, Langston, Okla.;
Mr. John Lawson Senior, Jr., Glenn L. Martin Co., Middle River, Md.;
Mr. Jules H. Sreb, Johns Hopkins University;
Dr. Andrew Vazsonyi, Graduate School of Engineering, Harvard University;
Professor Shu-Chi Wu, National Hunan University, China;
Rev. Alfred John Zanolar, St. Joseph's College, Collegeville, Ind.;
Sister M. Claudia Zeller, College of St. Francis, Joliet, Ill.

It was reported that the following had been elected as nominees on the Institutional Membership of the institution indicated:

California Institute of Technology: Messrs. Robert Francis Koeber Benton and Albert George Wilson.

The following appointments by President M. H. Stone were reported: as representative of the Society at the inauguration of Harry S. Devore as President of Central College (Fayette, Missouri) on May 25, 1944, Professor G. E. Wahlin; as a Committee on Arrangements for the 1944 Annual Meeting in Chicago on November 24–25, Professors L. R. Ford (Chairman), T. R. Hollcroft, A. S. Householder, B. W. Jones, R. G. Sanger, and H. S. Wall; as a committee to study certain problems in connection with the future of Mathematical Reviews, Professors Oswald Veblen (Chairman), G. D. Birkhoff, Dr. Warren Weaver, and Secretary J. R. Kline (ex officio); as a committee to study the role of the Society in the publication of mathematical research, Professors R. L. Wilder (Chairman), Garrett Birkhoff, R. E. Langer, J. D. Tamarkin, A. E. Taylor, Oscar Zariski and Secretary J. R. Kline (ex officio).

Times and places of meetings during 1945 were set as follows: February 24 in New York City; April 27–28 in New York City; April 27–28 in Chicago; April 28 in Stanford University; October 27 in New York City.

Certain invitations to give hour addresses were announced: Professor S. M. Ulam for the April, 1945, meeting in Chicago; Professor Gabor Szegö and Dr. František Wolf for the November, 1944, meeting in Los Angeles.

The Council adopted the following resolution on the death of Professor David Eugene Smith:

> David Eugene Smith, professor emeritus of mathematics at Teachers College in Columbia University, died on 29 July, 1944, aged 84 years. Probably no other member of the Society had as many friends in this country and where he had personally visited in almost every country of the world. His special studies were in the fields of Pedagogy and History of Mathematics, and his attractive personality, coupled with exceptional gifts of facile and interesting exposition, made him an ideal lecturer. The special Library which he accumulated contains books, pamphlets, scores of medals and medallions, and thousands of portraits of mathematicians, as well as thousands of manuscripts and tablets, and it will long be a source for research. He was a prolific and popular writer, and mathematical editor of many encyclopedias. Millions of copies of his books were sold.
>
> The Society is indebted to him for service as a member of the Editorial Committee of its Bulletin, and for the extraordinary development of its Library while Librarian during a period of nineteen years. He was also one of our Vice Presidents. We salute the memory of one with whom contact always left a beautiful memory. Resolved that this minute be spread on the records and that a copy be sent to his family.

President Stone, as Chairman of the War Policy Committee, presented a report on the activities of his Committee as prepared for the Rockefeller Foundation. As announced previously, the Rockefeller Foundation made a grant of $2,500 to the War Policy Committee for its activities during the period August 5, 1943, to August 1, 1944. Of this amount, $1,500 has been contributed to the support of the Office of Scientific Personnel in Washington. This Office, which is subsidized by several leading scientific societies, has been of great aid to the mathematicians in a number of problems, including those of deferment. The War Policy Committee now has three subcommittees: Committee on Available Teachers of Collegiate Mathematics; a Subcommittee on War Training Programs; a Subcommittee to advise the Examinations Staff of the United States Armed Forces Institute.

Since the inauguration of the War Policy Committee, the office of the Secretary of the Society has continued its policy of supplying to chairmen of departments of mathematics throughout the country, as quickly as possible, certain information regarding the proper use of mathematicians in the war effort and of regulations from Selective Service affecting the mathematicians. In this connection, nine memoranda have been distributed to some three hundred fifty chairmen of departments of mathematics. It is hoped that the information has helped chairmen in the solution of their personnel problems and has contributed to the most effective use of mathematicians in the war effort.

The Committee on Available Teachers of Collegiate Mathematics, with headquarters in the Secretary's office, was appointed in order to aid department chairmen in securing teachers for the Army and Navy programs. Since the inauguration of the Committee's work in April, 1943, two hundred seventy-four persons have registered. The Committee has answered requests from one hundred four different institutions. It is hoped that this Committee has been providing a worth while service in connection with the problem of the proper allocation of teachers of mathematics. Until the curtailment of the Army Specialized Training Program, there was an acute shortage of teachers of mathematics and department chairmen have taken advantage of the service and have found it valuable. This Committee will also be of service in aiding department chairmen in the solution of their postwar problems.

At the request of Dr. Ralph W. Tyler, Chief of the Examinations Staff of the Armed Forces Institute, a subcommittee of the War Policy Committee was appointed in June, 1944, to advise Dr. Tyler's staff. Members of this Committee are: Professors W. T. Reid (Chairman), Ralph Beatley, L. L. Dines, W. L. Hart, and C. C. MacDuffee. As colleges and universities will face the problem, after the war, of the proper credit to be assigned to courses taken under the auspices of the Armed Forces Institute, this subcommittee will perform a vital service to mathematicians in this connection.

The Subcommittee on War Training Programs has prepared careful analyses of the mathematical portions of the various service programs, including the College Training Program (Aircrew) of the Army Air Forces, the Army Specialized Training Program, the Navy College Training Program. Copies of the latter two reports were distributed by the Secretary's office to chairmen of departments of mathematics and a selected list of other interested persons. The Navy Program report contained some specific recommendations which, it is hoped, will be of aid in making certain improvements in the Program.

Titles and cross references to the abstracts of papers read are given below. The papers were read as follows: papers 1–6 in the section for Probability and Statistics, Sunday morning; papers 7–11 in the section for Analysis, Sunday morning; papers 12–16 in the general session, Monday morning; papers 17–22 in the session for Algebra and Applied Mathematics, Monday afternoon; papers 23–27 in the section for Geometry and Topology, Monday afternoon; and papers 28–77, whose abstract numbers are followed by the letter *t*, were read by title. Professor P. L. Hsu was introduced by Professor Will Feller, Mr. Hsiung by Professor J. A. Shohat, Sister Ingonda von Mezynski by Professor H. P. Pettit. Paper 1 was read by Dr. Wolfowitz, paper 4 by Professor Feller, paper 6 by Mr. Churchman, paper 8 by Professor Cameron, paper 19 by Dr. Sherman, paper 22 by Dr. Pate, paper 24 by Professor Eilenberg.

1. Abraham Wald and Jacob Wolfowitz: *Statistical tests based on permutations of the observations.* (Abstract 50-9-253.)

2. F. E. Satterthwaite: *Error control in matrix calculation.* (Abstract 50-9-251.)

3. Abraham Wald: *On cumulative sums of random variables.* (Abstract 50-9-252.)

4. P. L. Hsu: *The approximate distribution of the mean and of the variance of independent variates.* (Abstract 50-9-250.)

5. E. J. Gumbel: *Ranges and midranges.* (Abstract 50-9-249.)

6. C. W. Churchman and Benjamin Epstein: *Statistics of sensitivity data.* II. Preliminary report. (Abstract 50-9-248.)

7. P. C. Rosenbloom: *Interpolation and extremal problems for absolutely monotonic functions.* (Abstract 50-9-224.)

8. R. H. Cameron and W. T. Martin: *Evaluation of various Wiener integrals by use of certain Sturm-Liouville differential equations.* (Abstract 50-9-212.)

9. Otto Szász: *On the generalized jump of a function and Gibbs' phenomenon.* (Abstract 50-9-225.)

10. B. M. Ingersoll: *On singularities of solutions of linear partial differential equations.* II. (Abstract 50-9-217.)

11. Harry Pollard: *One-sided boundedness as a condition for the unique solution of certain heat equations.* (Abstract 50-9-223.)

12. A. W. Tucker: *Antipodal-point theorems proved by an elementary lemma.* (Abstract 50-9-259.)

13. R. P. Agnew: *Convergence fields of methods of summability.* (Abstract 50-7-180.)

14. W. J. Trjitzinsky: *Singular elliptic and hyperbolic partial differential equations.* (Abstract 50-9-226.)

15. Gordon Pall: *Note on factorization in quadratic fields.* (Abstract 50-9-205.)

16. C. N. Moore: *Convergence factors in general analysis.* I. (Abstract 50-9-221.)

17. Andrew Vazsonyi: *An existence theorem in the theory of compressible fluids.* Preliminary report. (Abstract 50-9-234.)

18. R. R. Stoll: *Primitive semigroups.* (Abstract 50-9-206.)

19. Seymour Sherman, Jane DiPaola, and H. F. Frissel: *Routh's discriminant, flutter, and ground resonance.* Preliminary report. (Abstract 50-7-190.)

20. B. W. Jones: *A canonical quadratic form for the ring of 2-adic integers.* (Abstract 50-9-203.)

21. Eric Reissner: *On the theory of bending of elastic plates.* (Abstract 50-9-232.)

22. R. J. Duffin and R. S. Pate: *Structure elements of quasigroups.* III. (Abstract 50-7-175.)

23. W. H. Gottschalk: *Totally disconnected sets and almost periodic properties.* (Abstract 50-9-257.)

24. Samuel Eilenberg and N. E. Steenrod: *Axiomatic approach to homology.* (Abstract 50-9-254.)

25. A. H. Wheeler: *One-sided polyhedra from the five regular solids.* (Abstract 50-7-193.)

26. Abraham Seidenberg: *Valuation ideals in polynomial rings.* (Abstract 50-7-192.)

27. Warren Ambrose: *Structure theorems for a special class of normed rings.* (Abstract 50-9-198.)

28. R. P. Agnew: *A family of bounded sequences summable M.* (Abstract 50-7-179-*t*.)

29. R. P. Agnew: *A genesis for Cesàro methods.* (Abstract 50-9-208-*t*.)

30. R. P. Agnew: *Criteria for completeness of orthonormal sets and summability of Fourier series.* (Abstract 50-9-209-*t*.)

31. R. P. Agnew: *Abel transforms of Tauberian series.* (Abstract 50-9-207-*t*.)

32. E. F. Beckenbach and R. H. Bing: *Concerning the vertex mean-value property of harmonic polynomials.* (Abstract 50-9-210-*t*.)

33. E. T. Bell: *Separable diophantine equations.* (Abstract 50-9-199-*t*.)

34. Stefan Bergman: *A class of nonlinear partial differential equations and their properties.* (Abstract 50-9-211-*t*.)

35. L. M. Blumenthal: *Note concerning an extension of the notion of matrix rank.* (Abstract 50-9-200-*t*.)

36. Herbert Busemann: *Local metric geometry.* (Abstract 50-9-235-*t*.)

37. R. H. Cameron and W. T. Martin: *The Wiener measure of Hilbert neighborhoods in the space of real continuous functions.* (Abstract 50-9-213-*t*.)

38. Claude Chevalley: *Intersections of algebraic and algebroid varieties.* (Abstract 50-9-236-*t*.)

39. Nathaniel Coburn: *The Kármán-Tsien pressure-volume relation in the two-dimensional supersonic flow of compressible fluids.* (Abstract 50-9-229-*t*.)

40. A. J. Coleman: *Phase space in Eddington's theory.* (Abstract 50-9-230-*t*.)

41. A. J. Coleman: *Red-shift in the sun.* (Abstract 50-9-231-*t*.)

42. John DeCicco: *Conformal maps with isothermal systems of scale curves.* (Abstract 50-9-237-*t*.)

43. J. J. Dennis: *Some points in the theory of positive definite J-fractions.* (Abstract 50-9-214-*t*.)

44. R. J. Duffin and R. S. Pate: *Structure elements of quasigroups.* II. (Abstract 50-7-174-*t*.)

45. Nelson Dunford and Einar Hille: *The differentiability and uniqueness of continuous solutions of addition formulas.* (Abstract 50-9-215-*t*.)

46. H. W. Eves: *Applications of some new matrix products to geometry.* (Abstract 50-9-238-*t*.)

47. H. W. Eves: *Skew curves setting up a null system in space.* (Abstract 50-9-239-*t*.)

48. H. W. Eves: *Some associated theorems on matrices and determinants.* (Abstract 50-9-201-*t*.)

49. Mariano Garcia: *Orbit-components and component orbits.* (Abstract 50-9-255-*t*.)

50. W. H. Gottschalk: *A note on pointwise nonwandering mappings.* (Abstract 50-9-256-*t*.)

51. W. H. Gottschalk: *Orbit-closure decompositions and almost periodic properties.* (Abstract 50-7-195-*t*.)

52. C. C. Hsiung: *A study on the theory of conjugate nets.* (Abstract 50-9-240-*t*.)

53. C. C. Hsiung: *The contact of conics and quadrics with two surfaces in space of n dimensions.* (Abstract 50-9-242-*t*.)

54. W. H. Ingram: *On the integral equations of continuous dynamical systems.* (Abstract 50-9-218-*t*.)

55. Nathan Jacobson: *Structure theory of simple rings without finiteness assumptions.* (Abstract 50-9-202-*t*.)

56. Edward Kasner: *Conformal symmetry and satellite theory for algebraic curves.* (Abstract 50-9-243-*t*.)

57. R. E. Lane: *The convergence and values of periodic continued fractions.* (Abstract 50-11-273-*t*.)

58. A. N. Lowan and H. E. Salzer: *Formulas for complex interpolation.* (Abstract 50-7-187-*t*.)

59. Sister Ingonda von Mezynski: *Projective description of some*

plane sextic curves derived from conics as base curves. (Abstract 50-9-244-*t*.)

60. D. S. Miller: *Some properties of Carathéodory linearly measurable plane point sets.* (Abstract 50-9-220-*t*.)

61. S. B. Myers: *Arcs and geodesics in metric spaces.* (Abstract 50-9-258-*t*.)

62. E. N. Nilson and J. L. Walsh: *On functions analytic in a region: approximation in the sense of least pth powers.* (Abstract 50-9-222-*t*.)

63. Moses Richardson: *The pressure distribution on a body in shear flow.* (Abstract 50-9-233-*t*.)

64. H. E. Salzer: *Coefficients for mid-interval numerical integration with central differences.* (Abstract 50-7-188-*t*.)

65. H. E. Salzer: *Table of coefficients for differences in terms of the derivatives.* (Abstract 50-7-189-*t*.)

66. Peter Scherk: *On the number of certain singularities of differentiable curves of order $n+1$ in real projective n-space.* (Abstract 50-9-245-*t*.)

67. O. F. G. Schilling: *Automorphisms of fields of formal power series.* (Abstract 50-7-176-*t*.)

68. O. F. G. Schilling: *Noncommutative valuations.* (Abstract 50-7-177-*t*.)

69. A. R. Schweitzer: *On functional equations with solutions containing arbitrary functions.* VI. (Abstract 50-7-183-*t*.)

70. A. R. Schweitzer: *On functional equations representing abstract groups.* (Abstract 50-7-184-*t*.)

71. A. R. Schweitzer: *Functional relations valid in the domains of abstract groups and Grassmann's space analysis.* (Abstract 50-7-185-*t*.)

72. W. J. Thron: *A family of simple convergence regions for continued fractions.* (Abstract 50-7-186-*t*.)

73. S. M. Ulam and C. J. Everett; *On ordered groups.* (Abstract 50-7-178-*t*.)

74. H. S. Wall: *Note on the expansion of a power series into a continued fraction.* (Abstract 50-9-227-*t*.)

75. P. A. White: *On additive properties of compact sets.* (Abstract 50-7-196-*t*.)

76. Y. C. Wong: *Family of totally umbilical hypersurfaces in Einstein 4-space.* (Abstract 50-9-246-*t*.)

77. Y. C. Wong: *Quasi-orthogonal ennuple of congruences in Riemannian space.* (Abstract 50-9-247-*t*.)

T. R. HOLLCROFT,
Associate Secretary

HARRIS HANCOCK—IN MEMORIAM

The death of Harris Hancock on March 16, 1944, at Charlottesville, Virginia, constituted a notable loss, both to the University of Cincinnati and to American mathematics. During his active career in Cincinnati, Hancock had always been a stout champion of scholarship at the highest possible level. After his retirement in 1937 he continued, in his letters to his friends and colleagues, his service in the good cause. In the last letter from him received by the writer of the present notice, written two days prior to his death, he expressed his great interest in our coming mathematical symposium and in addition made certain cogent suggestions as to the development and improvement of mathematical scholarship at our institution.

In choosing mathematics for a career Harris Hancock was motivated by an interest in the subject that was genuinely enthusiastic. This interest he succeeded in communicating to all his more able students and to many of his friends who were not professional mathematicians. He was at all times thoroughly convinced of the major importance of mathematics, both as a fundamental and virtually indispensable discipline in education at the secondary and collegiate level, and as a most powerful aid in mankind's unceasing quest to understand himself and the universe in which he lives. Hancock's constant stress on the human interest of mathematics bore substantial fruit in the inclusion of mathematics as one of eight departments of the University of Cincinnati to be aided by the Charles Phelps Taft Memorial Fund, a fund explicitly designated as being created in support of the humanities.

In his recent obituary notice on William Fogg Osgood, published in this Bulletin, Professor Koopman ventured the opinion that Osgood's excellent sense of balance in scientific fields was due in no small measure to his early training in the classics. As a former student under Hancock and Osgood and as one who has likewise profited by his own study of the classics, I wish to record here my concurrence in Professor Koopman's opinion and my further belief that a similar educational background had much to do with the broad and balanced viewpoint manifested by Harris Hancock in his approach to scientific and educational questions. Like Osgood, Hancock had received a thorough classical training in his early years. Later both men received the inspiration to be found at the great mathematical centers of Europe in the late eighties and the early nineties. Finally they both returned to this country imbued with a desire to gradually raise mathematical scholarship here to a level comparable with that of the leading European nations. Harris Hancock, in his own environment and in such larger circles to which the influence of his environment could extend, made notable contributions in this direction.

To properly evaluate the importance of his contribution one must take account of the fact that some strong educational trends of his time were of a type distinctly hostile to his scientific and educational ideals. To have breasted adverse tides that would have swept a weaker individual off his feet and to have made excellent progress in spite of such handicaps was a notable achievement.

Early in the century Hancock recognized the fact that there were developing certain pathological conditions in our educational system, for which the center of infection could be found in some of our colleges for teachers. Realizing that such conditions were a grave menace to the advancement of mathematical scholarship, he took time from the more congenial task of mathematical research and mathematical exposition to write a number of articles for the educational journals which dealt with the field of secondary education. The title of two of these: *The defective scholarship of our public schools—The pernicious influence of the colleges for teachers*, I and II (School and Society vol. 9 (1919) pp. 552–556; vol. 10 (1919) pp. 336–343) indicates clearly his readiness to take a courageous stand in matters of genuine importance. It should be recalled, in this connection, that at the time these articles were written a College for Teachers at the University of Cincinnati had been in existence for some thirteen years. In the face of strong administrative disapproval, Hancock had opposed this form of expansion, urging instead that there be created a Department of Pedagogy in the College of Liberal Arts.

In spite of the strong protests made by Hancock and some other mathematicians who shared his views, the general level of mathematical education in our secondary schools continued to fall throughout the period between World War I and World War II. The grave disadvantage which this state of affairs recently imposed on those charged with the responsibility for training officers for the Navy was clearly indicated in a letter from Admiral C. W. Nimitz (Mathematics Teacher vol. 35 (1942) pp. 88–89). Naturally, our deficiencies show up more clearly in time of war. However, they exist just as well in time of peace, since national efficiency depends to an important extent on mathematical education.

From the year 1919 on Harris Hancock could only rely on the College of Liberal Arts and the relatively small Graduate School of Arts and Sciences for the promotion of mathematical scholarship in accordance with his ideals, inasmuch as the teaching staff in the College of Engineering and Commerce was made a separate unit at that time. Since, when he first came to the University of Cincinnati in September, 1900, the entire mathematics staff consisted of himself, an assistant professor, and an instructor, one can readily see that he made notable progress in the face of many discouraging handicaps.

In spite of the time and effort spent on the development of his department and on the promotion of mathematical education at all levels, Hancock's unflagging industry and systematic methods of work enabled him to complete an impressive amount of published work during his mathematical career. This work included eight books, four of which were very extensive and authoritative treatises on the theories involved, thirty-three research papers, and seven papers on various educational questions. The books and research papers deal with a rather wide range of subjects in the fields of Analysis, Algebra, and the Theory of Numbers. Hancock's initial interest in these various topics had been enlisted during his period of European study, but he continued to work at them during his entire scientific career. His published works thus constitute the ripe harvest of many years of devoted labor.

Hancock's first books subsequent to his Berlin thesis (*Maxima and minima of functions of several variables*, 1903; *Lectures on the calculus of variations*, 1904) were published by the University of Cincinnati, which at that period had a press of its own. They were the outgrowth of a careful study of the profound work of Weierstrass and first hand contact with Weierstrass' disciple, H. A. Schwarz.

Beginning with the *Theory of elliptic functions* (John Wiley and Sons, 1910), we find the development of the theory showing the influence of much independent work by Hancock and his students. The same is characteristic of his *Elliptic integrals* (John Wiley and Sons, 1917), and in increasing measure of his *Foundations of the theory of algebraic numbers*, vols. I and II (Macmillan, 1931, 1932) and his *Development of the Minkowski geometry of numbers* (Macmillan, 1939). The date of publication of the last named work shows how the author continued his scientific labors, even after his retirement.

Most of Hancock's research papers were concerned with the same general fields treated in his books. As indicated above, some of this research, as well as other investigations, was later incorporated in his books. Other portions of it, however, are only available in the original papers. Space is lacking for a complete bibliography, but since the papers were published in well known journals, the interested student will have no difficulty in finding them.

In the case of one important activity of the American Mathematical Society, Hancock was in advance of his colleagues in other institutions. Shortly after the end of the first World War he urged strongly that an abstract journal, edited, managed, and financed in this country should be inaugurated. At that time there was little support for his idea. Many prominent mathematicians took the stand that the work of reviewing would take up so much time on the part of our younger American mathematicians that their own research would suffer. I think that experience in connection with Mathematical Reviews has shown that Hancock was right and that those who advocated the contrary view were wrong.

Hancock's efforts in behalf of better secondary school education brought about one important improvement in that field as far as the city of Cincinnati was concerned. A local committee, formed by his efforts, recommended strongly that one high school, devoted to college preparation, should be organized as part of our city school system. This recommendation was favorably acted upon by the local School Board, and from about 1920 on the Walnut Hills Classical High School, beginning college preparation in the 7th grade and continuing it through the 12th grade, was operated as an integral part of the Cincinnati Public Schools. The existence and success of such a school has undoubtedly raised the tone of education at the secondary level in the city of Cincinnati.

A certain number of Hancock's colleagues at the University of Cincinnati were in complete sympathy with his high scholarly ideals and his devoted efforts to raise the level of scholarship throughout our entire educational system. Others, however, seemed to think that his ambitions were too lofty for the institution in which he worked and that his efforts to improve scholarship in the secondary schools were foredoomed to frustration. It seems a pity that he could not have enlisted wider support for his eminently sane viewpoint. For under such circumstances there seems little doubt that his already impressive achievements would have been notably increased. However, his former students and that wider group of his friends and admirers, of which the first mentioned class constitutes a subgroup, may well find consolation in those immortal lines from Lowell's poem, *The present crisis*:

"Count me o'er earth's chosen heroes,—
they were souls that stood alone,
While the men they agonized for hurled
the contumelious stone."

CHARLES N. MOORE

AID TO LIBRARIES IN WAR AREAS

Up to the end of 1943, more than $160,000 has been spent for subscriptions to 325 scholarly and scientific journals, to be stored in this country, for distribution after the war to libraries in war areas. The money is provided by a grant from the Rockefeller Foundation, which has allotted from $50,000 to $70,000 annually for this purpose, since 1941.

Several factors influence orders placed for each title: the journal's foreign circulation before the war; the ratio of foreign circulation to total circulation; the number of free or exchange subscriptions; the number of copies available through gifts from subscribers in this country; and the relative importance of the journal to research.

In no case have purchases equalled more than half the number of paid institutional subscriptions in Europe and Asia discontinued because of the war. No funds are available for purchasing periodicals from individuals nor for the acquisition of volumes issued before 1939.

The funds from the Rockefeller Foundation are administered by the Committee on Aid to Libraries in War Areas of the American Library Association. The committee's periodical purchases are supplemented by gifts. In this way current issues and back files of important periodicals have been secured for use in restocking libraries that have been damaged or destroyed. Anyone wishing to contribute mathematical journals for this purpose should address inquiries to Miss Edith A. Wright, Committee on Aid to Libraries in War Areas, Library of Congress Annex, Study 251, Washington, D. C. Prospective donors are asked to report titles and dates of the journals available to the office of the committee. Shipping instructions will then be issued, indicating where and how shipment should be made.

NOTES

Dean T. M. Focke of the Case School of Applied Science has retired with the title emeritus.

Professor Solomon Lefschetz of Princeton University has been elected an honorary member of the Mexican Mathematical Society.

Professor I. F. Neff of Drake University has retired with the title emeritus.

Dr. O. S. Adams of the United States Coast and Geodetic Survey has retired.

Associate Professor Reinhold Baer of the University of Illinois has been promoted to a professorship.

Dr. T. A. Bancroft of Iowa State College has been promoted to an assistant professorship.

Dr. F. C. Biesele of the University of Utah has been promoted to an assistant professorship.

Associate Professor N. R. Bryan of the University of Maine has been promoted to a professorship.

Dr. R. E. Byrne has been appointed to an assistant professorship at the University of California at Los Angeles.

Dr. R. H. Cole of the University of Western Ontario has been promoted to an assistant professorship.

Assistant Professor Mary E. Decherd of the University of Texas has retired.

Assistant Professor H. L. Dorwart of Washington and Jefferson College has been promoted to an associate professorship.

Dr. C. J. Everett of the University of Wisconsin has been promoted to an assistant professorship.

Associate Professor W. M. Ewing of Lehigh University has been appointed to an associate professorship of geology at Columbia University.

Dr. Hans Fried of Swarthmore College has been appointed lecturer.

Associate Professor M. O. Gonzalez-Rodriguez of the University of Havana has been promoted to a professorship.

Mr. E. S. Grable of the University of Richmond has been promoted to an assistant professorship.

Dr. Mary C. Graustein has been appointed to an assistant professorship at Tufts College.

Associate Professor E. A. Hedberg of Baylor University, Waco, Texas, has been appointed visiting associate professor in electrical communications at Massachusetts Institute of Technology.

Dr. Olaf Helmer of the College of the City of New York has been appointed lecturer at the New School for Social Research, New York City.

Assistant Professor J. T. Hurt of the Agricultural and Mechanical College of Texas has been promoted to an associate professorship.

Associate Professor A. E. Johns of McMaster University has been promoted to a professorship.

Dr. R. E. Johnson has been appointed to an assistant professorship at Mount Holyoke College.

Assistant Professor Roberta F. Johnson of Wilson College, Chambersburg, Pennsylvania, has been promoted to an associate professorship.

Assistant Professor H. L. Krall of Pennsylvania State College has been promoted to an associate professorship.

Dr. P. E. Lewis of Oklahoma Agricultural and Mechanical College has been promoted to an assistant professorship.

Mr. H. W. Linscheid of the University of Nebraska has been appointed to an assistant professorship at Eastern New Mexico College.

Assistant Professor A. T. Lonseth of Iowa State College has been appointed to an assistant professorship at Northwestern University.

Dr. R. R. R. Luckey of Houghton College has been promoted to an associate professorship.

Dr. Leonard McFadden of Virginia Polytechnic Institute has been promoted to an associate professorship.

Professor E. L. Mackie of the University of North Carolina has been appointed Dean of Men.

Dr. E. J. Mickle of Ohio State University has been promoted to an assistant professorship.

Dr. C. G. A. Nordling of the University of Connecticut has been promoted to an assistant professorship.

Miss Elizabeth A. Oliphant of Texas College of Arts and Industries, Kingsville, Texas, has been promoted to an assistant professorship.

Assistant Professor T. S. Peterson of the University of Oregon has been promoted to an associate professorship.

Reverend J. H. Raymond of St. Martin's College, Lacey, Washington, has been promoted to an assistant professorship.

Dr. I. F. Ritter of New York University has been promoted to an assistant professorship.

Professor J. B. Rosenbach of the Carnegie Institute of Technology has been named secretary of the Division of Humanistic and Social Studies.

Dr. E. H. Rothe has been appointed to an assistant professorship at the University of Michigan.

Assistant Professor P. A. Samuelson of Massachusetts Institute of Technology has been promoted to an associate professorship.

Assistant Professor Edith R. Schneckenburger of Michigan State Normal College, Ypsilanti, Michigan, has been promoted to an associate professorship.

Dr. Abraham Schwartz of Pennsylvania State College has been promoted to an assistant professorship.

Associate Professor D. V. Steed of the University of Southern California has been promoted to a professorship.

Dr. B. M. Stewart has been appointed to an assistant professorship at Michigan State College.

Assistant Professor A. E. Taylor of the University of California at Los Angeles has been promoted to an associate professorship.

Dr. G. B. Thomas of Massachusetts Institute of Technology has been promoted to an assistant professorship.

Assistant Professor Abraham Wald of Columbia University has been promoted to an associate professorship.

Assistant Professor S. E. Warschawski of Washington University has been promoted to an associate professorship.

Associate Professor Antoni Zygmund of Mount Holyoke College has been promoted to a professorship.

The following appointments to instructorships have been announced: University of California: Dr. S. A. Schaaf; University of Chicago: Dr. R. M. Martin; Connecticut College: Dr. Josephine M. Mitchell; Cornell University: Dr. G. K. Kalisch; Drew University, Madison, New Jersey: Mr. Bernard Greenspan; University of Michigan: Dr. A. A. Grau; Northwestern University: Mr. Neil Lockwood; Ohio State University: Mr. P. S. Jones; Pennsylvania State College: Dr. L. P. Hutchison; Princeton University: Mr. Abraham Hillman; Smith College: Miss Aida Kalish; Sweet Briar College: Miss Helen S. Pollock; Wilson College, Chambersburg, Pennsylvania: Miss Alice E. A. Priestly.

Professor Ralph Fowler of the University of Cambridge died July 3, 1944, at the age of fifty-five years.

Professor P. M. Merino of the University of Havana died April 3, 1944.

Professor M. L. Drum of Bucknell University died July 2, 1944. He had been a member of the Society since 1933.

ABSTRACTS OF PAPERS

SUBMITTED FOR PRESENTATION TO THE SOCIETY

The following papers have been submitted to the Secretary and the Associate Secretaries of the Society for presentation at meetings of the Society. They are numbered serially throughout this volume. Cross references to them in the reports of the meetings will give the number of this volume, the number of this issue, and the serial number of the abstract.

Algebra and Theory of Numbers

260. R. A. Beaumont: *Groups with isomorphic proper subgroups.*

A completely reducible group is a group which is the direct product of groups of rank 1. Necessary and sufficient conditions that a completely reducible group have an isomorphic proper subgroup are found. If G is a completely reducible abelian group without elements of infinite order, then G has an isomorphic proper subgroup if, and only if, G is of infinite rank. If G is a completely reducible abelian group without elements of finite order, then G has an isomorphic proper subgroup if, and only if, either G has a direct factor which is essentially a proper subgroup of the additive group of all rational numbers or G is of infinite rank. (Received September 29, 1944.)

261. R. H. Bruck: *Quasigroup theory.* II. *The lower central series.* Preliminary report.

This abstract is a continuation of abstract 50-5-107. Let G be a loop with unit 1, $\mathfrak{G}$ be the group consisting of all products of a finite number of permutations R_x, L_y, R_x^{-1} and L_x^{-1}, where $xy = xR_y = yL_x$. R. Baer has posed the problem of determining an "inner mapping group" $\mathfrak{I} \subset \mathfrak{G}$ with the property that a subloop $H \subset G$ is normal if and only if invariant under $\mathfrak{I}$. It is shown here that $\mathfrak{I}$ is generated by the set of all permutations $R_{x,y} = R_x R_y R_{xy}^{-1}$, $M_{x,y} = R_y L_x R_{xy}^{-1}$; moreover $U \subset \mathfrak{G}$ is in $\mathfrak{I}$ if and only if $1U = 1$. If H is any subloop of G and $\mathfrak{M}$ any set of mappings of G which map H into itself, the H-subloop $H(\mathfrak{M})$ generated by all elements xAL_x^{-1} with $x \subset H$, $A \subset \mathfrak{M}$ is mapped into itself by $\mathfrak{M}$. In particular if H is a normal subloop of G then $H(\mathfrak{I})$ is normal both in G and in H, and $H/H(\mathfrak{I})$ is an abelian group. The lower central series of G may be defined by $H_0 = G$, $H_1 = G' = H_0(\mathfrak{I})$, $H_2 = H_1(\mathfrak{I})$, $\cdots$, and the notions of solubility and class are immediate. $\mathfrak{I}$ is a group of automorphisms of G only in special cases, as for example in the (non-trivial) case of a commutative Moufang loop. (Received August 21, 1944).

262. R. H. Bruck: *Quasigroup theory.* III. *Finite p-loops.* Preliminary report.

In this study a finite loop G is called a p-loop if and only if the associated group $\mathfrak{G}$ is a p-group. Also if H is a normal subloop of G then by definition $(H, G) = H(\mathfrak{I})$. (For explanation of these notations see abstract 50-11-261.) Finally a loop is said to be abelian if and only if it is a commutative group. It is then shown that many of the elementary theorems for p-groups, and even some of the proofs, remain valid when the word "group" is replaced throughout by "loop." For example, P. Hall's proof of the Burnside basis theorem survives without further change; the essential underlying theorems are valid but require new proofs. Again, the upper and lower central series exist and have equal length c, the class of the p-loop. However, as shown by construction, there exist metabelian p-loops of order p^2, for p an odd prime, in contrast with the case for groups. It is to be noted that although every p-loop has prime-power order, the converse is false; nevertheless every *p-group* is also a p-loop in the present sense. (Received September 21, 1944.)

263. A. L. Foster: *The idempotent elements of a commutative ring form a Boolean algebra; application to principal ideal rings; ring duality and general transformation theory.*

In a previous communication ([1] Foster and Bernstein, Duke Math. J. vol. 11 (1944) pp. 603–616), it was shown that the classical symmetry and duality of Boolean algebra is merely an instance of a symmetry-duality theory inherent in the general concept of ring. The present paper is mainly concerned with (a) an application of this theory to yield the extension of Stone's theorem mentioned in the title; (b) a specialization of this extension to the factor rings of principal ideal rings; (c) a remark on Boolean-like rings (a certain generalization of Boolean rings in which the ring addition has the same formal definition as in ordinary Boolean rings); (d) a preliminary location of the ring-duality theory of [1] within the framework of a more general transformation theory. (Received September 11, 1944.)

264. L. K. Hua: *On the distribution of quadratic non-residues and the Euclidean algorithm in real quadratic fields.* I.

This paper determines conditions under which the Euclidean algorithm cannot hold in a real quadratic field, by first obtaining an upper bound for the least quadratic non-residue q_1 of a given prime p. Let $p \geqq e^{250}$, and let q_1, q_2, and q_3 be the three least prime quadratic non-residues mod p. Then q_1 is at most $(60p^{1/2})^{0.425}$, while q_2 and q_3 have similar bounds with the coefficient 60 replaced by 240 and 720, respectively. These results are established by appeal to certain lemmas of Rosser (Amer. J. Math. vol. 63 (1941) pp. 211–232). Actually, similar results could be obtained under the weaker hypothesis $p \geqq e^{160}$, but the proof would be much more complicated. Using these inequalities it is proved that there is no Euclidean algorithm in the real quadratic field $R(d^{1/2})$ when d is square-free and greater than e^{250}. This extends results of Rédei and A. Brauer. (Received November 3, 1944.)

265. L. K. Hua and S. H. Min: *On the distribution of quadratic non-residues and the Euclidean algorithm in real quadratic fields.* II.

The problem of the existence of the Euclidean algorithm in quadratic fields $R(m^{1/2})$ is still unsolved only in the case that $m=p$ is a prime of form $8n+1$. In this paper it is proved that the algorithm does not exist if $p>137$ is a prime of form $24n+17$. (Received September 9, 1944.)

266. Nathan Jacobson: *A definition of the radical for an arbitrary ring.*

If A is an arbitrary ring with an identity, the radical N is defined to be the totality of elements z such that $1+za$ has a right inverse for every a in A. If A does not have an identity, the radical is defined by using the concept of a quasi-inverse due to Perlis. It is shown that N is a two-sided ideal and it coincides with the left radical, defined in a similar manner. If A has an identity, N is the intersection of the maximal right (left) ideals of A. The results of Stone and of McCoy are a consequence of this theorem. The author has also investigated the radical of an arbitrary algebra and of a normed ring and in the latter case has obtained the criterion: $z\in N$ if and only if $(za)^n\to 0$ for every a in A. (Received August 30, 1944.)

267. Jakob Levitzki: *Chain-conditions and nilpotency.*

Let S be an arbitrary ring. Denote by N_σ, N, and N_γ the sum of all nilpotent, semi-nilpotent and nil-ideals of S, respectively. R. Brauer (Bull. Amer. Math. Soc. vol. 48 (1942) pp. 752–758) by a simple argument proved that if the minimal condition holds for the ideals of S contained in N_σ, then N_σ is nilpotent. By a similar argument the author derived a characteristic condition for the nilpotency of N_σ (Duke Math. J. vol. 11 (1944) pp. 367–368). In the present note, characteristic minimal and maximal conditions are derived for the nilpotency of N_σ, N and N_γ. These results are corollaries of the following theorems: Denote by $R_1, R_2, \cdots$ resp. by $L_1, L_2, \cdots$ infinite sequences of right resp. left-ideals of a nil-subring T of S, then T is nilpotent if and only if: I. Each descending chain of the form $SL_1\supset SL_1L_2\supset\cdots$ and of the form $R_1S\supset R_2R_1S\supset\cdots$ is finite. II. Each ascending chain of the form $(O:L_1)_r\subset(O:L_1L_2)_r$ and of the form $(O:R_1)_l\subset(O:R_2R_1)_l\subset\cdots$ is finite. These theorems yield as a consequence various characteristic conditions for the semi-primarity of a ring. (Received September 7, 1944.)

ANALYSIS

268. R. P. Agnew: *A simple and natural notation for the theory of summability of series and sequences.*

It is proposed that methods of summability be regarded as *operators*, and that the operational (that is, functional) notation be employed in the theory of summability. Thus the statement that a given sequence $s_0, s_1, s_2, \cdots$ or $\{s_n\}$ is summable to σ by a given method A is represented by $\sigma=A\{s_0, s_1, \cdots\}$ or $\sigma=A\{s_n\}$. The statement that a series $u_0+u_1+\cdots$ or $\sum u_n$ is summable B to σ is abbreviated to $\sigma=B\{u_0+u_1+\cdots\}$ or $\sigma=B\{\sum u_n\}$. Discussions and examples are given to illustrate the notation which, the author believes, should have been universally adopted many years ago. (Received September 28, 1944.)

269. R. P. Agnew: *Spans in Lebesgue and uniform spaces of translations of step functions.*

It is shown that, for each $p>1$, the closure in the Lebesgue space L_p of the linear manifold determined by the translations of a given simple step function is the whole space L_p. An explicit formula is given for the approximation of one simple step function by linear combinations of translations of another. (Received September 23, 1944.)

270. E. F. Beckenbach: *Concerning the definition of harmonic functions.*

The following result, which may be compared with the Looman-Menchoff theorem concerning the Cauchy-Riemann first order partial differential equations, is established: If the real function $u(x, y)$ and its first order partial derivatives are continuous in the finite domain D, and if the second order partial derivatives $\partial^2u/\partial x^2$ and $\partial^2u/\partial y^2$ exist at all points of D except at most at the points of a denumerable set of points in D, are Lebesgue integrable in D, and satisfy the Laplace equation almost everywhere in D, then $u(x, y)$ is harmonic in D. (Received October 2, 1944.)

271. R. P. Boas and Harry Pollard: *Completeness of sets of functions.* I. *The Legendre functions on* $(-1,1)$. Preliminary report.

The authors discuss the completeness of sets of Legendre functions $P_{\lambda_n}(x)$ for values of the subscripts which are not necessarily integral. By use of Abel's integral equation and Young's inequality it is shown that these functions are complete $L^p(-1,1)$ if the functions $\{\cos \lambda_n x\}$ are complete $L^p(0,\pi)$ and only if $\{\sin \lambda_n x\}$ are complete $L_p(0,\pi)$. Further generalizations are obtained. (Received September 30, 1944.)

272. B. M. Ingersoll: *On singularities of solutions of linear partial differential equations.* III.

It is a well known result for second order linear partial differential equations of hyperbolic type, (1) $u_{xy}+a(x, y)u_x+b(x, y)u_y+c(x, y)u+d(x, y)=0$, that if a, b, c, d are continuous, a unique solution $u(x, y)$ of (1) is determined by the prescription of $u(x, 0)$, $u(0, y)$, if $u(x, 0)$ and $u(0, y)$ are both of class $C^{(1)}$ and $u(x, 0)_{x=0}=u(0, y)_{y=0}$. This paper contains a generalization of this classic boundary value problem. It is shown that if a, b, c, d are of class $C^{(k-1)}$ and if three determinants, the elements of which consist of linear combinations of the coefficients a, b, c, d and their derivatives evaluated at the origin, are not zero, then a unique solution $u(x, y)$ of (1) over a certain rectangular domain is determined by the prescription of $(\partial^k u(x, y)/\partial y^k)_{y=0}=f(x)$ and $(\partial^k u(x, y)/\partial x^k)_{x=0}=g(y)$, where $f(x)$ and $g(y)$ are subject to the conditions that they both be of class $C^{(k)}$ and that $f^{(k)}(0)=g^{(k)}(0)$. (Received September 28, 1944.)

273. R. E. Lane: *The convergence and values of periodic continued fractions.*

Two elementary results from the theory of linear fractional transformations are used to provide a simple, concise proof of a theorem (originally presented by Otto Stolz in a different form) which gives conditions necessary and sufficient for the convergence of the periodic continued fraction. (Received August 13, 1944.)

274. Raphael Salem: *On power series with integral coefficients.*

The paper contains results on the behavior of power series with integral coefficients and their connections with certain classes of algebraic integers. A new proof is given of the closure of the set of algebraic integers whose conjugates lie all inside the unit circle (see Duke Math. J. vol. 11 (1944) p. 103). It is proved that a power series with integral coefficients converging in the unit circle takes values arbitrarily close to any given number, unless it represents a rational function. This theorem is extended in various ways, in particular to monomorphic functions. Also a power series with integral coefficients can not be schlicht in the unit circle unless it is a rational function. The paper gives some properties of the class of algebraic integers having one conjugate inside the unit circle, and all others on the unit circle. The distribution of these numbers is studied, as well as their connection with power series with integral coefficients. (Received September 23, 1944.)

275. Otto Szász: *On Lebesgue summability and its generalization to integrals.*

Hardy and Littlewood proved that $(C, -\alpha)$ summability for some positive $\alpha<1$ implies Lebesgue summability. The author proves the following generalization: if a series is summable $(C, 1-\alpha)$ and the absolute values of the $(C, -\alpha)$ means are bounded $(C, 1)$, then the series is Lebesgue summable. This summation method is then generalized to integrals and to a general class of trigonometric series. (Received September 29, 1944.)

APPLIED MATHEMATICS

276. Stefan Bergman: *Representation of potentials of electric charge distributions.*

The author investigates the harmonic functions $V(x_k)$, $k=1, 2, 3$, obtained as potentials of charges of the linear density $F(\zeta)$ along curves $[x_k^*=\xi_k(\zeta),\ k=1, 2, 3,\ \zeta_0 \leqq \zeta \leqq \zeta_1]$, where F and the ξ_k are rational functions of ζ. $V(\zeta_0, \zeta_1; x_k)$ is a hyperelliptic integral, $\int_{\zeta_0}^{\zeta_1} F(\zeta)\left[\prod_\nu(\zeta-a_\nu)\right]^{-1/2}d\zeta$, where the a_ν are algebraic functions of the x_k. Generalizing his previous considerations (see Math. Ann. vol. 101 (1929) p. 534 and Bull. Amer. Math. Soc. vol. 49 (1943) p. 163) and using classical results, the author studies the $V(\zeta_0, \zeta_1; x_k)$ considered as functions of the x_k. Every V can be represented as a sum of the integrals of the first, second and third kind. The periods $\omega_{\alpha\beta}(x_k)$ of the integrals of the first kind are entire functions of the a_ν. The periods of the integrals of the second and third kinds can be expressed in a closed form using θ-functions in terms of the $\omega_{\alpha\beta}(x_k)$. Every $V(\zeta_0, \zeta_1; x_k)$ can be expressed in a closed form using θ-functions in terms of the $\omega_{\alpha\beta}(x_k)$ and the integrals of the first kind. Analogous results are obtained for the potentials V considered as functions of the x_k and some additional parameters Y_n, say the coefficients of the function $\xi_k(\zeta)$. (Received September 29, 1944.)

277. H. W. Eves: *Calculating machine computation forms for the three-point problem on a rectangular coordinate system.*

The three-point problem of surveying has been solved by plane table methods, mechanical methods, trigonometrical methods, geometrical construction methods, and coordinate geometry methods. In view of the general trend to tie surveys into the various state plane coordinate systems, and because of the increasing adoption by engineering concerns of calculating machine procedures to replace the former logarithmic ones, the last, or coordinate geometry methods, are considered the most important. In this paper four different geometrical construction solutions are utilized in building up four corresponding calculating machine forms for finding the coordinates of the observation point in terms of the coordinates of the three observed points and the two observed angles. These forms do not exhibit any very great differences in length or simplicity, running from one of forty entries to one of fifty-six entries, and involving more or less the same combinations of the given elements. One of the forms is essentially that developed and used by the Tennessee Valley Authority. The others are believed to be new. (Received September 11, 1944.)

278. I. J. Schoenberg: *An interpolation formula derived from heat-flow.* I.

Let $\{f_n\}$ $(-\infty<n<\infty)$ be a real sequence. Consider the function $F(x, 0)$ whose graph is the polygonal line of vertices (n, f_n) as a temperature distribution along the

x-axis at the time $t=0$. Let $F(x, t)$ denote the resulting temperature distribution at the time $t>0$ according to the equation $4\partial F/\partial t=\partial^2 F/\partial x^2$. For a fixed t, say $t=1/2$, $F(x, t)$ depending linearly on the parameters $\{f_n\}$ is now used as a family of interpolating functions. Indeed, for a given sequence of equidistant ordinates $\{y_n\}$ $(-\infty<n<\infty)$, the interpolation problem $F(n, t)=y_n$ $(-\infty<n<\infty)$ admits the following explicit solution: (1) $F(x, t)=\sum_{n=-\infty}^{\infty} y_n N(x-n, t)$. Here $N(x, t)=(1/2\pi) \cdot\int_{-\infty}^{\infty}\{\psi(u)/\phi(u)\}\cos ux\, du$ where $\psi(u)=(\sin 2^{-1}u/2^{-1}u)^2 \exp\{-t(2^{-1}u)^2\}$, $\phi(u) =\sum_{k=-\infty}^{\infty}\psi(u+2\pi k)$. For $t=0$, $N(x, 0)=1-|x|$ $(-1\leqq x\leqq 1)$, $N(x, 0)=0$ $(x>1$ or $x<-1)$, and (1) reduces to the *linear interpolation* of the ordinates $\{y_n\}$. For $t\rightarrow\infty$, $N(x, \infty)=\sin \pi x/\pi x$ and (1) reduces to the *cardinal interpolation series* (2) $F(x, \infty) =\sum_n y_n\{\sin \pi(x-n)/\pi(x-n)\}$. For a fixed finite t, such as $t=1/2$, formula (1) combines the smooth character of (2) $(t=\infty)$ with the computational advantages of linear interpolation $(t=0)$. The computational advantage of a finite t arises from the exponential damping of $N(x, t)$ as compared with the slow damping of $\sin \pi x/\pi x$. (Received October 2, 1944.)

279. I. J. Schoenberg: *On smoothing and subtabulation of empirical functions by means of heat-flow.* II.

Given a sequence of ordinates to be smoothed and subtabulated; the parameters $\{f_n\}$ of the interpolating function $F(x, t)$ of the previous paper are now determined so as to minimize $S=\sum_{n=-\infty}^{\infty}\{F(n, t)-y_n\}^2+\epsilon\sum_{n=-\infty}^{\infty}(f_n-y_n)^2$, where ϵ is a positive smoothing parameter. For $\epsilon=0$ the previous interpolation problem results. For $\epsilon=\infty$, $f_n=y_n$ and $F(x, t)$ is the smoothed version by heat-flow of the polygonal line $F(x, 0)$ of vertices (n, y_n). A compromise between strict interpolation $(\epsilon=0)$ and pure smoothing $(\epsilon=\infty)$ gives the explicit solution (1) $F(x, t, \epsilon)=\sum_{n=-\infty}^{\infty} y_n N(x-n, t, \epsilon)$, where $N(x, t, \epsilon)=(2\pi)^{-1}\int_{-\infty}^{\infty}\{(\epsilon+\phi(u))/(\epsilon+\phi^2(u))\}\psi(u) \cos ux\, du$. Clearly $N(x, t, 0)$ is identical with $N(x, t)$ of the previous paper. Eight-place tables have just been computed on punched card machines for the function $N(x, t, \epsilon)$ and its derivatives $N_x'(x, t, \epsilon)$, $N_x''(x, t, \epsilon)$ for $t=1/2$, $\epsilon=0, 0.1, 0.2, \cdots, 0.9, 1.0$ and the range $-18.5\leqq x\leqq 18.5$ (step 0.1) outside of which these functions vanish to 8 places. On increasing the smoothing parameter ϵ, the approximation (1) becomes smoother in the following sense: If $\sum y_n<\infty$ all integrals $\int_{-\infty}^{\infty}\{F^{(k)}(x, t, \epsilon)\}^2 dx$ $(k=0, 1, 2, \cdots)$ exist and each is a monotone decreasing function of ϵ in the range $0\leqq\epsilon<\infty$. This procedure is being applied to certain empirical functions for which very smooth tables are required. (Received October 2, 1944.)

280. Alexander Weinstein and J. R. Pounder: *On two elementary problems of mechanics and electromagnetic theory.*

It is shown that the problem of the motion of a heavy particle on a rotating earth and the problem of the motion of a point charge in uniform electric and magnetic fields are mathematically equivalent, except for a change of axes; whereas they are usually treated by different methods. (Received September 29, 1944.)

GEOMETRY

281. J. E. Wilkins: *A special class of surfaces in projective differential geometry.* II.

In this paper, which is a sequel to one appearing under the same title (Duke Math. J. vol. 10 (1943) pp. 667–675), a more intensive study is made of surfaces in three-dimensional projective space which satisfy the condition $\beta\psi^3-\gamma\phi^3=0$. The subclass of these surfaces which also satisfy the condition $\phi\psi k^2=9\beta\gamma$ with a constant k plays an important role. Applications are made to conjugal quadrics, D_k quadrics and curves, asymptotic section quadrics, chord section quadrics, the Segre-Darboux conjugate nets, deformable curves, projective curves, axial quadrics, and other projective differential concepts. (Received September 29, 1944.)

LOGIC AND FOUNDATIONS

282. A. E. Meder and Elizabeth Hallett: *The equipollence of two systems of axioms.*

It is proved that axioms 1–5 of the set given by Weisner (Trans. Amer. Math. Soc. vol. 38 (1935) pp. 474–484) for a "hierarchy" and the set of axioms given by Klein-Barmen (Math. Ann. vol. 111 (1935) pp. 596–621) for a "Verband" are equipollent, and that Weisner's axiom 6 is not derivable from Klein's axioms. (Received September 30, 1944.)

STATISTICS AND PROBABILITY

283. L. A. Aroian: *The frequency function of the product of two normally distributed variables.*

Let $z=xy/\sigma_1\sigma_2$, x and y normally distributed, the means be denoted by $\overline{X}$, $\overline{Y}$, standard deviations by σ_1, σ_2, coefficient of correlation by r_{xy}, and $\rho_1=\overline{X}/\sigma_1$, $\rho_2=\overline{Y}/\sigma_2$. C. C. Craig (Annals of Mathematical Statistics vol. 7 (1936) pp. 1–15) has obtained the probability function of the product as the difference of two integrals and also as the sum of an infinite series of Bessel functions of the second kind. The author proves that as ρ_1 and ρ_2 approach infinity in any manner the distribution of z approaches normality, $-1\leqq r\leqq 1$. The theorem holds in addition under the circumstances $\rho_1\rightarrow\infty$, $\rho_2\rightarrow-\infty$; $\rho_1\rightarrow-\infty$, $\rho_2\rightarrow-\infty$; $\rho_1\rightarrow-\infty$, $\rho_2\rightarrow\infty$; with $-1<r<1$. The distribution of z approaches normality, further, if ρ_1 is constant, $\rho_2\rightarrow\infty$, $-1\leqq r\leqq 1$; and the same variations in ρ_1 and ρ_2 apply as in the previous theorem. The special case $r=0$ is discussed in some detail. When ρ_1 and ρ_2 are small, the Gram-Charlier Type A series and the Pearson Type III function are excellent approximations in the proper regions. To determine the numerical values of the exact probability function of z, the author uses mechanical quadrature to evaluate the two integrals, since the series development converges very slowly even for values of ρ_1 and ρ_2 as small as 2. (Received September 25, 1944.)

TOPOLOGY

284. R. F. Arens: *Topologies for the class of continuous functionals on a completely regular space.*

Among the topologies that can be introduced into a class C of continuous mappings of one topological space A into another, B, those which are "admissible" are of especial interest. A topology for C is admissible if the expression $f(x)$, where f is an element of C and x is a point in A, depends continuously on f and x simultaneously (cf. abstract 49-1-89). The paper contains a proof that a weakest admissible topology for C always exists when A is completely regular and locally bicompact. (One topology is weaker than another if the open sets of the former are included among the open sets of the latter.) On the other hand, if A is completely regular and B contains a simple arc (in particular, if B is a real interval) and C is known to have a weakest admissible topology, it follows that A is locally bicompact. This result is interesting because it was once conjectured that a weakest admissible topology would always exist whether A were locally bicompact or not, and because it provides a new characterization for locally bicompact spaces. (Received August 28, 1944.)

285. P. J. Kelly: *Some properties of a certain interchange type of self-isometry.*

An isometric mapping of a set on itself, not the identity, which leaves each element invariant or else interchanges it with another is defined as a self-isometry of order two. It is shown that if each of two finite, isometric sets has the above type of self-isometry, and if the non-smaller of the two subsets left respectively invariant does not have this type of self-isometry, then there is an isometry of the original sets which maps one invariant subset on the other. This result is applied to prove that if the cartesian product sets E^2 and F^2 are formed from finite metric sets E and F, and are metrized by one of a certain class of functions, then E^2 isometric to F^2 implies E isometric to F provided that one of the diagonal sets of E^2 and F^2 fails to have a self-isometry of order two. (Received September 20, 1944.)

286. A. H. Stone: *On the normality of product spaces.* Preliminary report.

It is proved that a necessary and sufficient condition for a product of separable metric spaces to be normal is that all but a countable number (at most) of the factor spaces be compact. (Received August 7, 1944.)

287. P. A. White: *On regular transformations.*

A transformation $T(K)=K'$, where K is a compact metric space and T is continuous and single-valued, is called n-regular if for any sequence of points (y_i) of K' converging to y, one has $(T^{-1}(y_i))\to T^{-1}(y)$, r-regularly for all $r\leqq n$. (For definitions of r-regular convergence see G. T. Whyburn, *On sequences and limiting sets*, Fund. Math. vol. 25 (1935).) It is shown that if T is n-regular, then the n-dimensional Betti group of K is the direct sum of two groups, one of which is isomorphic with the n-dimensional Betti group of K', while the other is isomorphic with the n-dimensional Betti group of $T^{-1}(y)$ relative to K for any point $y\in K'$. It is also shown that the property of being an lc^n (locally connected for all dimensions not greater than n) is invariant under an $(n-1)$-regular transformation, and that this property is invariant under the inverse of an n-regular transformation. Finally it is shown that under n-regular transformations the property of being an n-dimensional closed Cantorian manifold is invariant provided the dimension is not increased and that the transform contains more than one point. (Received September 23, 1944.)

288. G. W. Whitehead: *On products in homotopy groups.*

This paper is devoted to an investigation of the multiplication in homotopy groups defined by J. H. C. Whitehead (Ann. of Math. vol. 42 (1941) p. 411). The principal tool is a new product, depending on four elements $\alpha\in\pi_p(X)$, $\beta\in\pi_q(X)$, $\phi\in\pi_{q-1}(F)$, and $\psi\in\pi_{p-1}(F')$, where F and F' are certain function spaces over X depending on α and β, and π_p denotes the pth homotopy group. The generalized products of J. H. C. Whitehead are characterized in terms of the ordinary products and certain isomorphisms between the homotopy groups of X and F. As a corollary to a general theorem on the structure of the function space X^{S^n} it is shown that the "Einhängung" (Freudenthal, Compositio Math. vol. 5 (1937) pp. 299–314) of a product is always inessential. Finally it is shown that if α is an element of a certain subgroup of $\pi_{p+q-1}(S^p)$ (the whole group if $q \leqq p$), and β, β' are elements of $\pi_p(X)$, then $(\beta+\beta')\alpha = [\beta\gamma, \beta'\delta] + \beta\alpha + \beta'\alpha$, where γ and δ are elements of $\pi_p(S^p)$ and $\pi_q(S^p)$ depending only on α. (Received August 23, 1944.)

NEW PUBLICATIONS

BARLAZ, J. See SZÁSZ, O.

BLAIR, M. M. Elementary statistics with general applications. New York, Holt, 1944. 14+690 pp. $3.50.

CHURCH, A. Introduction to mathematical logic. Part I. (Annals of Mathematics Studies, no. 13.) Princeton University Press, 1944. 6+118 pp. $1.75.

MORGENSTERN, O. See VON NEUMANN, J.

VON NEUMANN, J., and MORGENSTERN, O. Theory of games and economic behavior. Princeton University Press, 1944. 18+625 pp. $10.00.

SMITH, T. L. See STERNBERG, W. J.

STERNBERG, W. J., and SMITH, T. L. The theory of potential and spherical harmonics. (Mathematical Expositions, no. 3.) University of Toronto Press, 1944. 12+312 pp. $3.35.

SZÁSZ, O. Introduction to the theory of divergent series. Lectures by O. Szász, written by J. Barlaz. University of Cincinnati, 1944. vi+72 pp. $1.25.

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G. B. Price	Nelson Dunford	E. T. Bell
Tibor Radó	W. T. Martin	Richard Courant
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1944

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1942–1943—G. C. Evans, F. D. Murnaghan, M. H. Stone.
1943–1944—F. D. Murnaghan, M. H. Stone, Marston Morse.
1944–1945—M. H. Stone, Marston Morse, Warren Weaver.
1945–1946—Marston Morse, Warren Weaver, G. D. Birkhoff.

Representatives on the Council of the American Association for the Advancement of Science:

1943—E. J. Moulton, W. J. Simon.
1944—E. J. Moulton, W. J. Simon.
1945—R. P. Agnew, M. R. Hestenes.

Representatives on the Editorial Board of the Annals of Mathematics:

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Representatives on the Editorial Board of the Duke Mathematical Journal:

F. J. Murray, Morgan Ward.

Representative on the American Year Book:

J. F. Ritt.

Editorial Committee for Mathematical Surveys (to serve until December 31, 1944):

F. D. Murnaghan (Chairman), A. A. Albert, T. H. Hildebrandt.

Liaison officer in connection with editorial management of Quarterly of Applied Mathematics:

Einar Hille.

Colloquium Speakers:

1942—R. L. Wilder.
1943—E. J. McShane.
1944—Einar Hille.
1945—Tibor Radó.

Committee on Gibbs Lecturers:

J. H. Van Vleck (Chairman), B. O. Koopman, E. J. McShane.

Gibbs Lecturers:

1923—M. I. Pupin.
1924—Robert Henderson.
1925—James Pierpont.
1926—H. B. Williams.
1927—E. W. Brown.
1928—G. H. Hardy.
1929—Irving Fisher.
1930—E. B. Wilson.
1931—P. W. Bridgman.
1932—R. C. Tolman.
1934—Albert Einstein.
1935—Vannevar Bush.
1936—H. N. Russell.
1937—C. A. Kraus.
1939—Theodore von Kármán.
1941—Sewall Wright.
1943—Harry Bateman.
1944—John von Neumann.

Committee on Society Visiting Lectureship:

G. D. Birkhoff (Chairman), G. A. Bliss.

Visiting Lecturers:

1927–1928—Constantin Carathéodory, University of Munich.
1928–1929—Hermann Weyl, Zurich Technical School.
1929–1930—Enrico Bompiani, University of Rome.
1930–1931—Wilhelm Blaschke, University of Hamburg.
1931–1932—R. L. Moore, University of Texas.
1936–1937—Thirjkkannapuram Vijayaraghavan, University of Dacca.

Auditors for 1944:

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Committee on the Moore Fund:

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Committee on Publication of Moore Manuscript:

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For Eastern Sectional Meetings: T. R. Hollcroft, E. J. McShane, Saunders MacLane.
For Western Sectional Meetings: A. A. Albert, W. L. Ayres, R. E. Langer.
For Far Western Sectional Meetings: Harry Bateman, A. D. Michal, W. M. Whyburn.

Committee on Places of Meetings:

W. B. Carver (Chairman), C. R. Adams, E. W. Chittenden.

War Policy Committee:

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Subcommittee to advise Examinations Staff of the United States Armed Forces Institute: W. T. Reid (Chairman), Ralph Beatley, L. L. Dines, W. L. Hart, C. C. MacDuffee.

Emergency Executive Committee of the International Congress of Mathematicians:

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E. H. Moore, 1901–1902.
T. S. Fiske, 1903–1904.
W. F. Osgood, 1905–1906.
H. S. White, 1907–1908.
Maxime Bôcher, 1909–1910.
H. B. Fine, 1911–1912
E. B. Van Vleck, 1913–1914.
E. W. Brown, 1915–1916.
L. E. Dickson, 1917–1918.
Frank Morley, 1919–1920.
G. A. Bliss, 1921–1922.
Oswald Veblen, 1923–1924.
G. D. Birkhoff, 1925–1926.
Virgil Snyder, 1927–1928.
E. R. Hedrick, 1929–1930.
L. P. Eisenhart, 1931–1932.
A. B. Coble, 1933–1934.
Solomon Lefschetz, 1935–1936.
R. L. Moore, 1937–1938.
G. C. Evans, 1939–1940.
Marston Morse, 1941–1942.

ENDOWMENT FUND

In 1923 an Endowment Fund was collected to meet the greater demands on the Society's publication program caused by the ever increasing number of important mathematical memoirs. Of this fund, which now amounts to some $65,500, a considerable proportion was contributed by members of the Society. Under the terms of the will of the late Robert Henderson, for many years a Trustee of the Society, the Society receives approximately $4,000 yearly. Upon the death of the other legatees, the Society will receive the entire principal of the estate for its Endowment Fund.

SPECIAL FUNDS

The Bôcher Memorial Prize.

This prize was founded in memory of Professor Maxime Bôcher. It is awarded every five years for a notable research memoir in analysis which has appeared during the preceding five years in a recognized journal published in the United States or Canada; the recipient must be a member of the Society, and not more than fifty years old at the time of publication of his memoir.

First (Preliminary) Award, 1923: To G. D. Birkhoff, for his memoir *Dynamical systems with two degrees of freedom.*

Second Award, 1924: to E. T. Bell, for his memoir *Arithmetical paraphrases,* and to Solomon Letschetz, for his memoir *On certain numerical invariants with applications to abelian varieties.*

Third Award, 1928: To J. W. Alexander, for his memoir *Combinatorial analysis situs.*

Fourth Award, 1933: To Marston Morse, for his memoir *The foundations of a theory of the calculus of variations in the large in m-space,* and to Norbert Wiener, for his memoir *Tauberian theorems.*

Fifth Award, 1938: To John von Neumann, for his memoir *Almost periodic functions and groups.*

Sixth Award, 1943: To Jesse Douglas, for his memoirs *Green's function and the problem of Plateau*, *The most general form of the problem of Plateau*, and *Solution of the inverse problem of the calculus of variations.*

The Frank Nelson Cole Prize in Algebra.
The Frank Nelson Cole Prize in the Theory of Numbers.

These prizes were founded in honor of Professor Frank Nelson Cole, on the occasion of his retirement as Secretary of the American Mathematical Society and editor of the Bulletin, after twenty-five years of service; the fund was later doubled by his son, Charles A. Cole. They are awarded at five-year intervals for contributions to algebra and the theory of numbers, respectively, under restrictions similar to those for the Bôcher prize.

First Award, 1928: To L. E. Dickson, for his book *Algebren und ihre Zahlentheorie*, Zurich, 1927.

Second Award, 1931: to H. S. Vandiver, for his several papers on Fermat's last theorem published in the Transactions of the American Mathematical Society and the Annals of Mathematics during the last five years, with special reference to a paper entitled *On Fermat's last theorem*, which appeared in volume 31 of the Transactions.

Third Award, 1939: to A. A. Albert, for his papers on the construction of Riemann matrices, published in volumes 35 and 36 of the Annals of Mathematics.

Fourth Award, 1941: To Claude Chevalley, for his paper entitled *La théorie du corps de classes*, published in volume 41 of the Annals of Mathematics.

The Eliakim Hastings Moore Fund.

This fund was founded in 1922 in honor of Professor Eliakim Hastings Moore, on the occasion of the twenty-fifth anniversary meeting of the Chicago Section of the American Mathematical Society. The income from the fund is to be used at the discretion of the Council of the Society, for the publication of important mathematical books or memoirs, or the award of prizes.

The Marion Reilly Fund.

Dean Marion Reilly willed to the American Mathematical Society a portion of her estate to be used for the advancement of research in pure mathematics. The principal of this fund is $23,500.

The Ernest William Brown Fund.

From the estate of Professor Ernest William Brown, a fund of $1000 is available, the interest on which, at the discretion of the Council, can be used for the furtherance of such mathematical interests as (a) the publication of important mathematical books, memoirs, and periodicals, and (b) lectures to be delivered on special occasions by invited guests of the Society.

The Robert Henderson Fund.

Upon his retirement from the Board of Trustees on December 31, 1940, Dr. Robert Henderson presented to the Society an unrestricted gift of $1,000 which the Trustees have set aside as the Robert Henderson Fund.

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Immaculata College, Immaculata, Pa.
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Ohio State University, Columbus 10, Ohio.
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Oregon State College, Corvallis, Ore.
Pennsylvania State College, State College, Pa.
University of Pennsylvania, Philadelphia 4, Pa.
University of Pittsburgh, Pittsburgh 13, Pa.
Purdue University, Lafayette, Ind.
University of Rochester, Rochester 3, N.Y.
Rutgers University, New Brunswick, N.J.
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Stanford University, Stanford University, Calif.
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MEMBERS OF THE SOCIETY

November 1, 1944

The following list includes the name of every person who is a member of the Society as of November 1, 1944.

So far as information is available the listings are made up as follows: (a) Surname. A dagger before a surname indicates life membership. (b) Initials, or the given name when there is but one, except that the first given name of every woman member is given in full. Parentheses enclosing any part of a name indicate that that part is not generally used in correspondence. (c) Official title or position. (d) Name and address of institution or organization with which item (c) is associated. (e) The address to be used for mail if this is different from (d). The name of the city and postal district number, where necessary, are not repeated if they are the same as in (d). (f) Following the regular listing, information re military service or temporary appointments is given in italic type. This is as brief as possible in general. However, if this italic portion is complete as to city and state, this is the address to which mail should be sent.

Ablow, C. M. 1943 N. Van Ness Ave., Los Angeles 28, Calif. *Lt., U.S.N.R.*

Abramowitz, Milton. Asso. Math., Math. Tables Project, National Bur. of Standards, Dept. of Commerce, 150 Nassau St., New York 7, N.Y. 832 Ocean Ave., Brooklyn 7, N.Y.

Adams, C. R. Prof., Brown Univ., Providence 12, R. I. 60 Intervale Rd., Providence 6.

Adams, E. P. Prof. Emeritus, Princeton Univ., Princeton, N.J. Edgerstoune Rd.

Adams, G. J. Junior Civil Eng., State Highway Dept., Linden, Ala. 2411 14th Ave., Sheffield, Ala.

Adams, Louise. Asst. Prof., High Point Coll., High Point, N.C.

Adams, Dr. Rachel B. (Mrs. C. R.). 60 Intervale Rd., Providence 6, R.I.

Ader, Dr. O. B. Instr., Univ. of Arizona, Tucson, Ariz. 908 E. 4th St.

Adkisson, V. W. Prof., Univ. of Arkansas, Fayetteville, Ark. 236 Buchanan St.

Adler, Claire F. (Mrs. E. H.). Asst. Prof., New York Univ., New York, N.Y. 189–21 Tioga Dr., St. Albans, N.Y.

Adler, F. T. Visiting Asst. Prof., Physics Dept., Univ. of Wisconsin, Madison 6, Wis. *Consultant, National Research Council of Canada.*

Adshead, J. G. Asso. Prof., Dalhousie Univ., Halifax, N.S., Canada.

Agard, H. L. Prof., Williams Coll., Williamstown, Mass. Box 49.

Agnew, Dr. Jeanne L. (Mrs. Lee). Junior Research Physicist, National Research Council of Canada. 4860 Queen Mary Rd., Montreal, Que., Canada.

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Alaoglu, Dr. Leonidas. Instr., Purdue Univ., Lafayette, Ind. 419 N. 4th St.

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Albert, G. E. Asst. Prof., Ohio State Univ., Columbus 10, Ohio. *Research Math., Lukas-Harold Corp., Indianapolis 6, Ind. Box 988.*

Albert, O. W. Prof., Univ. of Redlands, Redlands, Calif. 629 Buena Vista St.

Alder, H. L. Asso., Univ. of California, Berkeley 4, Calif. 854 Santa Barbara Rd., Berkeley 1. *Private, U.S.A.*

Alexander, J. W. Prof., Inst. for Advanced Study, Princeton, N.J. 29 Cleveland Lane.

Alger, P. L. Staff Asst. to Vice Pres., General Elec. Co., Schenectady 5, N.Y. 1758 Wendell Ave., Schenectady 8. *Consultant, N.D.R.C.*

Alfieri, F. A. *Private, U.S.A. No address.*

Allen, E. B. Prof., Rensselaer Polytech. Inst., Troy, N.Y. 4 Sheldon Ave. *Research Math., Watervliet Arsenal.*

Allen, E. S. Prof., Iowa State Coll. of Agric. and Mech. Arts, Ames, Iowa.

Allen, Dr. Florence E. Instr., Univ. of Wisconsin, Madison 6, Wis. 219 Lathrop St., Madison 5.

Allen, Harriet W. Asso. Prof., Hollins Coll., Hollins College, Va.

Allen, W. R. *2d Lt., A.U.S. 1261 Air Force Base Unit, N.A.F.D., Air Transport Command, A.P.O. 396, New York, N. Y.*

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Alsberg, Julius. Chemical Eng., Office of Chief of Chemical Warfare Service, War Dept., Washington, D.C. R.F.D. 2, Peekskill, N.Y.
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Ames, L. D. Prof., Univ. of Southern California, Los Angeles 7, Calif.
Amundson, N. R. Asst. Prof., Univ. of Minnesota, Minneapolis 14, Minn. Fellow, Brown Univ., Providence 12, R.I. The Cabinet.
Ananda-Rau, K. Prof., Presidency Coll., Madras, India.
Anastasia Maria, Sister. Registrar, Immaculata Coll., Immaculata, Pa.
Andersen, C. C. 1917 Warren Blvd., Chicago 12, Ill. *1st Lt., U.S.A.*
Anderson, E. W. Asso. Prof., Iowa State Coll. of Agric. and Mech. Arts, Ames, Iowa.
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Anderson, Dr. P. H. Stat. Analyst, Distribution Div., Bur. of Foreign and Domestic Commerce, Dept. of Commerce, Washington 25, D.C.
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Anderson, W. E. Prof., Miami Univ., Oxford, Ohio. 112 E. Walnut St.
Andree, R. V. Grad. Asst., Univ. of Wisconsin, Madison 6, Wis. North Hall.
Andrews, T. B. Junior Instr., Johns Hopkins Univ., Baltimore 18, Md. *Math., N.A.C.A. 78 Elizabeth Rd., Hampton, Va.*
Annenberg, Lawrence. *Private, U.S.A. Co. A, 43 Bln., A.P.O. 711, San Francisco, Calif.*
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Apostle, H. G. Asst. Prof., Univ. of Rochester, Rochester 3, N.Y.
Archibald, H. F. Prof., Keuka Coll., Keuka Park, N.Y.
†**Archibald, R. C.** Prof. Emeritus, Brown Univ., Providence 12, R.I.
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Arnold, H. E. Asso. Prof., Wesleyan Univ., Middletown, Conn.
†**Arnold, J. W.** 37 Washington Sq., New York 11, N.Y. *Lt. Col., U.S.A.*
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Arnoldy, Sister M. Nicholas. Head, Math. Dept., Marymount Coll., Salina, Kan.
Aroian, Dr. L. A. Instr., Hunter Coll., New York 21, N.Y. 247 Wadsworth Ave., New York 33.
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Artin, Emil. Prof., Indiana Univ., Bloomington, Ind. 715 S. Fess Ave.
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Askovitz, S. I. Instr., Univ. of Pennsylvania, Philadelphia 4, Pa. 932 N. Franklin St., Philadelphia 23. *Private, A.U.S.*
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Astrachan, Max. Asso. Prof., Antioch Coll., Yellow Springs, Ohio. *Stat., U.S.A.*
Atanasoff, J. V. Prof., Iowa State Coll. of Agric. and Mech. Arts, Ames, Iowa. *Official Investigator, N.D.R.C.*
Atchison, Dr. W. F. Smithfield, Ky. *Ens., U.S.N.R.*
Atkins, H. P. Instr., Univ. of Rochester, Rochester 3, N.Y.
Avann, Dr. S. P. Instr., Oregon State Coll., Corvallis, Ore.
Avramy, Dr. Melvin. Instr., Physics Dept., Columbia Univ., New York 27, N.Y. Pupin Hall. *Consultant, O.S.R.D.*

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Ayres, W. L. Prof., Purdue Univ., Lafayette, Ind. 126 W. Lutz Ave. *Consultant, U.S.A.*

Babcock, R. W. Dean, School of Arts and Sciences, Kansas State Coll. of Agric. and Applied Science, Manhattan, Kan.
Babcock, Wealthy. Asso. Prof., Univ. of Kansas, Lawrence, Kan.
†**Bacon, Clara L.** Prof. Emeritus, Goucher Coll., Baltimore 18, Md. 2316 N. Calvert St.
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Bagemihl, Frederick. Instr., Univ. of Rochester, Rochester 3, N.Y.
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Bailey, E. A. Dean, Registrar, and Prof., LaGrange Coll., LaGrange, Ga.
Bailey, H. W. Asso. Prof. and Dir., Personnel Bur., Univ. of Illinois, Urbana, Ill.
Bailey, R. P. Asst. Prof., Lafayette Coll., Easton, Pa. *Lt. (j.g.), U.S.N.R. Room 103A, Bldg. 3, Naval Training Station, Great Lakes, Ill.*
Baker, E. G. Asso. Prof., Newark Coll. of Eng., Newark 2, N.J. 41 Yantacaw Pl., Nutley 10, N.J. *Member of Tech. Staff, American Bur. of Shipping.*
Baker, Frances E. Asso. Prof., Vassar Coll., Poughkeepsie, N.Y.
Baker, G. A. Asst. Prof. and Asst. Stat., Coll. of Agric., Univ. of California, Davis, Calif.
Bal, Y. K. Columbia Univ., New York 27, N.Y. 500 Riverside Dr.
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Southard
East Lansing
Childs
Michigan State Coll.
Beth
Bissinger
Elveback
Frame
Grove
Herzog
Hill
Jones
Mitchell
Nordhaus
Plant
Powell
Sheedy
Speeker
Stewart
Wells
Welmers
Zimmer
Fenton
Tryon
Ferndale
Jones
Highland Park
Spiro
Kalamazoo
Northam
Walton
Marquette
Spooner
Mt. Pleasant
LaRue
Ypsilanti
Schneckenburger

MINNESOTA

Collegeville
Kalinowski
Duluth
Cothran
Forth Snelling
Yagi
Minneapolis
Gilbert
Norris
Polansky
Varineau
Univ. of Minnesota
Amundson
Brink
Bussey
Campaigne
Carlson
Colson
Gibbens
Hart
Hill
Jackson
Jensen
Koehler
McEwer
Munro
Ness
Olmsted
Scammon
Shumway
Turrittin
Underhill
Moorhead
Anderson
Northfield
Norlie
Rochester
Hickman
St. Paul
Berg
Clamons
Coll. of St. Thomas
Bush
Landshoff
Spitzbart
Macalester Coll.
Camp
Winona
French
Seiler

MISSISSIPPI

Blue Mountain
Trott
Cleveland
Walters
Ward
Decatur
Dearman
Jackson
McCoy
State College
Mississippi State Coll.
Ollivier
Pettis
Walker
Welch
University
Hume

MISSOURI

Cape Girardeau
Michal
Clayton
Rosskopf
Columbia
Haynes
Univ. of Missouri
Betz
Blumenthal
Ewing
Ferguson
Shanks
Wahlin
Wehausen
Westfall
Fayette
Central Coll.
Burcham
Helton
Rodd
Fulton
Sloss
Kansas City
Basye
Lawler
Melcher
Rockhurst Coll.
Doyle
St. Theresa's Coll.
Lackay
Maryville
Godfrey
Roanoke
Ferguson
Rolla
Carpenter
Erkiletian
St. Louis
Cronvich
Green
Shreve
Fontbonne Coll.
Creane
Maryville Coll.
Kernaghan
St. Louis Coll. of Pharmacy
DeFoe
St. Louis Univ.
Case
Regan
Ross
Washington Univ.
Dunkel
Rider
Roever
Warschawski
Springfield
Erickson
H'Doubler
Drury Coll.
Finkel

MONTANA

Bozeman
Hurst
McKinsey
Butte
Smith
Garrison
Canning
Missoula
Montana State Univ.
Carey
Chatland
Dubisch
Lennes
Merrill

NEBRASKA

Harvard
Fouch
Lincoln
Union Coll.
Ogden
Univ. of Nebraska
Basoco
Brenke
Camp
Candy
Collins
Doole
Gaba
Hull
Pierce
Runge
Omaha
Earl

NEVADA

Reno
Univ. of Nevada
Beesley
Palm
Wood

NEW HAMPSHIRE

Concord
Conwell
Durham
Univ. of New Hampshire
Fulton
Giddings
Lewis
Slobin
Solt
Hanover
Clark School
Morgan
Dartmouth Coll.
Brown
Mathewson
Perkins
Robinson
Silverman
Street
Wilder

NEW JERSEY

Avon by the Sea
Daum
Bayonne
Yood
Belleplain
Durell
Belman
Greenwald
Caldwell
Fowler
Convent
O'Neill
East Orange
Molina
Fairlawn
De Prima

Hightstown
Litterick
Hoboken
Stevens Inst. of Tech.
Davis
Deimel
Korn
Olmstead
Irvington
Capecelatro
Jersey City
Markow
Linden
Scanlan
Long Branch
Rosenthal
Madison
Greenspan
Maplewood
Gutmann
Hazeltine
Montclair
Jacobus
Mountain Lakes
Hotelling
Murray Hill
Shewhart
New Brunswick
Rutgers Univ.
Brasefield
Bunyan
Fender
Grant
LeLeiko
Meder
Morris
Nelson
Robertson
Starke
Walter
Worth
Newark
Bulloch
Fenichel
Levinson
Ott
Rauch
Rhodes
Thompson
Widmark
Newark Coll. of Eng.
Baker
Fithian
Prudential Ins. Co. of America
Bartleson
Mosesson
Princeton
Brown
Epstein
Thron
Inst. for Advanced Study
Alexander
Bargmann
Gödel
Grau
Mayer
Morse
von Neumann
Straus
Veblen
Weyl
Princeton Univ.
Adams
Anderson
Bochner
Bohnenblust
Chevalley
Church
Eisenhart
Flood
Gillespie
Hillman
Lefschetz
Rauch
Singleton
Snapper
Tompkins
Tuan
Tucker
Tukey
Wedderburn
Wigner
Wilks
South Orange
Stanwick
Summit
Bode
Tenafly
Williams
Trenton
Shuster
Whitney
Upper Montclair
Campbell
New Jersey State Teachers Coll.
Davis
Mallory
West Orange
Edison
Wannemacher

NEW MEXICO

Albuquerque
Univ. of New Mexico
Barker
Gentry
Hove
Larsen
Rosenthal
Las Vegas
Rodgers
Portales
Linscheid
Silver City
Kiefer
Michelson
State College
Swingle
Wells

NEW YORK

Albany
Roland
Thompson
New York State Coll. for teachers
Beaver
Birchenough
DoBell
Lester
Stokes
Alfred
Lowenstein
Aurora
Hollcroft
Wells Coll.
Clement
Hollcroft
Rusk
Avon
Gilbert
Bethpage
Tripp
Brockport
Stopher
Bronxville
Strodt
Brooklyn
Bearman
Berry
Brock
Fately
Hopkins
Kramer-Lassar
McCarthy
Supnick
Treuenfels
Vitale
Wynne
Brooklyn Coll.
Boeker
Borofsky
Boyer
Douglas
Fleisher
Griffin
Harkin
Hurwitz
Johnson
Karlin
Kennison
Kormes
Landers
MacNeish
Maria, A. J.
Maria, D. H.
Moore
Prenowitz
Richardson
Singer
Wolfe
Woodbridge
Long Island Univ.
Lieber, H. G.
Lieber, L. R.
Polytech. Inst. of Brooklyn
Fenn
Foster
Montroll
Reagen
Weber
Whitford
Pratt Inst.
Cowles
Thompson
St. John's Univ.
Cassidy
Tolle
St. Joseph's Coll. for Women
Larkin
Buffalo
Podmele
Bell Aircraft Corp.
Giese
Pell
Shephard
Curtiss-Wright Corp.
Carner
Cramer
Keller
Komm
D'Youville Coll.
Maloney

Spencer Lens Co.
Kavanagh
Luneburg
Univ. of Buffalo
Civin
Gehman
Montague
Pound
Canton
Bates
Clinton
Hamilton Coll.
Ferry
Patterson
Williams
Corning
Lavin
Elmhurst
Eckweiler
Elmira
Suffa
Farmingdale
Goodman
Harrison
Far Rockaway
Moore
Flushing
Queens Coll.
Archibald
Brown
Cairns
Cope
Eaton
Feld
Goodman
Hartman
Hoffmann
Raudenbush
Sard
Sullivan
Williamson
Zippin
Garden City
Bowden
Stewart
Geneva
Hobart Coll.
Bruns
Durfee
Hubbs
Hamilton
Munshower
Wardwell
Hempstead
Ollmann
Stabler
Houghton
Luckey
Ithaca
Miller
Cornell Univ.
Agnew
Beinert
Carver
Curtiss
Firestone
Flexner
Goodier
Gunderson
Hurwitz
Jones
Kac
Kalisch
Karapetoff
Lawrence
LeVeque
Malti
Rosser
Scott
Sharpe
Smith
Snyder
Stephan
Sternberg
Turquette
Ventriglia
Walker
Jackson Heights
Friedman
Keuka Park
Archibald
Kew Gardens
Rosenthal
Loudonville
Mullen
Malverne
Luginbuhl
Maspeth
Gelbaum
Mineola
Huntley
Mt. Kisco
Blake
Mt. Vernon
Wetzel
New Lebanon
Pflaum
New York
Allen
Alton
Arnold, J. W.
Arnold, K. J.
Arrow
Ascher
Bigelow
Blau
Boehm
Bowker
Brodie
Burgess, L.
Burgess, R. W.
Byerly
Carpenter
Churchill
Cohn
Coleman
Colvin
Crane
Dutka
Eachus
Frey
Gordon
Greitzer
Hammer
Hastings
Holló
Hopper
Hutchinson
Ingram
Jablon
Joffe
Kagno
Kaplansky
Katz
Keyser
Kimball
Kiss
Kormes
Laderman
Ling
Lorch
Lorenzen
McGrath
Makarov
Manring
Mathews
Miller
Mills
Mitchell
Mittleman
Morrison
Neuman
Nodelman
Noether
Parsons
Pollard
Preston
Raisbeck
Reynolds
Ritt
Robinson
Roos
Roosevelt
Savage
Sherman
Shook
Simon
Snook
Solomon
Steinhardt
Stibitz
Sutherland
Swift
Tanzola
Uluçay
Wayne
Weaver
Wernick
Williams
Wolfowitz
Wurtele
Zelinsky
American Math. Society
Hull
Kellar
Zucker
American Telephone and Telegraph Co.
Belcher
Halbert
Hodgkinson
Bell Telephone Labs.
Bari
Barnes
Burton
Dietzold
Fry
Grant
Gray
Harvey
Hebbert
Hoyt
Jones
MacColl
Mason
Mead
Riordan
Schelkunoff
Shannon
Sunde
Vaage
Wise
Zobel
Coll. of the City of New York
Fagerstrom
Garrison
Gill
Grove
Hubert
Linehan
Malin
Philip
Post
Robinson
Shiffman
Wirth
Wright
Coll. of Mt. St. Vincent
Brady
Conlan
O'Brien
Columbia Univ.
Avramy

Bal
Biot
Bowden
Cole
Comer
Ewing
Fialkow
Fite
Gotz
Gourin
Hotelling
Kasner
Keyser
Koopman
Ladue
Levene
Levi
Lopata
Lorch
Martin
Mirick
Mullins
Murray
Nagel
Norman
Northcott
Poor
Ramsey
Ritt
Siceloff
Smith
Strodt
Upton
Wald
Williams
Cooper Union
Lehmann
Levenson
MacDonald
Miller
Roth
Equitable Life Assurance Society of the U. S.
Murphy
Steinhaus
Fordham Univ.
Kirby
Kubis
Lynch
Murtaugh
Hunter Coll.
Anderson
Aroian
Bradley
Bushey, H.
Bushey, J. H.
Cooper
Darkow
Guggenbühl
Hall
Hill
Landers
Levin
Miller
Norris
Rees
Simons
Tuller
Weisner
Whelan
Kellex Corp.
Henkin
Lehner
Pitts
Manhattan Coll.
Hett
Welch
Manhattan Life Ins. Co.
Porter
Manhattanville Coll. of the Sacred Heart
Kirby
Mira
Math. Tables Project
Abramowitz
Blanch
Salzer
Metropolitan Life Ins. Co.
Bassford
Davis
Herman
Hohaus
Lotka
Mutual Life Ins. Co.
Monsky
New School for Social Research
Gumbel
Helmer
Kogbetliantz
New York Telephone Co.
Beard
New York Univ.
Adler
Bernstein
Cohn
Coler
Cooley
Courant
Flanders
Friedrichs
Graham
Kline
Payne
Peck
Peters
Pohle
Putnam
Reddick
Rehberg
Ritter
Robbins
Stoker
Tilley
Wasow
Yanosik
St. Francis Xavier School
McCauley
Teachers Ins. and Annuity Asso.
Robbins
U. S. Life Ins. Co.
Grossman
Yeshiva Coll.
Ginsburg
Lowan
Matz
Niagara Falls
Lagerstrom
Morkovin
Niagara University
Banks
Oneonta
Callahan
Perry
Walker
Poughkeepsie
Vassar Coll.
Baker
Durand
McDonald
Newton
Smith
Wells
Queens Village
Raney
Rochester
Betz
LaSalle
Wyland
Eastman Kodak Co.
Chesna
Herzberger
Univ. of Rochester
Apostle
Atkins
Bagemihl
Bernstein
Gale
Ginkel
Green
Huff
Kieval
Lengyel
Lohwater
Long
Seidel
Watkeys
Wundheiler
Rocky Point
Carter
St. Bonaventure
Scheier
Schenectady
Crawford
General Elec. Co.
Alger
Poritsky
Union Coll.
Burkett
Farrell
Fox
Garis
Morse
Snyder
Syracuse
Roe
Syracuse Univ.
Carroll
Cowgill
Decker
Eves
Fox
Gelbart
Halmos
Ludwig
Martin
Ryan
Samelson
Scheffé
Troy
Rensselaer Polytech. Inst.
Allen
Ames
Basch
Brown
Campbell
Hall
Huston
Rempfer
Russell Sage Coll.
Reschovsky
West Coxsackie
Van Schaack
West Haverstraw
Nicholl
West Point
U. S. Military Acad.
Collier
Farnell
Ingersoll
Jones
Kibbey
White Plains
Downey
Willseyville
Seely
Wyoming
Hartnell
Yonkers
Miller

NORTH CAROLINA

Chapel Hill
Carroll
Univ. of North Carolina
Brauer
Browne
Cameron
Hobbs
Hurewicz
Kattsoff
Lasley
Mackie
Williams
Wong
Durham
Duke Univ.
Carlitz
Dressel
Elliott
Gergen
Hickson
Maker
Pirkle
Roberts
Thomas
Wade
Greensboro
Bennett Coll.
Stanfield
Greensboro Coll.
Pegram
Agric. and Tech. Coll. of North Carolina
Thaxton
Greenville
ReBarker
High Point
Adams
Raleigh
Efird
North Carolina State Coll.
Bullock
Cell
Fisher
Miles
Mumford
Nahikian
Park
Winton
Univ. of North Carolina
Clarkson
Salisbury
Dearborn
Tryon
Ripandelli

NORTH DAKOTA

Fargo
Smith

OHIO

Akron
Gravalos
Miller
Athens
Ohio Univ.
Marquis
Reid
Starcher
Berea
Albers
Bowling Green
Ogg
Overman
Bryan
Gardner
Cincinnati
Barlaz
Marx
Stubbe
Our Lady of Cincinnati Coll.
Belai
Univ. of Cincinnati
Barnett
Brand
Dunholter
Farnau
Horvay
Justice
Kersten
Lipsich
Lubin
Merriman
Miller
Moore
Rogers
Smith
Szász
Taylor
White
Yowell
Western and Southern Life Ins. Co.
Folz
Learson
Ryan
Xavier Univ.
Marcaccio
Stechschulte
Cleveland
Brainard
Burwell
Kraft
Latshaw
Livingood
Case School of Applied Science
Brown
Burington
Focke
Green
Guenther
Healy
Johannes
Morris
Rinehart
Thomas
Torrance
Fenn Coll.
Tolar
Topp
Western Reserve Univ.
Boyce
Musselman
Simon
Columbus
Bell
Ohio State Univ.
Albert
Bamforth
Bareis
Blumberg
Helsel
Jones
Kuhn
LaPaz
Mann
Mickle
Miller
Radó
Rasor
Rice
Strehler
Synge
Toops
Wylie
Coshocton
Morgan
Dayton
Gorn
Midgley
Univ. of Dayton
Schraut
Defiance
MacCullough
Delaware
Rowland
Gambier
Bumer
MacNeille
Granville
Denison Univ.
Kato
Ladner
Wiley
Hiram
Clarke
Kent
Kent State Univ.
Brooks
Harshbarger
Manchester
Olson
Stelson
Laurelville
Reichelderfer
Marietta
Bennett
Sandt
Marion
Leach
Mt. St. Joseph
Corona
Dimond
North Baltimore
Blackall
Oberlin
Oberlin Coll.
Cairns
Newsom
Randolph
Rodabaugh
Vance
Wagner
Yeaton, C. H.
Yeaton, M. J.
Oxford
Miami Univ.
Anderson
Pollard
Western Coll. for Women
Tappan

Painesville
Peters
South Euclid
Garvin
Springfield
Eagle
Tripp
Toledo
Brandeberry
Wellington
Ruger
Wilberforce
Davidson
Wooster
Coll. of Wooster
Hildner
Williamson
Yanney
Yellow Springs
Astrachan
Youngstown
McKissock

OKLAHOMA

Bartlesville
Morris
Langston
Rouce
Norman
LaFon
Univ. of Oklahoma
Brixey
Court
Dixon
Fullerton
Hassler
McFarland
Meacham
Reaves
Springer
Stillwater
Oklahoma Agric. and Mech. Coll.
Barnett
Hamilton
Lewis
Oboukhoff
Smith
Tulsa
Ellis
Hodell
Ricker
Sharpe
Woolson

OREGON

Corvallis
Oregon State Coll.
Avann
Hammer
Hostetter
Manning
Milne
Pomeroy
Sobczyk
Eugene
Univ. of Oregon
DeCou
Ghent
Kossack
Moursund
Peterson
Wood
Portland
Rigby
Reed Coll.
Griffin
Rosenbaum
Univ. of Portland
Vassallo
Salem
Luther
Olshen

PENNSYLVANIA

Allentown
Cedar Crest Coll.
Butz
Muhlenberg Coll.
Deck
Keck
Koehler
Altoona
Swope
Annville
Black
Bethlehem
Miesse
Lehigh Univ.
Beale
Bergmann
Cutler
Fort
Lamson
Pitcher
Raynor
Shook
Smail
Smiley
Bruceton
Brown
Bryn Mawr
Bryn Mawr Coll.
Lehr
Morrow
Oxtoby
Wheeler
California
Bernstorff
Carlisle
Ayres
Craver
Chambersburg
Wilson Coll.
Davison
Disert
Johnson
Priestley
Chester
Williams
Clarion
Carey
Dallas
Garrahan
Drexel Hill
McDonough
East Pittsburgh
Westinghouse Elec. and Mfg. Co.
Assadourian
Epstein
Rose
Slepian
Easton
Bond
Fuller
Lafayette Coll.
Bailey
Greeley
Smith
Gettysburg
Arms
Glenside
Patterson
Greensburg
McNeil
Harrisburg
McKee
Whited
Haverford
Haverford Coll.
Allendoerfer
Gummere
Oakley
Wilson
Immaculata
Anastasia Maria
Beamish
Kutztown
Bordner
Knedler
Lancaster
Murray
Latrobe
Yahner
Lewisburg
Gold
Miller
Lincoln University
Haviland
Meadville
Smith
Steen
Middletown
Rawhouser
Mill Hall
Kirk
Narberth
Robertson
Newtown
Sutton
New Wilmington
Black
Philadelphia
Brown
Campbell, R. C.
Campbell, W. B.
Constable
Dressler
Fernbach
Fudge
Groat
Hemmingsen
Jaffe
Kratz
Laden
Leifer
Levenberg
Munster
Rosenbloom
Ruley

Schoenfeld
Sherman
Smith
Tornheim
Drexel Inst. of Tech.
Davis
McNeary
Tartler
Temple Univ.
Gleason
Lawton
Rorer
Stokes
Univ. of Pennsylvania
Askovitz
Beal
Brainerd
Burks
Caris
Chapline
Clarkson
Evans
Goff
Gottschalk
Huskey
Kline
Knobelauch
Lufkin
Rademacher
Safford
Schoenberg
Shugert
Wallace
Whaples
Witmer
Wong
Pittsburgh
Boggs
Briant
Frankel
Whitman
Carnegie Inst. of Tech.
Clippinger
Dines
Hestenes
Landau
Morehead
Moskovitz
Neeley
Olds
Peach
Rosenbach
Saibel
Smith
Whitman
Pennsylvania Coll. for Women
Calkins
Univ. of Pittsburgh
Culver
Foraker
Hovey
Knipp
Taylor
Scranton
Walton
Selinsgrove
Robison
State College
Pennyslvania State Coll.
Cohen
Cunningham
Curry
Dotterer
Frink, A. H.
Frink, O.
Gordon
Graves
Hagen
Harrington
Herpel
Johnson
Kinney
Krall
Odle
Owens, F. W.
Owens, H. B.
Rose
Schwartz
Sheffer
Thorpe
Umberger
Wagner
Swarthmore
Franklin Inst.
Schwartz
Swann
Swarthmore Coll.
Brinkmann
Dresden
Fried
Marriott
Wallingford
Miller
Washington
Dorwart
Shaub
Wayne
Feather

PUERTO RICO

Mayagüez
Univ. of Puerto Rico
García
Morales
Sánchez-Díaz
Rio Piedras
Bobonis
Kuschke

RHODE ISLAND

Kingston
Pease
Stauffer
Portsmouth
Taliaferro
Providence
Adams
Brown Univ.
Adams
Archibald
Barron
Bartholomay
Bennett
Bergman
Bers
Carlen
Chernoff
Díaz
Erickson
Feller
Gaskell
Gilman
Greenberg
Handelman
Hay
Lindsay
Manning
Monroe
Neugebauer
Prager
Protter
Richardson
Roettinger-Kaplan
Rosenbloom
Schmidt
Slud
Smiley
Stoner
Tamarkin
Tolsted
Western
Providence Coll.
McKinney
West Warwick
Brant

SOUTH CAROLINA

Charleston
Dye
Hedge
Clemson
Coker
Columbia
Jackson
Williams
Rock Hill
Stokes
Spartanburg
Fort
Pettis
Wofford Coll.
Pettis

SOUTH DAKOTA

Brookings
South Dakota State Coll.
MacDougal
Walder
Wente
Sioux Falls
Lintvedt
Vermillion
Beaty
Sealander

TENNESSEE

Knoxville
Rich
Univ. of Tennessee
Cooley
Eaves
Ficken
Lee
Snyder
Memphis
MacQueen
Nashville
Tennessee Agric. and Industrial State Teachers Coll.
Boswell
Vanderbilt Univ.
Dyer-Bennet
Miser
Sarratt
Oak Ridge
Bennett
Dismuke

TEXAS

Abilene
Abilene Christian Coll.
Mullings
Hardin-Simmons Univ.
Burnam
Clarkson
Austin
Decherd
Univ. of Texas
Batchelder
Beckenbach
Bing
Butler
Coburn
Craig
Ettlinger
Greenwood
Hurt
Jones
Lubben
Moore
Porter, G. H.
Porter, M. B.
Titt
Vandiver
College Station
Agric. and Mech. Coll. of Texas
Brewer
Chaney
Edmondson
Hurt
Klipple
Moore
Commerce
Wright
Dallas
McDermott
Montgomery
Southern Methodist Univ.
Huff
Palmquist
Starr
Denton
Parrish
North Texas State Teachers Coll.
Beeman
Brown
Hanson
Texas State Coll. for Women
Miller
Dumas
Eberman
Houston
Glass
Jackson
Rust
Slotnick
Rice Inst.
Bray
Dean
Leighton
Lovett
MacLane
Newhouse
Ulrich
Wilson
Univ. of Houston
Hutchinson
Parson
Huntsville
Querry
Kingsville
Kennedy
Oliphant
Liberty
Brunk
Lubbock
Texas Tech. Coll.
Bowman
Michie
Thompson
Webb
Whyburn
Marlin
Falconer
Mercedes
Shaw
Midland
Rosskopf
San Antonio
Bright
Coleman
Incarnate Word Coll.
Gough
Our Lady of the Lake Coll.
Hill
San Antonio Junior Coll.
Hurry
Trinity Univ.
Newton
San Marcos
Cude
Waco
Williams

UTAH

Logan
Bird
Provo
Marshall
Salt Lake City
Woodbury
Univ. of Utah
Biesele
Gibson
Hayes
Henriques
Horsfall
Pehrson
Smith

VERMONT

Bennington
Dewing
Ellis
Burlington
Univ. of Vermont
Bullard
Schatten
Swift
Whitman
Middlebury
Ballou
Bowker
Springfield
Terry
Winooski Park
Lorenz

VIRGINIA

Alexandria
Forsythe
Arlington
Getchell
Marston
Blacksburg
McFadden
Charlottesville
Cooley
Patten
Univ. of Virginia
Botts
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Lytle
McShane
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McMillan, B.
Pierce
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Falls Church
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Schuettler
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Hollins College
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Fazel
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Virginia Military Inst.
Byrne
Knox
Washington and Lee Univ.
Paxton
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Sandidge
Randolph-Macon Woman's Coll.
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Wiggin
Quantico
Cole
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Ballantine, J. P.
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Sokolnikoff, I. S.
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Baños
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Nápoles-Gándara
PERU
Lima
Catholic Univ. of Peru
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García
Rosenblatt
URUGUAY
Montevideo
Laguardia

BY-LAWS OF THE AMERICAN MATHEMATICAL SOCIETY

ARTICLE I

OFFICERS

Section 1. There shall be a President, three Vice Presidents, a Secretary, four Associate Secretaries, a Treasurer, and a Librarian.

Section 2. It shall be a duty of the President to deliver an address before the Society at the close of his term of office or within one year thereafter.

Section 3. The Librarian shall have charge of arrangements for the exchange of the Society's publications.

ARTICLE II

BOARD OF TRUSTEES

Section 1. There shall be a Board of Trustees consisting of five trustees elected by the Society, in accordance with Article V.

Section 2. The function of the Board of Trustees shall be to receive and administer the funds of the Society, to have full legal control of its investments and properties, to make contracts and, in general, to conduct all business affairs of the Society.

Section 3. The Board of Trustees shall have the power to appoint a manager and such assistants and agents as may be necessary or convenient to facilitate the conduct of the affairs of the Society, and to fix the terms and conditions of their employment. The Board may delegate to the officers of the Society duties and powers normally inhering in their respective corporative offices, subject to supervision by the Board. The Board of Trustees may appoint committees to facilitate the conduct of the business of the Society and delegate to such committees such powers as may be necessary or convenient for the proper exercise of those powers. Agents appointed, or members of committees designated, by the Board of Trustees need not be members of the Board.

Nothing herein contained shall be construed to empower the Board of Trustees to divest itself of responsibility for, or legal control of, the investments, properties and contracts of the Society.

ARTICLE III

EDITORIAL COMMITTEES

Section 1. There shall be six Editorial Committees—one of four members for the Bulletin, one of three members for the Transactions, one of three members for the Mathematical Reviews, one of three members for the Colloquium Publications, one of three members for the Mathematical Surveys, and one consisting of three representatives of the Society on the Board of Editors of the American Journal of Mathematics.

ARTICLE IV

COUNCIL

Section 1. The Council shall consist of fifteen members at large, and the following *ex officio* members: the officers of the Society specified in Article I, the members of the Editorial Committees, ex-secretaries who have served as secretaries for ten years or more, and ex-presidents for a period of six years after the completion of their respective presidential terms.

Section 2. The Council shall formulate and administer the scientific policies of the Society and shall act in an advisory capacity to the Board of Trustees.

Section 3. All members of the Council shall be entitled to be present at any meeting of the Council and to take part in the discussion at that meeting.

Section 4. In the absence of the Secretary from any meeting of the Council, one of the Associate Secretaries present may be designated as Acting Secretary for the meeting, either by written authorization of the Secretary, or, failing that, by majority agreement among the Associate Secretaries present.

Section 5. The voting members at any meeting of the Council shall be the following: the members at large of the Council; the President; the Vice Presidents; the Secretary; the Acting Secretary for the meeting, if one is designated in the absence of the Secretary; the Treasurer; and a member of each Editorial Committee chosen annually by the committee, provided that if such voting member is absent at any meeting of the Council an alternate designated by him from among the membership of the committee shall have the power to vote.

Section 6. The method for settling matters before the Council at any meeting shall be by majority vote of the voting members present. If the result of such a vote is challenged at the time by any member of the Council, it shall be the duty of the Secretary—or of the Acting Secretary for the meeting—or of a member present who shall be designated by the Council for that purpose—to prepare a list of the voting members present, on the basis of which the presiding officer shall determine the true vote by a roll call.

Section 7. A quorum for the transaction of business at any meeting of the Council shall consist of five voting members as defined in Section 5 of this Article.

Section 8. Between meetings of the Council, business may be transacted by a mail vote. Voting members for this purpose shall be the members at large of the Council, the President, the Vice Presidents, the Secretary, the Treasurer, and the voting member of each Editorial Committee as designated in Section 5. An affirmative vote on any proposal by mail shall require at least a majority of the voting members, and at least three-quarters of the total number of votes transmitted to the Secretary before the time announced for the closing of the polls. If two or more voting members request postponement at the time of voting, action on the matter at issue shall be postponed until the next meeting of the Council, unless at the discretion of the Secretary the question is made the subject of a second vote by mail, in connection with which brief statements of reasons, for and against, are circulated.

ARTICLE V

ELECTIONS AND TERMS OF OFFICERS

Section 1. The term of office shall be two years in the case of the trustees, the President, the Vice Presidents, the Secretary, the Associate Secretaries, and the Treasurer; three years in the case of the Librarian and the members of the Editorial Committees. The term of office for members at large of the Council shall be three years, five of the members at large retiring annually. In every case, however, the officials specified in Articles I, II, III, and IV shall continue to serve until their successors have been duly elected and qualified.

Section 2. The President and Vice Presidents shall not be eligible for immediate re-election to their respective offices. A member at large or an *ex officio* member of the Council shall not be eligible for immediate election (or re-election) as a member at large of the Council.

Section 3. Election of officers specified in Article I, trustees, members of Editorial Committees, and members at large of the Council shall be by ballot at the Annual Meeting. An official ballot shall be sent to each member by the Secretary at least one month before the Annual Meeting, and such ballots, if returned to the Secretary in envelopes bearing the name of the voter and received prior to the closing of the polls, shall be counted at the Annual Meeting. Each ballot shall contain one or more names proposed by the Council for each office to be filled, with blank spaces in which the voter may substitute other names. A plurality of all votes cast, whether cast in person or by mail, shall be necessary for election. In case of failure to secure a plurality for any office, the members present at the Annual Meeting shall choose by ballot between the two having the highest number of votes.

Section 4. If the President of the Society die or resign before the expiration of his term of office, the Council, may, with the approval of the trustees, designate one of the Vice Presidents to serve as President until the next Annual Meeting, when the Society shall elect a President for a two-year term. Such vacancies as may occur at any time in the group consisting of the Vice Presidents, the Secretary, the Associate Secretaries, the Treasurer, the Librarian, and the members of the Editorial Committees may be filled by the Council with the approval of the trustees.

Section 5. If any trustee die or resign during his tenure of office, the vacancy thus created shall be filled for his unexpired term by the Board of Trustees.

Section 6. If any member at large of the Council die or resign more than one year before the expiration of his term, the vacancy for the unexpired term shall be filled by the Society at the next Annual Meeting.

ARTICLE VI

Election of Members

Section 1. Election of members shall be by vote of the Council.

Section 2. There shall be four classes of members, ordinary, contributing, institutional contributing, and sustaining.

Section 3. A firm, corporation, institution, association or individual interested in the support of mathematics, may be elected to sustaining membership. A sustaining member shall receive the *Bulletin*, the *Transactions*, and *Mathematical Reviews* (see Article IX), and shall have the privilege of nominating one or more persons for election by the Council to ordinary membership in the Society, the number to be determined by the Council. Such nominations shall not be acted upon until at least thirty days after their presentation to the Council (at a meeting or by mail).

Section 4. A firm, corporation, institution, or association interested in the support of mathematics may be elected to institutional contributing membership. An institutional contributing member shall receive the *Bulletin* or have the privilege, for each twenty-five dollars contributed, of nominating one person to ordinary membership in the Society. Such nominations shall not be acted upon until at least thirty days after their presentation to the Council (at a meeting or by mail).

Section 5. Except in the case of nominees of sustaining members and of institutional contributing members, application for admission to ordinary membership shall be made by the applicant, on a blank provided by the Secretary, and shall be approved by two members of the Society. Such applications shall not be acted upon until at least thirty days after their presentation to the Council (at a meeting or by mail), except in the case of members of other societies entering under special action of the Council.

Section 6. An ordinary member may become a contributing member by paying the dues for such membership. (See Article VII, Section 4.)

Section 7. An ordinary member or a contributing member shall receive the *Bulletin*, except in the case of members whose dues are remitted under Article VII, Section 10.

ARTICLE VII

DUES

Section 1. Persons elected to ordinary membership in the Society by the Council, under the provisions of Article VI, Section 5, shall be admitted to membership upon the payment, within sixty days of the date of their election, of an initiation fee of five dollars.

Section 2. The annual dues of persons elected by the Council to ordinary membership under the provisions of Article VI, Section 5, shall be eight dollars, with the following exceptions: (1) during the first three years of membership, the annual dues shall be six dollars; a person shall be considered to have completed his first year of membership on January 1 following his election; (2) the amount of dues may be altered by reciprocity agreements with other societies; (3) the Council may make special rulings in exceptional cases, with the approval of the Board of Trustees. Each new member shall pay in proportion to the unexpired fraction of the year at the time of his election.

Section 3. Four dollars and fifty cents of the annual dues of those members who receive the *Bulletin* under the provisions of Article VI shall be allocated in payment therefor.

Section 4. The minimum dues for a contributing member shall be fifteen dollars per year. Members may, upon their own initiative, pay larger dues.

Section 5. The dues for an institutional contributing member for any year shall not be less than twenty-five dollars. Institutions may pay larger dues.

Section 6. The dues of a sustaining member for any year shall not be less than one hundred dollars. A sustaining member who contributes annually at least five hundred dollars shall be designated as a Patron of the Society.

Section 7. Persons elected to ordinary membership as nominees of sustaining members or institutional contributing members under the provisions of Article VI, Sections 3–4, shall not be required to pay an initiation fee. They shall not be required to pay dues, so long as it is agreed that they are designated by the sustaining member or institutional contributing member under these provisions. If a nominee of a sustaining member or an institutional contributing member later becomes a dues-paying member, he shall pay dues at the rate of six dollars a year for the remainder (if any) of his first three years of membership, and eight dollars a year thereafter.

Section 8. If the annual dues of any member remain unpaid beyond a reasonable time, the Board of Trustees shall remove his name from the list of members, after due notice.

Section 9. Any member not in arrears of dues may become a life member on the payment of a sum to be determined in accordance with actuarial principles. A life member shall have for life the status and privileges of an ordinary member, without further payment of dues. After October 25, 1941, no additional applications for life memberships will be accepted.

Section 10. After retirement from active service on account of age, any member who is not in arrears of dues and with membership extending over at least twenty years may—by giving proper notification to the Secretary—have his dues remitted, on the understanding that he will thereafter receive the programs of the meetings, but not the *Bulletin*.

ARTICLE VIII

MEETINGS

Section 1. The Annual Meeting of the Society shall be held between the fifteenth of December and the fifteenth of January next following. Notice of the time and place of this meeting shall be mailed by the Secretary or an Associate Secretary to the last known post office address of each member of the Society. The times and places of the Annual and other meetings of the Society shall be designated by the Council. No matter of general business shall be considered at any meeting of the Society except the Annual Meeting, without the recommendation of the Council.

Section 2. The President may call a meeting of the Council whenever the affairs of the Society make it desirable; he shall call a meeting of the Council on written request of five of its members.

Section 3. The Board of Trustees shall have its regular Annual Meeting on the first day of the Annual Meeting of the Society. Special meetings of the Board of Trustees may be called by the Chairman of the Board upon three days' notice of such meeting mailed to the last known post office address of each trustee. He shall call a meeting upon the receipt of a written request of two of the trustees. Meetings for the transaction of business may also be held by common consent of all the trustees.

Section 4. Papers intended for presentation at any meeting of the Society shall be passed upon in advance by a program committee appointed by or under the authority of the Council; and only such papers shall be presented as shall have been approved by such committee. Papers in form unsuitable for publication, if accepted for presentation, shall be referred to on the program as preliminary communications or reports.

ARTICLE IX

PUBLICATIONS

Section 1. The Society shall publish an official organ called the *Bulletin of the American Mathematical Society*. It shall publish a journal called the *Transactions of the American Mathematical Society*, the object of which shall be to make known important researches presented at meetings of the Society. It shall publish a periodical called *Mathematical Reviews* containing abstracts or reviews of current mathematical literature. It shall publish a series of volumes called *Colloquium Publications* which shall embody in book form new mathematical developments. It shall publish a series of monographs called *Mathematical Surveys* which shall furnish expositions of the principal methods and results of particular fields of mathematical research. It shall also cooperate in the conduct of the *American Journal of Mathematics*.

Section 2. The editorial management of the *Bulletin*, *Transactions*, *Mathematical Reviews*, *Colloquium Publications*, and *Mathematical Surveys*, and the participation of the Society in the management of the *American Journal of Mathematics* shall be in charge of the respective Editorial Committees as provided in Article III, Section 1.

ARTICLE X

AMENDMENTS

These By-Laws may be amended or suspended at any meeting of the Society on recommendation of the Council and by a two-thirds vote of the members present, provided notice of such proposed action and of its general nature shall have been given in the call for such meeting.

AN ELEMENTARY PROOF OF THE STRONG FORM OF THE CAUCHY THEOREM

LYNN H. LOOMIS

A difficult step in the derivation of the strong forms of the Cauchy theorem, Green's lemma, and related theorems from the corresponding weak forms is the construction, for a given rectifiable Jordan curve J, of a sequence of Jordan polygons lying interior to J, converging to J, and having uniformly bounded lengths. This note presents what the author believes to be a simpler elementary construction of this sequence than any hitherto available (see [1, 2][1] and the bibliographies at the ends of these papers). We shall illustrate its use by proving the strong Cauchy theorem.

We assume the Jordan separation theorem together with its elementary consequences, as is usual in proofs of this kind, and we assume the weak form of the Cauchy theorem ($\int_P f(z)dz=0$ if $f(z)$ is analytic in a region containing P and its interior) for Jordan polygons P whose edges lie on lines of the form $x=m2^{-N}$, $y=n2^{-N}$, where $m, n=0, \pm 1, \cdots$. The polygon P and its interior is of course the sum of a finite number of squares of the network of closed squares (with sides 2^{-N}) into which these lines cut up the plane, so that we need to assume the weak Cauchy theorem only for single squares.

Let X be a fixed interior point of the given rectifiable Jordan curve J with coordinates *not* of the form $m2^{-N}$, and let C be a fixed closed square lying interior to J, containing X and having sides parallel to the axes. Let I be the set of squares of the above network lying interior to J; for sufficiently large N every square of I which contains a point of C lies interior to J. The vertical ray from X cuts a first edge l_1 of a square not in I. We form a polygon P by proceeding to the left along l_1 from its right-hand end point p_0, and at any vertex p_n choosing as l_{n+1} from the three remaining edges the counter-clockwise-most one which has a square of I on its left and a square not in I (and so containing a point of J) on its right. We shall call the latter square S_{n+1}. The reader will find the following argument trivial if he will sketch the four possible configurations at p_n. In tracing P a first vertex p_n must be repeated, $p_n=p_{n+M}$. If $n>0$, then l_n and l_{n+M} approach p_n from opposite directions, $S_{n+1}=S_{n+M}$, and the Jordan polygon l_{n+1} to l_{n+M} separates S_{n+1} from S_n and so separates points of J, an impossibility. Thus $n=0$, $p_M=p_0$ and P is a Jordan polygon. Note

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[1] Numbers in brackets refer to the bibliography at the end of the paper.

that P depends on N, although it is not explicit in the notation.

Let q_n be a point of J in S_n nearest to l_n, and let q_nt_n be the perpendicular dropped from q_n to l_n (t_n on l_n). Because of the definition of the q's no two distinct segments q_nt_n and q_mt_m can intersect except possibly in a common end point; in particular, the order of the t_n on P is the same as that of the q_n on J.

As a general remark, the squares touching any S_n have a total of 24 edges, so that, for some $k \leqq 24$, S_{n+k} does not touch S_n and the length of the arc q_nq_{n+k} of J is at least 2^{-N}. Thus if indices n_i are chosen so that $24 \leqq n_{i+1} - n_i < 48$ for $i = 0, \cdots, K-1$ and $t_{n_0} = t_{n_K} = t_0$, then the length of the arc $q_{n_i}q_{n_{i+1}}$ of J is at least 2^{-N} and the length of the arc $t_{n_i}t_{n_{i+1}}$ of P is at most $48/2^N$. If C_i is the Jordan curve consisting of these two arcs and the two segments $q_{n_i}t_{n_i}$, $q_{n_{i+1}}t_{n_{i+1}}$, we have that $L(C_i) \leqq 51 \cdot L(q_{n_i}q_{n_{i+1}})$, and

$$\sum_{i=0}^{K-1} L(C_i) \leqq 51 \cdot L(J).$$

We can now apply the device of Newman [3, p. 156] to prove the strong Cauchy theorem.

THEOREM. *If $f(z)$ is analytic interior to J and continuous on J plus its interior, then $\int_J f(z)dz = 0$.*

The distance from q_{n_i} to $q_{n_{i+1}}$ is for every i at most $50/2^N$, and since J is rectifiable it follows that given δ there is an N such that for every i one of the two arcs determined on J by q_{n_i} and $q_{n_{i+1}}$ has length at most δ. But this must be the arc we have called $q_{n_i}q_{n_{i+1}}$ (containing q_j for $n_i \leqq j \leqq n_{i+1}$); for the length of its complementary arc on J is at least 1/48th of the length of the major arc of the points t_{n_i} and $t_{n_{i+1}}$ on P, and since the square C lies interior to P this length is bounded below by one-half the length of the perimeter of C. Thus the maximum diameter of the curves C_i approaches 0 as $N \rightarrow \infty$ and since $f(z)$ is uniformly continuous on J plus its interior, there exists for every ϵ an N such that $|f(z) - f(t_{n_i})| < \epsilon$ for all z on C_i. Now $\int_J f(z)dz = \sum_0^{K-1} \int_{C_i} f(z)dz$, for the difference between these two integrals consists of $\int_P$, which is 0 by hypothesis, and the integrals over the segments $p_{n_i}t_{n_i}$, which are traced once in each direction and so cancel. Since $\int_{C_i} f(t_{n_i})dz = f(t_{n_i})\int_{C_i} dz = 0$, this implies that

$$\left| \int_J f(z)dz \right| = \left| \sum_{i=0}^{K-1} \int_{C_i} (f(z) - f(t_{n_i}))dz \right| \leqq \epsilon \cdot 51L(J)$$

for every ϵ; thus $\int_J f(z)dz = 0$.

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HARVARD UNIVERSITY

ON A CERTAIN TYPE OF NONLINEAR INTEGRAL EQUATIONS

MARK LOTKIN

1. **Introduction.** The object of this paper is to prove that the nonlinear integral equation

$$\phi(x) = \lambda \Big[f(x) + \sum_{i=1}^{m} \int_a^b \cdots \int_a^b K_i(x, s_1, \cdots, s_i) \cdot F_i(s_1, \cdots, s_i, \phi(s_1), \cdots, \phi(s_i)) ds_1 \cdots ds_i \Big] \tag{1}$$

has at least one eigenvalue, provided the functionals

$$G_i(x, v) = \int_a^b \cdots \int_a^b K_i(x, s_1, \cdots, s_i) \cdot F_i(s_1, \cdots, s_i, v(s_1), \cdots, v(s_i)) ds_1 \cdots ds_i \tag{2}$$

are fully continuous, and the F_i satisfy a certain linear integrodifferential equation. The solution of (1) is shown to be equivalent to that of a variational problem containing infinitely many parameters. The latter problem, however, can be solved easily by the method of Rayleigh-Ritz, which consists in approaching the solution of the variational problem by a sequence of variational problems containing only a finite number of parameters. The convergence of this procedure is assured by a convergence theorem of Friedrich Riesz.

2. **Preparatory remarks.** Let I be the closed interval $a \leq x \leq b$, and L^2 the class of all functions having Lebesgue integrable squares on I with a norm not larger than N^2. Let, further, $\{v_n(x)\}$ $(n = 1, 2, 3, \cdots)$

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be a set of functions in L^2 and $\bar{v}(x)$ a function such that[1]

$$\lim_{n\to\infty} \int v_n(x)w(x)dx = \int \bar{v}(x)w(x)dx \tag{3}$$

for any arbitrary w of integrable square on I or, what is equivalent, for any arbitrary $w\in L^2$; that is, we assume the weak convergence of $\{v_n\}$, in the following denoted by $W\text{-}\lim_{n\to\infty} v_n = \bar{v}$. We next show that $\bar{v}\in L^2$. Since the right-hand side of (3) is assumed to exist for every $w\in L^2$, it follows by a known result (see, for example, Kaczmarz-Steinhaus, *Theorie der Orthogonalreihen*) that $\bar{v}$ is of integrable square over I. Hence, if $w=\bar{v}$, (3) becomes

$$\int \bar{v}^2(x)dx = \lim_{n\to\infty} \int v_n(x)\bar{v}(x)dx,$$

and since by Schwarz's inequality

$$\left[\int v_n(x)\bar{v}(x)dx\right]^2 \leqq N^2 \left[\int \bar{v}^2(x)dx\right],$$

we obtain, as claimed,

$$\int \bar{v}^2(x)dx \leqq N^2. \tag{4}$$

We now assume that the functional $G_i(x, v)$ be fully continuous, that is, that

$$\lim_{n\to\infty} G_i(x_n, v_n) = G_i(\bar{x}, \bar{v}), \qquad i = 1, 2, \cdots, m, \tag{5}$$

for any $\{x_n\}\in I$ and $v_n\in L^2$ for which $\lim_{n\to\infty} x_n = \bar{x}$ and $W\text{-}\lim_{n\to\infty} v_n = \bar{v}$.

The introduction of a closed orthonormal system of functions $\{w_\nu(x)\}\in L^2$ associates with each v_n an infinite sequence of numbers

$$c_{n\nu} = \int v_n(x)w_\nu(x)dx, \qquad \nu = 1, 2, \cdots,$$

such that

$$\sum_{\nu=1}^{\infty} c_{n\nu}^2 = \int v_n^2(x)dx \leqq N^2$$

for every $n \geqq 1$. The class L^2 of functions v then corresponds to a class $\mathfrak{H}^2$ of vectors $\mathfrak{v} \equiv (c_1, c_2, \cdots)$ with $c_\nu = \int v(x)w_\nu(x)dx$. The transition from L^2 to $\mathfrak{H}^2$ implies the substitution in $G_i(x_n, v_n)$ of $v_n(x)$ by its equivalent $\mathfrak{v}(x) \sim \sum_{\nu=1}^{\infty} c_{n\nu} w_\nu(x)$, and (5) now changes to

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[1] All integrations are to be extended over the interval I.

$$\lim_{n\to\infty} G_i(x_n, v_n) = G_i(\bar{x}, \bar{v}) \equiv P_i(\bar{x}, \bar{c}_1, \bar{c}_2, \cdots)$$

for any $\{x_n\} \in I$ and $\{v_n\} \in \mathfrak{H}^2$ for which $\lim_{n\to\infty} x_n = \bar{x}$ and $\lim_{n\to\infty} c_{n\nu} = \bar{c}_\nu$, $\nu = 1, 2, \cdots$. From the full continuity of the G_i thus follows the full continuity (*Vollstetigkeit*)[2] of the P_i. It is just as easily seen that the converse also holds true.

3. **Construction of a solution of the integral equation.** In the G_i we now admit as arguments v_n only aggregates of the form $v_n(x) = \sum_{\nu=1}^{n} c_{n\nu} w_\nu(x)$ with $\sum_{\nu=1}^{n} c_{n\nu}^2 = N^2$ for every fixed n. The functional

$$(6) \qquad J(v_n) = 2\left[\int f(x) v_n(x) dx + \sum_{i=1}^{m} e_i \int G_i(x, v_n) v_n(x) dx\right]$$

—here the e_i denote finite numbers to be determined later—is now a continuous function of the $c_{n\nu}$ and so has at least one minimum. Let $c_{n\nu} = a_{n\nu}$ ($\nu = 1, 2, \cdots, n$) be the minimal coordinates:

$$(7) \quad \min J(v_n) = d_n = J(\phi_n), \quad \phi_n(x) = \sum_{\nu=1}^{n} a_{n\nu} w_\nu(x), \qquad \sum_{\nu=1}^{n} a_{n\nu}^2 = N^2.$$

As a consequence of (7) we have[3]

$$(8) \quad \frac{\partial}{\partial c_{n\nu}}\left[J(v_n) + \frac{1}{\lambda_n}\left(N^2 - \sum_{\mu=1}^{n} c_{n\mu}^2\right)\right]_{c_{n\nu}=a_{n\nu}} = 0, \qquad \nu = 1, 2, \cdots, n.$$

However,

$$\frac{\partial J(v_n)}{\partial c_{n\nu}} = 2\left[\int f w_\nu dx + \sum_i e_i \int\int \cdots \int K_i \left(v_n \frac{\partial F_i}{\partial c_{n\nu}} + w_\nu F_i\right) dx ds_1 \cdots ds_i\right].$$

We must now make the following assumption: *The F_i satisfy the linear integrodifferential equations*

$$(9) \quad \int\int \cdots \int K_i \left[e_i v_n \frac{\partial F_i}{\partial c_{n\nu}} - (1 - e_i) w_\nu F_i\right] dx ds_1 \cdots ds_i = 0$$

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[2] See D. Hilbert, *Grundzüge einer allgemeinen Theorie der linearen Integralgleichungen*, 1912, p. 177. A fully continuous function $P(x, c_1, c_2, \cdots)$, where $x \in I$ and $\sum_{r=1}^{\infty} c_r \leqq N^2$, is bounded.

[3] In (8), $1/\lambda_n$ denotes Lagrange's multiplier for the extremum problem under consideration. As will be shown subsequently, $\lambda_n \neq 0$ for every n.

for all arguments $v_n = \sum_{(\nu)} c_{n\nu} w_\nu$, *identically in the* $c_{n\nu}$.

In this case we obtain

$$\frac{\partial J(v_n)}{\partial c_{n\nu}} = 2\left[\int\left(f + \sum_i \int \cdots \int K_i F_i ds_1 \cdots ds_i\right) w_\nu dx\right],$$

and (8) leads to

$$a_{n\nu} = \lambda_n \int G(x, \phi_n) w_\nu dx, \qquad \nu = 1, 2, \cdots, n, \tag{10}$$

with

$$G(x, \phi_n) = f(x) + \sum_i \int \cdots \int K_i(x, s_1, \cdots, s_i) \cdot F_i(s_1, \cdots, s_i, \phi_n(s_1), \cdots, \phi_n(s_i)) ds_1 \cdots ds_i.$$

On account of $a_{n\nu} = \int \phi_n w_\nu dx$ the relations (10) may be written as

$$\int (\phi_n - \lambda_n G(x, \phi_n)) w_\nu dx = 0 \qquad \text{for } \nu = 1, 2, \cdots, n. \tag{11}$$

Equations (10) show that the $|\lambda_n|$ have a common positive lower bound: multiplication of (10) by $a_{n\nu}$ and summation for $\nu = 1, \cdots, n$ result in

$$N^2 = \lambda_n \int G(x, \phi_n) \phi_n dx. \tag{12}$$

But since $G(x, \phi_n)$ is a fully continuous function of the $a_{n\nu}$ for $x \in I$ and $\phi_n \in H^2$ it is bounded: there exists a $\delta > 0$ such that

$$|G(x, \phi_n)| \leqq N/(b - a)^{1/2}\delta \quad \text{for every } n.$$

Therefore

$$\left|\int G(x, \phi_n)\phi_n dx\right| \leqq \left[\int G^2(x, \phi_n) dx\right]^{1/2} \left[\int \phi_n^2 dx\right]^{1/2} \leqq N^2/\delta,$$

whence $|\lambda_n| \geqq \delta > 0$ for every n.

Now the $\int \phi_n^2 dx$ all have the same value N^2. This property of the sequence $\{\phi_n\}$ guarantees the existence of a $\bar{\phi}(x)$—defined almost everywhere in I and possessing a Lebesgue integrable square—which is the W-lim of a suitably chosen subsequence $\{\phi_{\bar{n}}\}$ of $\{\phi_n\}$,[4]

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[4] Friedrich Riesz, *Untersuchungen über Systeme integrierbarer Funktionen*, Math. Ann. vol. 69 (1910) p. 467. The sequence $\{\bar{n}\}$ is determined by Hilbert's diagonal method (see, for example, Hellinger-Toeplitz, Encyklopädie der mathematischen Wissenschaften vol. II C 13, p. 1405).

$$W\text{-}\lim_{\bar{n}\to\infty} \phi_{\bar{n}}(x) = \bar{\phi}(x); \tag{13}$$

because the system $\{w_\nu\}$ is closed $\bar{\phi}$ is determined uniquely almost everywhere in I. On account of (4), $\int \bar{\phi}^2(x)dx \leqq N^2$.

We are now going to show that

$$\lim_{\bar{n}\to\infty} \int G(x, \phi_{\bar{n}})\phi_{\bar{n}}dx = \int G(x, \bar{\phi})\bar{\phi}dx. \tag{14}$$

Since

$$\left| \int G(x, \bar{\phi})\bar{\phi}dx - \int G(x, \phi_{\bar{n}})\phi_{\bar{n}}dx \right| \leqq \left| \int G(x, \bar{\phi})(\bar{\phi} - \phi_{\bar{n}})dx \right|$$
$$+ \left| \int (G(x, \bar{\phi}) - G(x, \phi_{\bar{n}}))\phi_{\bar{n}}dx \right|,$$

and the first expression on the right hand side—by (13)—may be made as small as desired by taking $\bar{n}$ sufficiently large, only the second term remains to be considered. Now

$$\left| \int (G(x, \bar{\phi}) - G(x, \phi_{\bar{n}}))\phi_{\bar{n}}dx \right| \leqq N \left[\int (G(x, \bar{\phi}) - G(x, \phi_{\bar{n}}))^2 dx \right]^{1/2},$$

and so (14) will be proved if we can show that $\lim_{\bar{n}\to\infty}\int(G(x, \bar{\phi}) - G(x, \phi_{\bar{n}}))^2dx = 0$. This, however, follows immediately from the convergence theorem of Lebesgue.[5] The sequence $L_{\bar{n}} \equiv (G(x,\bar{\phi}) - G(x, \phi_{\bar{n}}))^2$ obviously satisfies the conditions of that theorem: (a) $L_{\bar{n}}$ is Lebesgue integrable; (b) Since $|G(x,v)| \leqq N/(b-a)^{1/2}\delta$, $|L_{\bar{n}}| \leqq 4N^2/(b-a)\delta^2$ for every $\bar{n}$; (c) Because of the full continuity of $G(x, v)$, $\lim_{\bar{n}\to\infty}(G(x, \bar{\phi}) - G(x, \phi_{\bar{n}})) = 0$. Therefore $\bar{L} = 0$, which proves (14).

We must now distinguish between these two cases:

I. There exists a $\delta' > 0$ such that $|\int G(x, \bar{\phi})\bar{\phi}dx| \geqq N^2/\delta'$;

II. $\int G(x, \bar{\phi})\bar{\phi}dx = 0$.

CASE I. By (12), $\lambda_{\bar{n}} = N^2/\int G(x, \phi_{\bar{n}})\phi_{\bar{n}}dx$, so that by (14)

$$\lim_{\bar{n}\to\infty} \lambda_{\bar{n}} = \bar{\lambda} = \frac{N^2}{\int G(x, \bar{\phi})\bar{\phi}dx} \tag{15}$$

exists; because of I, $|\bar{\lambda}| \leqq \delta'$.

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[5] If a sequence of Lebesgue integrable functions $L_n(x)$ possessing a common bound has a limit function $\bar{L}(x)$, then $\bar{L}$, too, is Lebesgue integrable and $\lim_{n\to\infty}\int L_n(x)dx = \int \bar{L}(x)dx$.

If we now apply equations (11) to indices $\bar{n}$ only and then take the $\lim_{\bar{n}\to\infty}$ we obtain

$$\int (\bar{\phi}(x) - \bar{\lambda}G(x, \bar{\phi}))w_\nu(x)dx = 0 \qquad \text{for } \nu = 1, 2, \cdots .$$

Since the system of the $\{w_\nu\}$ is closed we may deduce $\bar{\phi}-\bar{\lambda}G(x, \bar{\phi}) = 0$, that is,

$$\bar{\phi}(x) = \bar{\lambda}\left[f(x) + \sum_{i=1}^{m}\int \cdots \int K_i(x, s_1, \cdots, s_i) \cdot F_i(s_1, \cdots, s_i, \bar{\phi}(s_1), \cdots, \bar{\phi}(s_i))ds_1 \cdots ds_i\right]$$

almost everywhere in I. We have thus obtained a solution $\bar{\phi}(x)$ of (1) belonging to the finite eigenvalue $\bar{\lambda}$.

The previously derived relationship $\int\bar{\phi}^2(x)dx \leqq N^2$ may now be improved: replacing $G(x, \bar{\phi})$ in (15) by its equal $(1/\bar{\lambda})\bar{\phi}$ leads to $\int\bar{\phi}^2(x)dx = N^2$.

CASE II. We write equations (10) for indices $\bar{n}$ only:

$$\int G(x, \phi_{\bar{n}})w_\nu dx = \frac{1}{\lambda_{\bar{n}}}a_{\bar{n}\nu}, \qquad \nu = 1, 2, \cdots, \bar{n},$$

and increasing $\bar{n}$ beyond any bound we obtain, since $\lim_{\bar{n}\to\infty}\lambda_{\bar{n}} = \infty$ and $|a_{\bar{n}\nu}| \leqq N$,

$$\int G(x, \bar{\phi})w_\nu dx = 0, \qquad \nu = 1, 2, \cdots .$$

In this case $\bar{\phi}(x)$ may be considered a solution of (1) belonging to $\lambda = \infty$.

4. **The variational problem.** We see, then, that $\bar{\phi}$ is always a solution of (1). This function possesses another important property: If $\bar{\mathfrak{H}}^2$ denotes the class of all $\mathfrak{v}(x) \sim \sum_{(\nu)} c_\nu w_\nu(x)$ with $\sum_{(\nu)} c_\nu^2 = N^2$, and

$$\bar{\mathfrak{v}}(x) \sim \sum_{\nu=1}^{\infty} \bar{d}_\nu w_\nu(x) \quad \text{with} \quad \bar{d}_\nu = \int \bar{\phi}(x)w_\nu(x)dx,$$

then $\bar{\mathfrak{v}}$ minimizes $J(\mathfrak{v})$.

To prove this we notice first that $J(\mathfrak{v}_n)$ results from $J(\mathfrak{v}_{n+1})$ if we put $c_{n+1} = 0$. Let d_n be the minimum of $J(\mathfrak{v}_n)$ in $\bar{\mathfrak{H}}^2$. Then obviously $d_n \geqq d_{n+1}$. Let, further, d be the minimum of $J(\mathfrak{v})$ for $\mathfrak{v} \in \bar{\mathfrak{H}}^2$; then $d_n \geqq d$ for every n. Therefore, if $\bar{d} = \lim_{\bar{n}\to\infty} d_{\bar{n}} = J(\bar{\mathfrak{v}})$, we get $\bar{d} \geqq d$ or $\bar{d} = d + \eta$ with $\eta \geqq 0$. We shall show that $\eta = 0$.

Since d is the lower bound of $J(\mathfrak{v})$ in $\bar{\mathfrak{H}}^2$ there exists a $\mathfrak{p}(x) \sim \sum_{(\nu)} p_\nu w_\nu(x)$ in $\bar{\mathfrak{H}}^2$ so that $J(\mathfrak{p}) = \bar{d} - \theta\eta$ with $0 < \theta \leqq 1$. If, now, $\epsilon > 0$ be chosen as small as desired, there is, because of the full continuity of $J(\mathfrak{v})$, a $\delta > 0$ and an index r such that $|J(\mathfrak{v}) - J(\mathfrak{p})| < \epsilon$ for every $\mathfrak{v} \in \bar{\mathfrak{H}}^2$, so long as $|c_\nu - p_\nu| < \delta$ for $\nu = 1, 2, \cdots, r$. We take $\epsilon = \theta\eta$ and choose $r' \geqq r$ large enough to have $\sum_{\nu=1}^{r'} p_\nu^2 = N'^2 > N^2 \cdot N^2/(N+\delta)^2$. Then the vector $\bar{\mathfrak{p}}(x) = \sum_{\nu=1}^{r'} \bar{p}_\nu w_\nu(x)$ with $\bar{p}_\nu = (N/N')p_\nu$ belongs to $\bar{\mathfrak{H}}^2$, and since $N' > N^2/(N+\delta)$,

$$| \bar{p}_\nu - p_\nu | = | p_\nu | \cdot (N/N' - 1) \leqq N(N/N' - 1) < \delta$$

for $\nu = 1, 2, \cdots, r'$. We may, therefore, conclude that $|J(\bar{\mathfrak{p}}) - J(\mathfrak{p})| < \theta\eta$ or $J(\bar{\mathfrak{p}}) < J(\mathfrak{p}) + \theta\eta = \bar{d}$. But $d_{r'} \leqq J(\bar{\mathfrak{p}})$, and so $d_{r'} < \bar{d}$.

By now choosing $\bar{n}$, $\bar{n} \geqq r'$, such that $d_{\bar{n}} \leqq d_{r'}$ we get $d_{\bar{n}} \leqq \bar{d}$, a relation which contradicts the fact that the sequence d_n converges to $\bar{d}$ from above. Thus we see that $\eta = 0$ or $\bar{d} = J(\bar{\mathfrak{v}})$.

5. Solution of the integrodifferential equation. It is easy to verify that equations (9) are fulfilled if we put $e_i = 1/(i+1)$, K_i continuous and

$$K_i(x, s_1, \cdots, s_k, \cdots, s_i) = K_i(s_k, s_1, \cdots, x, \cdots, s_i), \qquad k = 1, 2, \cdots, i,$$

$$F_i(s_1, \cdots, s_i, u_1, \cdots, u_i) = a_i u_1 \cdots u_i$$

for $i = 1, 2, \cdots, m$.

It remains to be shown that functionals of the type

$$Q(x, v) = \int \cdots \int K(x, s_1, \cdots, s_i) v(s_1) \cdots v(s_i) ds_1 \cdots ds_i$$

are fully continuous for $x \in I$ and $v \in L^2$. Let us, therefore, assume that $\{x_n\} \in I$, $\{v_n\} \in L^2$, $\lim_{n\to\infty} x_n = \bar{x}$, and W-$\lim_{n\to\infty} v_n = \bar{v}$. Then

$$\begin{aligned} | Q(\bar{x}, \bar{v}) - Q(x_n, v_n) | &\leqq | Q(\bar{x}, \bar{v}) - Q(\bar{x}, v_n) | + | Q(\bar{x}, v_n) - Q(x_n, v_n) | \\ &= \left| \int \cdots \int K(\bar{x}, s_1, \cdots, s_i) [\bar{v}(s_1) \cdots \bar{v}(s_i) \right. \\ &\qquad \left. - v_n(s_1) \cdots v_n(s_i)] ds_1 \cdots ds_i \right| \\ &\quad + \left| \int \cdots \int [K(\bar{x}, s_1, \cdots, s_i) \right. \\ &\qquad \left. - K(x_n, s_1, \cdots, s_i)] v_n(s_1) \cdots v_n(s_i) ds_1 \cdots ds_i \right|. \end{aligned}$$

Because of the continuity of K the second term on the right-hand side may be made arbitrarily small by choosing n sufficiently large. In order to show that the same applies also to the first term we continue as follows:

$$\left| \int \cdots \int K[\bar{v}(s_1) \cdots \bar{v}(s_i) - v_n(s_1) \cdots v_n(s_i)]ds_1 \cdots ds_i \right|$$

$$= \left| \int \cdots \int K \sum_{k=1}^{i} \bar{v}(s_1) \cdots \bar{v}(s_{k-1})[\bar{v}(s_k) - v_n(s_k)] \cdot v_n(s_{k+1}) \cdots v_n(s_i) ds_1 \cdots ds_i \right|$$

$$\leqq \sum_{k=1}^{i} \left| \int \cdots \int \left(\int K[\bar{v}(s_k) - v_n(s_k)]ds_k \right) \cdot \bar{v}(s_1) \cdots \bar{v}(s_{k-1}) v_n(s_{k+1}) \cdots v_n(s_i) ds_1 \cdots ds_i \right|$$

$$\leqq \sum_{k=1}^{i} \left\{ \int \cdots \int \left(\int K[\bar{v}(s_k) - v_n(s_k)]ds_k \right)^2 \cdot ds_1 \cdots ds_{k-1} ds_{k+1} \cdots ds_i \right\}^{1/2} \cdot N^{i-1}.$$

Since $W\text{-}\lim_{n\to\infty} v_n = \bar{v}$, $\lim_{n\to\infty}(\int K[\bar{v}(s_k) - v_n(s_k)]ds_k) = 0$, and since $|\int K(\bar{x}, s_1, \cdots, s_i)[\bar{v}(s_k) - v_n(s_k)]ds_k| \leqq 2N(b-a)\cdot\max|K|$, we see that the sequence of Lebesgue integrable functions $(\int K(\bar{x}, s_1, \cdots, s_i) \cdot [\bar{v}(s_k) - v_n(s_k)]ds_k)^2$ has a common bound and the limit function zero. By Lebesgue's convergence theorem we may conclude that

$$\lim_{n\to\infty} \int \cdots \int \left(\int K[\bar{v}(s_k) - v_n(s_k)]ds_k \right)^2 ds_1 \cdots ds_{k-1} ds_{k-1} \cdots ds_i = 0$$

for $k = 1, 2, \cdots, i$, so that the proof of the full continuity of $Q(x, v)$ is now complete.

6. A special case. The deductions of §4 are therefore applicable to the integral equation

$$\phi(x) = \lambda \left[f(x) + \sum_{i=1}^{m} a_i \int \cdots \int K_i(x, s_1, \cdots, s_i) \cdot \phi(s_1) \cdots \phi(s_i) ds_1 \cdots ds_i \right].$$

If we assume that $a_m=1$, $a_i=0$ for $i=1, 2, \cdots, m-1$, that is, if we consider

$$\phi(x) = \lambda\left[f(x) + \int \cdots \int K_m(x, s_1, \cdots, s_m) \cdot\phi(s_1) \cdots \phi(s_m)ds_1 \cdots ds_m\right],$$

we know that it has at least one solution, and that this solution may belong to a finite or an infinite eigenvalue. The homogeneous equation

$$\phi(x) = \lambda \int \cdots \int K_m(x, s_1, \cdots, s_m)\phi(s_1) \cdots \phi(s_m)ds_1 \cdots ds_m,$$

however, has always at least one finite eigenvalue. In this case namely (see (6))

$$J(v_n) = 2e_m \int G_m(x, v_n)v_n(x)dx;$$

so that

$$d_n = \frac{2}{m+1}\int G_m(x, \phi_n)\phi_n(x)dx = \frac{2}{m+1}\cdot\frac{N^2}{\lambda_n}$$

or

$$\lambda_n \cdot d_n = (2/(m+1))N^2.$$

But since the functional

$$\frac{1}{2(m+1)}J(v) = \int \cdots \int K_m(x, s_1, \cdots, s_m) \cdot v(s_1) \cdots v(s_m)dxds_1 \cdots ds_m,$$

$K_m \neq 0$, $v \in \overline{H}^2$, certainly has a minimum $\bar{d}$ differing from zero,

$$\bar{\lambda} = \lim_{n\to\infty} \lambda_n = \frac{2N^2}{\bar{d}(m+1)}$$

is finite.

CARLETON COLLEGE

THE SOLUTION OF THE DIFFERENTIAL EQUATION $(a^2\, \partial^2/\partial t^2-\Delta)(\partial^2/\partial t^2-\Delta)u=f(x, y, z, t)$ BY HADAMARD'S METHOD

J. P. KORMES

1. **Introduction.** Hadamard [1][1] generalized and extended Riemann's [2] method of solution of the initial value problem to any hyperbolic linear differential equation of the second order in any number of variables. The essential part of Hadamard's work is his discovery of the significance of an elementary solution which is a function of the geodetic distance determined by the characteristic manifold of the differential equation. Given the differential equation $L[u]=f$ and the initial data on a manifold C, let $\Gamma=0$ be the equation of the characteristic manifold of the differential equation and $M[u]$ the adjoint expression of $L[u]$. The first step is the determination of an elementary solution V, that is, a solution of $M[V]=0$ which is singular on $\Gamma=0$. To obtain the solution of the initial value problem, Hadamard considers the domain bounded by the characteristic "conoid" through a point P, and the initial manifold $C=0$. Since the elementary solution is singular on the characteristic conoid, in order to apply Green's formula Hadamard uses an interior domain $G_{\epsilon,\delta}$ which approaches the original domain as $\epsilon\rightarrow 0$ and $\delta\rightarrow 0$. For a fixed δ each term of Green's formula has then the following form:

$$B(\epsilon) = b + (1/\epsilon^{(n-3)/2})(b_0 + b_1\epsilon + \cdots + b_{(n-3)/2}\epsilon^{(n-3)/2}) + (\epsilon),$$

where (ϵ) tends to zero as $\epsilon\rightarrow 0$ and the b_i are independent of ϵ. The term b, the "finite part" of the integral, has the property to remain invariant under all transformations of the parameter ϵ. It follows from Green's formula that the sum of all such "finite parts" is equal to zero, which gives a relation for the function u. As we let the parameter δ approach zero we obtain the desired expression for $u(P)$. Hadamard, and independently Friedrichs [3], showed that in the case of an even number of variables each term of Green's formula will have an element containing $\log \epsilon$. The coefficients of $\log \epsilon$ have the property to remain invariant under all transformations of the parameter ϵ and it follows from Green's formula that the sum of all coefficients of $\log \epsilon$ is equal to zero, which gives the desired relation for the function u.

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[1] Numbers in brackets refer to the Bibliography at the end of the paper.

The purpose of this paper is to show by a significant example that this method can be extended to problems of a more complex type.

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2. **Elementary solution, Green's formula and domain of integration.** We consider the initial value problem:

$$(2.1)\quad \begin{aligned} L[u] &\equiv (a^2\partial^2/\partial t^2 - \Delta)(\partial^2/\partial t^2 - \Delta)u = f(x, y, z, t); \qquad a > 1;\\ u(x, y, z, 0) &= u_t(x, y, z, 0) = u_{tt}(x, y, z, 0)\\ &= u_{ttt}(x, y, z, 0) = 0, \end{aligned}$$

where Δ is the Laplace operator: $\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2$, and the function $f(x, y, z, t)$ is assumed to have continuous derivatives up to the fourth order. The characteristic manifold is given by:

$$[(\tau - t)^2 - r^2]\cdot[(\tau - t)^2/a^2 - r^2] = 0,$$

where $r^2 = (\xi - x)^2 + (\eta - y)^2 + (\zeta - z)^2$. We introduce the functions Γ and Γ':

$$\Gamma = (\tau - t)^2 - r^2; \qquad \Gamma' = (\tau - t)^2/a^2 - r^2.$$

The characteristic manifold consists of two conical sheets, the exteror: $\Gamma = 0$, and the interior: $\Gamma' = 0$.

As the elementary solution V of the self-adjoint equation $L[V] = 0$ we take:

$$(2.2)\qquad V = -\frac{1}{2r}\log\frac{(\tau - t) - r}{(\tau - t) + r} + \frac{a}{2r}\log\frac{(\tau - t)/a - r}{(\tau - t)/a + r}\cdot$$

In order to obtain Green's formula for equation (2.1) we consider:

$$Q[u, V] = a^2 u_{tt}V_{tt} - a^2\sum_i u_{tt}V_{ii} - \sum_i V_{tt}u_{ii} + \sum_{i,k} u_{ii}V_{kk}$$
$$(i,\ k = x,\ y,\ z).$$

The integration of $Q[u, V]$ over a four-dimensional domain G gives rise to two types of integrals:

$$\text{(a)}\quad \iiiint_G u_{ii}V_{tt}\,dx\,dy\,dz\,dt \quad\text{and}\quad \text{(b)}\quad \iiiint_G u_{ii}V_{kk}\,dx\,dy\,dz\,dt$$
$$(i,\ k = x,\ y,\ z,\ t).$$

Integrating by parts and collecting the results we obtain the following Green's formula:

$$(2.3)\quad \iiiint_G \{VL[u] - uL[V]\}\,dg = \iiint_S \left\{ V\frac{\partial N[u]}{\partial s_2} - u\frac{\partial M[V]}{\partial s_1} - N[u]\frac{\partial V}{\partial s_2} + M[V]\frac{\partial u}{\partial s_1}\right\} d\sigma,$$

where:

$$M = a^2\partial^2/\partial t^2 - \Delta; \qquad N = \partial^2/\partial t^2 - \Delta$$

and where

$$\partial/\partial s_1 = t_\nu\partial/\partial t - x_\nu\partial/\partial x - y_\nu\partial/\partial y - z_\nu\partial/\partial z$$

and

$$\partial/\partial s_2 = a^2 t_\nu\partial/\partial t - x_\nu\partial/\partial x - y_\nu\partial/\partial y - z_\nu\partial/\partial z$$

are the transversal differentiations, x_ν, y_ν, z_ν, t_ν being the direction coefficients of the outward normal to S. If in $Q[u, V]$ we interchange u and V we obtain a *second form* of Green's formula:

$$(2.4)\quad \iiiint_G \{VL[u] - uL[V]\}\,dg = \iiint_S \left\{ V\frac{\partial M[u]}{\partial s_1} - u\frac{\partial N[V]}{\partial s_2} - M[u]\frac{\partial V}{\partial s_1} + N[V]\frac{\partial u}{\partial s_2}\right\} d\sigma.$$

For the elementary solution V (2.2) we find readily:

$$(2.5)\quad M[V] = 2(a^2 - 1)(\tau - t)/\Gamma^2; \; N[V] = 2(a^2 - 1)(\tau - t)/a^2\Gamma'^2.$$

We consider the domain G bounded by the hyperplane $t=0$ and the exterior sheet $\Gamma=0$ through a point $P(\xi, \eta, \zeta, \tau)$. The domain G is characterized by the inequalities: $\Gamma \geqq 0$; $0 \leqq t \leqq \tau$. In accordance with Hadamard's method we truncate the domain G by a plane $t = \tau - \delta$ and obtain the domain G_δ for which $\Gamma \geqq 0$, $0 \leqq t \leqq \tau - \delta$, $\delta > 0$.

Since Γ' vanishes in the interior of G_δ, the elementary solution has there a singularity. In order to enable us to apply Green's formula, we subdivide G_δ into the following domains:

(a) the conical ring $G^1_{\epsilon_1\epsilon_2\delta}$:

$$0 \leqq t \leqq \tau - \delta;$$

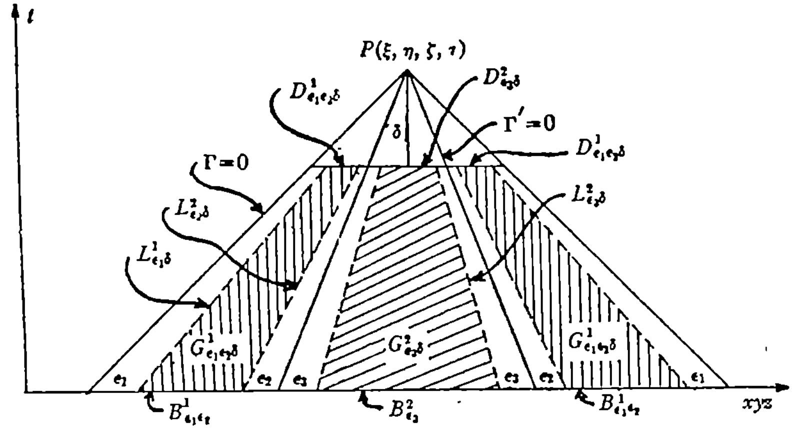
$$(1 - \epsilon_1)^2(\tau - t)^2 - r^2 > 0; \qquad 0 < \epsilon_1 < 1 - 1/a,$$

$$(1/a + \epsilon_2)^2(\tau - t)^2 - r^2 < 0; \qquad 0 < \epsilon_2 < 1 - 1/a,$$

(b) the truncated interior cone $G^2_{\epsilon_3\delta}$:

$$(1 - \epsilon_3)^2((\tau - t)/a)^2 - r^2 > 0; \qquad 0 < \epsilon_3 < 1.$$

The ring domain is bounded by the volume $B^1_{\epsilon_1\epsilon_2}$, the lateral surfaces $L^1_{\epsilon_1\delta}$ and $L^2_{\epsilon_2\delta}$, and the volume $D^1_{\epsilon_1\epsilon_2\delta}$. Similarly the domain $G^2_{\epsilon_3\delta}$ is bounded by the sphere $B^2_{\epsilon_3}$, the lateral surface $L^2_{\epsilon_3}$, and the sphere $D^2_{\epsilon_3\delta}$ (see diagram).



3. **Integration over the domain $G^1_{\epsilon_1\epsilon_2\delta}$.** In $G^1_{\epsilon_1\epsilon_2\delta}$ where $(\tau-t)>r$ and $(\tau-t)/a<r$, the elementary solution is:

$$V = -\frac{1}{2r}\log\frac{(\tau-t)-r}{(\tau-t)+r}+\frac{a}{2r}\log\frac{r-(\tau-t)/a}{r+(\tau-t)/a}\cdot$$

Since $L[V]=0$, Green's formula (2.3) may be written

$$\begin{aligned}(3.1)\qquad -\iiiint_{G^1_{\epsilon_1\epsilon_2\delta}} Vfdg + \iiint_{D^1_{\epsilon_1\epsilon_2\delta}} Ado + \iiint_{L^1_{\epsilon_1}} Ado \\ + \iiint_{L^2_{\epsilon_2\delta}} Ado + \iiint_{B^1_{\epsilon_1\epsilon_2}} Ado = 0\end{aligned}$$

where

$$A = V\partial N[u]/\partial s_2 - u\partial M[V]/\partial s_1 - \bar{N}[u]\partial V/\partial s_2 + M[V]\partial u/\partial s_1.$$

The last integral of (3.1) vanishes due to initial conditions and in order to express the remaining integrals as functions of ϵ and δ we introduce new coordinates, σ, μ; α, β, γ by the transformation

$$(3.2)\qquad \begin{aligned} x &= \xi + \sigma(1-\mu)\alpha, & y &= \eta + \sigma(1-\mu)\beta, \\ z &= \zeta + \sigma(1-\mu)\gamma, & t &= \tau - \sigma, \end{aligned}$$

where α, β, γ are parameters on the unit sphere $\alpha^2+\beta^2+\gamma^2=1$. The domain $G^1_{\epsilon_1\epsilon_2\delta}$ is then determined by the inequalities $\delta \leqq \sigma \leqq \tau$ and $\epsilon_1 \leqq \mu \leqq 1 - 1/a - \epsilon_2$, and in the new coordinates:

$$r = \sigma(1 - \mu); \quad dxdydzdt = \sigma^3(1 - \mu)^2 d\mu d\sigma d\omega \quad [d\omega = (1/\gamma)d\alpha d\beta],$$

$$\Gamma = \sigma^2\mu(2 - \mu); \quad \Gamma' = \sigma^2[1/a^2 - (1 - \mu)^2].$$

We are interested only in the coefficients of log ϵ_i of each integral which we shall indicate by placing bent bars with the index i: $\langle i \rangle$ in front of the integral sign.

For the first term of (3.1) we obtain the resolution

$$\begin{aligned} &- \langle 1, 2\rangle \iiiint_{G^1_{\epsilon_1\epsilon_2\delta}} V f dg \\ &\quad = \langle 1, 2\rangle \iiiint_{G^1_{\epsilon_1\epsilon_2\delta}} \left\{ \frac{1}{2\sigma(1 - \mu)} \log \frac{\mu}{2 - \mu} \right. \\ (3.3) &\qquad \left. - \frac{a}{2\sigma(1 - \mu)} \log \frac{1 - \mu - 1/a}{1 - \mu + 1/a} \right\} f\sigma^3(1 - \mu)^2 d\mu d\sigma d\omega \\ &\quad = \frac{1}{2} \int_\delta^\tau \sigma^2 d\sigma \iint_\rho d\omega \langle 1, 2\rangle \int_{\epsilon_1}^{1-1/a-\epsilon_2} \{(1 - \mu) \log \mu \\ &\qquad - a(1 - \mu) \log (1 - \mu - 1/a)\} f d\mu, \end{aligned}$$

where ρ is the surface of the intersection of the conical ring $G^1_{\epsilon_1\epsilon_2\delta}$ with the plane $\sigma = \tau - t$. It can be readily seen that upon μ-integration the coefficients of log ϵ_1 and log ϵ_2 will have ϵ_1 and ϵ_2 respectively as factors and will therefore tend to zero as ϵ_1 and ϵ_2 approach zero. Therefore,

$$(3.4) \quad \lim_{\delta \to 0,\, \epsilon_1 \to 0,\, \epsilon_2 \to 0} \left(- \langle 1, 2\rangle \iiiint_{G^1_{\epsilon_1\epsilon_2\delta}} V f dg \right) = 0.$$

On $D^1_{\epsilon_1\epsilon_2\delta}$: $t_\nu = 1$, $x_\nu = y_\nu = z_\nu = 0$ and the second term of (3.1) can be written:

$$\begin{aligned} (3.5) \quad &\iiint_{D^1_{\epsilon_1\epsilon_2\delta}} \left\{ a^2 V \frac{\partial N[u]}{\partial t} - u \frac{\partial M[V]}{\partial t} \right. \\ &\qquad \left. - a^2 N[u] \frac{\partial V}{\partial t} + M[V] \frac{\partial u}{\partial t} \right\} dxdydz. \end{aligned}$$

We consider each of the four integrals of (3.5) separately. The first integral contains V and a resolution similar to that of (3.3) shows that the coefficients of log ϵ_1 and log ϵ_2 tend to zero as ϵ_1 and ϵ_2 approach zero. Observing that $M[V]=2(a^2-1)(\tau-t)/\Gamma^2$, the second integral of (3.5) is:

$$-\int\int\int_{D^1_{\epsilon_1\epsilon_2\delta}} u\frac{\partial M[V]}{\partial t}\,dxdydz$$

$$= -(a^2-1)\int\int\int_{D^1_{\epsilon_1\epsilon_2\delta}} u\left\{\frac{8(\tau-t)^2}{\Gamma^3}-\frac{2}{\Gamma^2}\right\}dxdydz$$

$$= -(a^2-1)\int\int\int_{D^1_{\epsilon_1\epsilon_2\delta}} u\left\{\frac{8(1-\mu)^2}{\sigma\mu^3(2-\mu)^3}-\frac{2(1-\mu)^2}{\sigma\mu^2(2-\mu)^2}\right\}d\mu d\omega.$$

We note that only terms containing $1/\mu$ will give terms involving log ϵ_i after integration. We develop u into a Taylor series in terms of $\bar{u}$, that is, the value of u on the boundary of $D^1_{\epsilon_1\epsilon_2\delta}$. Since the distance of a point $(\sigma, \mu, \alpha, \beta, \gamma)$ from the corresponding boundary point $(\sigma, \epsilon_1, \alpha, \beta, \gamma)$ is $\sigma(1-\epsilon_1)-\sigma(1-\mu)=\sigma(\mu-\epsilon_1)$, we have $u=\bar{u}-\sigma(\mu-\epsilon_1)\bar{u}_r+(\sigma^2(\mu-\epsilon_1)^2/2)\bar{u}_{rr}-\cdots=\bar{u}-\sigma\mu\bar{u}_r+(\sigma^2\mu^2/2)\bar{u}_{rr}-\cdots+(\epsilon_1)$, where (ϵ_1) denotes terms which tend to zero with ϵ_1 and $\bar{u}_r=\alpha\partial u/\partial x+\beta\partial u/\partial y+\gamma\partial u/\partial z$. Since the fourth and higher terms in the Taylor development of u contain $\sigma^3\mu^3$, the difference of u and the first three terms of its development will not give any coefficient of log ϵ_i. For the purpose of determining coefficients of log ϵ_i, we may write therefore

$$-\langle 1, 2\rangle\int\int\int_{D^1_{\epsilon_1\epsilon_2\delta}} u\frac{\partial M[V]}{\partial t}\,dxdydz$$

$$= -\frac{a^2-1}{\sigma}\int\int_{\rho}\bar{u}d\omega\ \langle 1, 2\rangle\int_{\epsilon_1}^{1-1/a-\epsilon_2}\left\{\frac{8(1-\mu)^2}{\mu^3(2-\mu)^3}-\frac{2(1-\mu)^2}{\mu^2(2-\mu)^2}\right\}d\mu$$

$$+(a^2-1)\int\int_{\rho}\bar{u}_r d\omega\ \langle 1, 2\rangle\int_{\epsilon_1}^{1-1/a-\epsilon_2}\left\{\frac{8(1-\mu)^2}{\mu^2(2-\mu)^3}-\frac{2(1-\mu)^2}{\mu(2-\mu)^2}\right\}d\mu$$

$$-\frac{\sigma(a^2-1)}{2}\int\int_{\rho}\bar{u}_{rr}d\omega\ \langle 1. 2\rangle\int_{\epsilon_1}^{1-1/a-\epsilon_2}\left\{\frac{8(1-\mu)^2}{\mu(2-\mu)^3}-\frac{2(1-\mu)^2}{(2-\mu)^2}\right\}d\mu.$$

The coefficient of log ϵ_1 is zero in the first integral above, 1 in the second and $-1/2$ in the third. Since the integration does not yield $\log \epsilon_2$ we have:

$$\lim_{\epsilon_1\to 0,\ \epsilon_2\to 0}\left(-\langle 1, 2\rangle \iiint_{D^1_{\epsilon_1\epsilon_2\delta}} u\,\frac{\partial M[V]}{\partial t}\,dxdydz\right)$$

$$= (a^2-1)\iint_{\sigma\leqq\delta}\bar{u}_r d\omega + \frac{\delta(a^2-1)}{2}\iint_{\sigma=\delta}\bar{u}_{rr}d\omega.$$

But

$$\iint_{\sigma=\delta}\bar{u}_r d\omega = \frac{1}{\delta^2}\iint_{\sigma\leqq\delta}\bar{u}_r d\Omega = \frac{1}{\delta^2}\iiint_{r=\delta}\Delta u\,dxdydz,$$

where the last integral is an infinitesimal of the order δ^3 when δ approaches zero. We find therefore

$$(3.6)\quad \lim_{\delta\to 0}\ \lim_{\epsilon_1\to 0,\ \epsilon_2\to 0}\left\{-\langle 1, 2\rangle\iiint_{D^1_{\epsilon_1\epsilon_2\delta}} u\,\frac{\partial M[V]}{\partial t}\,dxdydz\right\} = 0.$$

The third integral of (3.5) is

$$-a^2\iiint_{D^1_{\epsilon_1\epsilon_2\delta}} H[u]\,\frac{\partial V}{\partial t}\,dxdydz$$

$$= -a^2\iiint_{D^1_{\epsilon_1\epsilon_2\delta}} N[u]\left\{\frac{1}{\Gamma}-\frac{1}{\Gamma'}\right\}dxdydz$$

$$= -\sigma a^2\iiint_{D^1_{\epsilon_1\epsilon_2\delta}} N[u]\left\{\frac{(1-\mu)^2}{\mu(2-\mu)}-\frac{(1-\mu)^2}{[1/a^2-(1-\mu)^2]}\right\}d\mu d\omega$$

and as we let $\sigma=\delta$ approach zero, this integral vanishes.

The fourth integral of (3.5) is

$$\langle 1,2\rangle\iiint_{D^1_{\epsilon_1\epsilon_2\delta}} M[V]u_t\,dxdydz$$

$$= (a^2-1)\langle 1\rangle\iiint_{D^1_{\epsilon_1\epsilon_2\delta}}\frac{2(\tau-t)}{\Gamma^2}\,u_t\,dxdydz$$

$$= (a^2-1)\langle 1\rangle\iiint_{D^1_{\epsilon_1\epsilon_2\delta}}\frac{2u_t(1-\mu)^2}{\mu^2(2-\mu)^2}\,d\mu d\omega$$

$$= (a^2-1)\iint_{\sigma=\delta}\bar{u}_t d\omega\,\langle 1\rangle\int_{\epsilon_1}^{1-1/a-\epsilon_2}\frac{2(1-\mu)^2}{\mu^2(2-\mu)^2}\,d\mu$$

$$= \frac{a^2-1}{2}\iint_{\sigma=\delta}\bar{u}_t d\omega.$$

Whence

$$
\lim_{\delta\to 0,\ \epsilon_1\to 0,\ \epsilon_2\to 0}\left\{\langle 1\rangle \iiint_{D^1_{\epsilon_1\epsilon_2\delta}} M[u]\frac{\partial u}{\partial t}\,dxdydz\right\} = 2\pi(a^2-1)u_t(\xi,\eta,\zeta,\tau). \tag{3.7}
$$

On the lateral surface $L^1_{\epsilon_1\delta}$, μ is constant and since $M[V]$ and $\partial M[V]/\partial s_1$ contain no logarithms, we need only consider

$$
\begin{aligned}
\langle 1\rangle S^1_{\epsilon_1\delta} &= \langle 1\rangle \iiint_{L^1_{\epsilon_1\delta}}\left\{V\frac{\partial N[u]}{\partial s_2} - N[u]\frac{\partial V}{\partial s_2}\right\}do \\
&= \langle 1\rangle \iiint_{L^1_{\epsilon_1\delta}}\left\{\left[-\frac{1}{2\sigma(1-\mu)}\log\frac{\mu}{2-\mu}\right.\right. \\
&\qquad \left.\left. + \frac{a}{2\sigma(1-\mu)}\log\frac{1-\mu-1/a}{1-\mu+1/a}\right]\frac{\partial N[u]}{\partial s_2} - N[u]\frac{\partial V}{\partial s_2}\right\}do.
\end{aligned}
$$

On the lateral surface $(\partial V/\partial s_2)do = [(1-\epsilon_1)a^2V_t - \alpha V_x - \beta V_y - \gamma V_z]\cdot(1-\epsilon_1)^2\sigma^2 d\sigma d\omega$. Since $V_t = 1/\Gamma - 1/\Gamma'$, it does not contribute any coefficient of $\log \epsilon_1$,

$$
V_x = \frac{\alpha}{2\sigma^2(1-\mu)^2}\log\frac{\mu}{2-\mu} - \frac{a\alpha}{2\sigma^2(1-\mu)^2}\log\frac{1-\mu-1/a}{1-\mu+1/a} + \cdots,
$$

where the dots indicate terms which do not contain logarithms. Therefore

$$
\begin{aligned}
\langle 1\rangle\frac{\partial V}{\partial s_2}do &= \langle 1\rangle\left[-\frac{1}{2\sigma^2(1-\epsilon_1)^2}(\alpha^2+\beta^2+\gamma^2)\right](1-\epsilon_1)^2\sigma^2 d\sigma d\omega \\
&= -\,d\sigma d\omega/2.
\end{aligned}
$$

Thus the coefficient of $\log \epsilon_1$ for the lateral surface $L^1_{\epsilon_1\delta}$ as $\epsilon_1\to 0$ is

$$
\lim_{\epsilon_1\to 0}\left(\langle 1\rangle S^1_{\epsilon_1\delta}\right) = -\frac{1}{2}\iiint_{L^1_\delta}\left\{\frac{1}{r}\frac{\partial N[u]}{\partial s_2} - N[u]\frac{\partial(1/r)}{\partial s_2}\right\}do.
$$

In the limit $\delta\to 0$, we obtain

$$
\lim_{\delta\to 0,\ \epsilon_1\to 0}\left(\langle 1\rangle S^1_{\epsilon_1\delta}\right) = -\frac{1}{2}\iiint_{L^1}\left\{\frac{1}{r}\frac{\partial N[u]}{\partial s_2} - N[u]\frac{\partial(1/r)}{\partial s_2}\right\}do.^{2} \tag{3.8}
$$

---

[2] Since $\partial(1/r)/\partial s_2$ is not defined at the point $P(\xi,\eta,\zeta,\tau)$, this integral is to be taken in the limiting sense.

Similarly on the lateral surface $L^2_{\epsilon_2\delta}$ we need only consider

$$\langle 2\rangle S^2_{\epsilon_2\delta} = \langle 2\rangle \iiint_{L^2_{\epsilon_2\delta}} \left\{ V\frac{\partial N[u]}{\partial s_2} - N[u]\frac{\partial V}{\partial s_2}\right\} do.$$

We have $\langle 2\rangle\ V=a/2r$; $\langle 2\rangle\ \partial V/\partial s_2=(a/2)\partial(1/r)/\partial s_2$. Therefore the coefficient of $\log \epsilon_2$ for the lateral surface $L^2_{\epsilon_2\delta}$ is, when $\epsilon_2\to 0$,

$$\lim_{\epsilon_2\to 0} (\langle 2\rangle S^2_{\epsilon_2\delta}) = \frac{a}{2}\iiint_{L^2_\delta} \left\{ \frac{1}{r}\frac{\partial N[u]}{\partial s_2} - N[u]\frac{\partial(1/r)}{\partial s_2}\right\} do.$$

In the limit $\delta\to 0$, we obtain:

$$\lim_{\delta\to 0}\lim_{\epsilon_2\to 0} (\langle 2\rangle S^2_{\epsilon_2\delta}) = \frac{a}{2}\iiint_{L^2} \left\{ \frac{1}{r}\frac{\partial N[u]}{\partial s_2} - N[u]\frac{\partial(1/r)}{\partial s_2}\right\} do. \tag{3.9}$$

Summarizing the results obtained by the use of the first form of Green's formula (2.3), we have the equation

$$\left[2\pi(a^2-1)u_i(P) - \frac{1}{2}\iiint_{L^1} \left\{ \frac{1}{r}\frac{\partial N[u]}{\partial s_2} - N[u]\frac{\partial(1/r)}{\partial s_2}\right\} do\right]\log \epsilon_1$$
$$+\left[\frac{a}{2}\iiint_{L^2} \left\{ \frac{1}{r}\frac{\partial N[u]}{\partial s_2} - N[u]\frac{\partial(1/r)}{\partial s_2}\right\} do\right]\log \epsilon_2 = 0.$$

Since ϵ_1 and ϵ_2 are independent, the coefficients of $\log \epsilon_1$ and $\log \epsilon_2$ must vanish, which gives the following relations:

$$2\pi(a^2-1)u_i(P) = \frac{1}{2}\iiint_{L^1} \left\{ \frac{1}{r}\frac{\partial N[u]}{\partial s_2} - N[u]\frac{\partial(1/r)}{\partial s_2}\right\} do, \tag{3.10}$$

$$\frac{a}{2}\iiint_{L^2} \left\{ \frac{1}{r}\frac{\partial N[u]}{\partial s_2} - N[u]\frac{\partial(1/r)}{\partial s_2}\right\} do = 0. \tag{3.11}$$

4. Application of the second form of Green's formula and integration over the domain $G^2_{\epsilon_2\delta}$. In a manner quite analogous to that in the preceding section we perform the integration of the second form of Green's formula (2.4) over the domains $G^1_{\epsilon_1\epsilon_2\delta}$ and $G^2_{\epsilon_2\delta}$ and obtain the following additional relations:

$$\frac{1}{2}\iiint_{L^1} \left\{ \frac{1}{r}\frac{\partial M[u]}{\partial s_1} - M[u]\frac{\partial(1/r)}{\partial s_1}\right\} do = 0, \tag{4.1}$$

$$2\pi a(a^2-1)u_i(P) = -\frac{a}{2}\iiint_{L^2} \left\{ \frac{1}{r}\frac{\partial M[u]}{\partial s_1} - M[u]\frac{\partial(1/r)}{\partial s_1}\right\} do. \tag{4.2}$$

Subtracting (4.2) from (3.10) we obtain an expression for u_t:

$$
\begin{aligned}
&2\pi(1-a)(a^2-1)u_t(P)\\
&\quad=\frac{1}{2}\iiint_{L^1}\left\{\frac{1}{r}\frac{\partial N[u]}{\partial s_2}-N[u]\frac{\partial(1/r)}{\partial s_2}\right\}do\\
&\qquad+\frac{a}{2}\iiint_{L^2}\left\{\frac{1}{r}\frac{\partial M[u]}{\partial s_1}-M[u]\frac{\partial(1/r)}{\partial s_1}\right\}do.
\end{aligned}
\tag{4.3}
$$

5. Transformation of lateral surface integrals into volume integrals. Let us consider the self-adjoint expression $M[u]=a^2\partial^2u/\partial t^2-\Delta u$. Green's formula for this expression of the second order is

$$
\begin{aligned}
&\iiiint_G\{VM[u]-uM[V]\}dg\\
&\qquad=\iiint_S\left\{V\frac{\partial u}{\partial s_2}-u\frac{\partial V}{\partial s_2}\right\}do,
\end{aligned}
\tag{5.1}
$$

where S is the three-dimensional boundary of G. If we put $V=1/r$ and $u=N[u]$, we cannot apply (5.1) over the interior of the surface $\Gamma=0$ because V has a singularity on the line $r=0$. To overcome this difficulty we again truncate the cone $\Gamma=0$ by the plane $t=\tau-\delta$ and remove the line $r=0$ by cutting out a small cone $\epsilon^2(\tau-t)^2-r^2=0$. Over this new domain, which we denote by $G^1_{\epsilon\delta}$, we apply formula (5.1):

$$
\begin{aligned}
&\iiiint_{G^1_{\epsilon\delta}}\left\{\frac{1}{r}M[N[u]]-N[u]M\left[\frac{1}{r}\right]\right\}dg\\
&\qquad=\iiint_{S^1_{\epsilon\delta}}\left\{\frac{1}{r}\frac{\partial N[u]}{\partial s_2}-N[u]\frac{\partial(1/r)}{\partial s_2}\right\}do.
\end{aligned}
$$

Since $M[1/r]=0$ and, by (2.1), $M[N[u]]=f(x, y, z, t)$, we have

$$
\begin{aligned}
\iiiint_{G_{\delta}}\frac{1}{r}f\,dg&=\iiint_{L^1_{\delta}}\left\{\frac{1}{r}\frac{\partial N[u]}{\partial s_2}-N[u]\frac{\partial(1/r)}{\partial s_2}\right\}do\\
&\quad+\iiint_{D^1_{\epsilon\delta}}\left\{\frac{a^2}{r}\frac{\partial N[u]}{\partial t}-a^2N[u]\frac{\partial(1/r)}{\partial t}\right\}dxdydz\\
&\quad+\iiint_{L_{\epsilon\delta}}\left\{\frac{1}{r}\frac{\partial N[u]}{\partial s_2}-N[u]\frac{\partial(1/r)}{\partial s_2}\right\}do\\
&\quad-\iiint_{B^1_{\epsilon}}\left\{\frac{a^2}{r}\frac{\partial N[u]}{\partial t}-a^2N[u]\frac{\partial(1/r)}{\partial t}\right\}dxdydz.
\end{aligned}
\tag{5.2}
$$

The last integral of (5.2) vanishes because of initial conditions. Since

$$\iiint_{D^1_{\epsilon\delta}} \left\{ \frac{a^2}{r} \frac{\partial N[u]}{\partial t} - a^2 N[u] \frac{\partial(1/r)}{\partial t} \right\} dxdydz$$

$$= \sigma^3 \iiint_{D^1_{\epsilon\delta}} \left\{ \frac{a^2}{r} \frac{\partial N[u]}{\partial t} - a^2 N[u] \frac{\partial(1/r)}{\partial t} \right\} (1-\mu)^2 d\mu d\omega,$$

the integral over $D^1_{\epsilon\delta}$ vanishes in the limit $\epsilon \to 0$, $\sigma = \delta \to 0$. In order to find the limiting value of the integral over $L_{\epsilon\delta}$, we make the following transformation of the coordinates:

$$x = \xi + \epsilon\sigma\alpha, \qquad y = \eta + \epsilon\sigma\beta, \qquad z = \zeta + \epsilon\sigma\gamma, \qquad t = \tau - \sigma.$$

In the new coordinates:

$$r = \epsilon\sigma, \quad do = \epsilon^2\sigma^2(1+\epsilon^2)^{1/2} d\sigma d\omega, \quad \partial(1/r)/\partial s_2 = -1/r^2(1+\epsilon^2)^{1/2},$$

and we obtain:

$$\iiint_{L_{\epsilon\delta}} \left\{ \frac{1}{r} \frac{\partial N[u]}{\partial s_2} - N[u] \frac{\partial(1/r)}{\partial s_2} \right\} do$$

$$= \iiint_{L_{\epsilon\delta}} \left\{ \frac{1}{\epsilon\sigma} \frac{\partial N[u]}{\partial s_2} + N[u] \frac{1}{\epsilon^2\sigma^2(1+\epsilon^2)^{1/2}} \right\} \epsilon^2\sigma^2(1+\epsilon^2)^{1/2} d\sigma d\omega.$$

In the limit $\epsilon \to 0$, $\delta \to 0$ we obtain

$$\lim_{\epsilon\to 0,\, \delta \to 0} \left\{ \iint_{r=\delta\epsilon} d\omega \int_\delta^\tau N[u] d\sigma \right\} = -4\pi \int_0^\tau N[u] dt.$$

Therefore as we let ϵ and δ approach zero, (5.2) gives

$$\iiiint_{G^1} \frac{1}{r} f dg = \iiint_{L^1} \left\{ \frac{1}{r} \frac{\partial N[u]}{\partial s_2} - N[u] \frac{\partial(1/r)}{\partial s_2} \right\} do - 4\pi \int_0^\tau N[u] dt. \tag{5.3}$$

Interchanging $N[u]$ with $M[u]$ we obtain a similar expression, but, since by (4.1) the integral over L^1 vanishes, we have the relation

$$\iiiint_{G^1} \frac{1}{r} f dg = -4\pi \int_0^\tau M[u] dt. \tag{5.4}$$

The same procedure applied to the domain G^2 consisting of the interior of the surface $\Gamma' = 0$ yields the following relations:

$$
\begin{aligned}
(5.5)\qquad \iiiint_{G^2} \frac{1}{r} f dg = \iiint_{L^2} \left\{ \frac{1}{r} \frac{\partial M[u]}{\partial s_1} - M[u] \frac{\partial(1/r)}{\partial s_1} \right\} do \\
- 4\pi \int_0^\tau M[u] dt,
\end{aligned}
$$

$$
(5.6)\qquad \iiiint_{G^2} \frac{1}{r} f dg = - 4\pi \int_0^\tau N[u] dt.
$$

6. Solution of the initial value problem. The relations of the preceding section enable us to obtain the solution of the initial value problem (2.1). Combining (5.3) and (5.6) we obtain

$$
\begin{aligned}
(6.1)\qquad \iiint_{L^1} \left\{ \frac{1}{r} \frac{\partial N[u]}{\partial s_2} - N[u] \frac{\partial(1/r)}{\partial s_2} \right\} do \\
= \iiiint_{G^1} \frac{1}{r} f dg - \iiiint_{G^2} \frac{1}{r} f dg,
\end{aligned}
$$

and combining (5.4) and (5.5) gives the relation

$$
\begin{aligned}
(6.2)\qquad \iiint_{L^2} \left\{ \frac{1}{r} \frac{\partial M[u]}{\partial s_1} - M[u] \frac{\partial(1/r)}{\partial s_1} \right\} do \\
= \iiiint_{G^2} \frac{1}{r} f dg - \iiiint_{G^1} \frac{1}{r} f dg.
\end{aligned}
$$

Substitution of the expressions (6.1) and (6.2) for the lateral surface integrals in (4.3) gives

$$
\begin{aligned}
4\pi(1 - a)(a^2 - 1) u_t(P) = (1 - a) \iiiint_{G^1} \frac{1}{r} f dg \\
- (1 - a) \iiiint_{G^2} \frac{1}{r} f dg
\end{aligned}
$$

or

$$
(6.3)\qquad 4\pi(a^2 - 1) u_t(P) = \iiiint_{G^1} \frac{1}{r} f dg - \iiiint_{G^2} \frac{1}{r} f dg.
$$

Therefore the solution of the initial value problem (2.1) is

$$
\begin{aligned}
(6.4)\qquad 4\pi(a^2 - 1) u(\xi, \eta, \zeta, \tau) = \iiiint_{G^1} \frac{\tau - t - r}{r} f dg \\
- \iiiint_{G^2} \frac{\tau - t - ar}{r} f dg.
\end{aligned}
$$

In order to show that the right-hand side of (6.4) has the required derivatives we apply thereto transformation (3.2) and obtain

$$\int\!\!\int\!\!\int\!\!\int_{G^1} f\sigma^3\mu(1-\mu)d\sigma d\mu d\omega$$
$$-\int\!\!\int\!\!\int\!\!\int_{G^2} f\sigma^3[1-a(1-\mu)](1-\mu)d\sigma d\mu d\omega.$$

Because of our assumptions concerning the function f both integrands are regular and our proof is complete.

7. **The limiting case $a=1$.** In order to determine what happens when $a=1$, we observe that the solution (6.4) for $a>1$ can be written in the form

$$\begin{aligned}4\pi(a^2-1)u(P) &= \int_0^\tau dt\int\!\!\int\!\!\int_{r\leqq\tau-t}\frac{\tau-t-r}{r}fdxdydz\\ &\quad-\int_0^\tau dt\int\!\!\int\!\!\int_{r\leqq(\tau-t)/a}\frac{\tau-t-ar}{r}fdxdydz\\ &=F(1)-F(a).\end{aligned}$$

Therefore $4\pi(a+1)u(P)=-[F(1)-F(a)]/(1-a)$ and in the limit $a=1$, $8\pi u(P)=-F'(1)$. But differentiation of $F(a)$ with respect to a gives

$$F'(a)=-\int_0^\tau dt\int\!\!\int\!\!\int_{r\leqq(\tau-t)/a}fdxdydz,$$

whence

$$F'(1)=-\int_0^\tau dt\int\!\!\int\!\!\int_{r\leqq\tau-t}fdxdydz$$

and the solution of the iterated wave equation is

$$8\pi u(P)=\int\!\!\int\!\!\int\!\!\int_{G^1}fdxdydzdt. \tag{7.1}$$

8. **Nonhomogeneous initial conditions.** We consider now equation (2.1) with the following nonhomogeneous initial conditions:

$$\begin{aligned}u(x,y,z,0)&=\phi_0(x,y,z), & u_t(x,y,z,0)&=\phi_1(x,y,z),\\ u_{tt}(x,y,z,0)&=\phi_2(x,y,z), & u_{ttt}(x,y,z,0)&=\phi_3(x,y,z).\end{aligned} \tag{8.1}$$

The introduction of the initial conditions (8.1) does not present anything new with regard to our method and we content ourselves in writing the solution which satisfies such conditions:

$$
\begin{aligned}
&4\pi(a^2-1)u(\xi,\eta,\zeta,\tau)\\
&\quad=\iiiint_{G^1}\frac{\tau-t-r}{r}f\,dxdydzdt\\
&\quad\quad-\iiiint_{G^2}\frac{\tau-t-ar}{r}f\,dxdydzdt\\
&\quad\quad+\iiint_{r\leqq\tau}\frac{1}{r}\Big\{(\tau-r)\left[a^2\phi_3-(a^2+1)\Delta\phi_1\right]\\
&\qquad\qquad\qquad\qquad+\left[a^2\phi_2-(a^2+1)\Delta\phi_0\right]\Big\}dxdydz\\
(8.2)\quad&\quad\quad-\iiint_{r\leqq\tau/a}\frac{1}{r}\Big\{(\tau-ar)\left[a^2\phi_3-(a^2+1)\Delta\phi_1\right]\\
&\qquad\qquad\qquad\qquad+\left[a^2\phi_2-(a^2+1)\Delta\phi_0\right]\Big\}dxdydz\\
&\quad\quad+\frac{a^3}{\tau}\iiint_{r\leqq\tau}\Delta\phi_0dxdydz-\frac{a}{\tau}\iiint_{r\leqq\tau/a}\Delta\phi_0dxdydz\\
&\quad\quad+\frac{a^2}{\tau}\iint_{r=\tau}\left(\phi_1+\frac{1}{\tau}\phi_0\right)dS\\
&\quad\quad-\frac{a^2}{\tau}\iint_{r=\tau/a}\left(\phi_1+\frac{1}{\tau}\phi_0\right)dS.
\end{aligned}
$$

In the limiting case $a=1$, the above result reduces to

$$
\begin{aligned}
&8\pi u(\xi,\eta,\zeta,\tau)\\
(8.3)\quad&\quad=\iiiint_{G^1}fdg+\iiint_{r\leqq\tau}\left(\phi_3-\Delta\phi_1+\frac{2}{\tau}\Delta\phi_0\right)dxdydz\\
&\quad\quad+\iint_{r=\tau}\left(\phi_2+\frac{2}{\tau}\phi_1+\frac{2}{\tau^2}\phi_0-\Delta\phi_0\right)dS.
\end{aligned}
$$

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ON UNIFORM CONVERGENCE OF TRIGONOMETRIC SERIES

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1. Introduction. The following theorems have been proved previously.[1]

THEOREM I. *If the function $\phi(t)$ is throughout continuous, periodic of period 2π, $\phi(t)=\phi(-t)=\phi(2\pi+t)$,*

$$\phi(t) \sim \frac{a_0}{2} + \sum_1^\infty a_n \cos nt, \tag{1.1}$$

and if

$$na_n > -K, \tag{1.2}$$

for some constant K, and all n, then the series (1.1) *is uniformly convergent* (*on the real axis*).

THEOREM II. *If $f(t)$ is everywhere continuous, periodic of period 2π, $f(t)=-f(-t)$,*

$$f(t) \sim \sum_1^\infty b_n \sin nt, \tag{1.3}$$

and if

$$nb_n > -K, \qquad n = 1, 2, 3, \cdots. \tag{1.4}$$

then the series (1.3) *is uniformly convergent.*

THEOREM III (CHAUNDY AND JOLLIFFE). *The Fourier series* (1.3) *is uniformly convergent, if*

$$b_n \geqq b_{n+1} > 0, \quad \text{and if} \quad nb_n \to 0. \tag{1.5}$$

Note that here no explicit assumption is made on $f(t)$.

THEOREM IV. *If $\phi(t)$ is continuous at $t=0$, and if*

$$\lim_{\lambda \downarrow 1} \limsup_{n\to\infty} \sum_n^{\lambda n} (|a_\nu| - a_\nu) = 0, \tag{1.6}$$

then the series (1.1) *is uniformly convergent at $t=0$.* (*That is, $s_n(t_n)\to s$ whenever $t_n\to 0$.*)

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[1] Cf. [2] and the references given there; numbers in brackets refer to the literature cited at the end of this paper.

THEOREM V. *If $f(t)$ is continuous at $t=0$, and if*

$$\lim_{\lambda\downarrow 1}\limsup_{n\to\infty}\sum_{n}^{\lambda n}(|b_\nu|-b_\nu)=0, \tag{1.7}$$

then $\sum_1^n \nu b_\nu=o(n)$, and the series (1.3) *is uniformly convergent at $t=0$.*

Some more general results are given in the present paper. In particular:

THEOREM 1. *Under the assumptions of Theorem* IV *the series* (1.1) *converges uniformly at each point of continuity of $\phi(t)$.*

THEOREM 2. *Under the assumptions of Theorem* V *the series* (1.3) *converges uniformly at each point of continuity of $f(t)$.*

Clearly Theorems 1 and 2 include Theorems I and II respectively. Either of the following two theorems includes Theorem III.

THEOREM 3. *Suppose that*

$$\sum_{n}^{2n}|b_\nu-b_{\nu+1}|=O(n^{-1})\quad\text{as}\quad n\to\infty, \tag{1.8}$$

and that

$$(1-r)\sum_{1}^{\infty}nb_nr^n\to 0\quad\text{as}\quad r\uparrow 1; \tag{1.9}$$

then the trigonometric series $\sum b_n$ sin nt is uniformly convergent.

Note that the assumptions refer solely to the coefficients b_n.

THEOREM 4. *Suppose, for some constants $p\geqq 0$, $q\geqq 0$,*

$$nb_n+p=B_n\geqq 0, \tag{1.10}$$

that

$$B_{n+1}\leqq(1+n^{-1}q)B_n,\quad\textit{for all large } n, \tag{1.11}$$

and that (1.9) *holds. Then $nb_n\to 0$ and the trigonometric series $\sum b_n$ sin nt is uniformly convergent.*

We also give (in §§5 and 6) analogous theorems for cosine series; here the partial sums $\sum_1^n a_\nu=s_n$ play a similar role as the sequence $\{nb_n\}$ for the sine series. However convergence of the series $\sum a_\nu$ does not carry as far as existence of the limit lim nb_n. It is for this reason that no such theorems have been established hitherto for cosine series. For details see §§5, 6 and 7.

2. Proof of Theorems 1 and 2. We have proved [2] that under the assumptions of Theorem IV

$$\lim_{\lambda\downarrow 1}\limsup_{n\to\infty}\sum_{n}^{\lambda n}|a_\nu| = 0. \tag{2.1}$$

If $\phi(t)$ is continuous at t_0, then the Fourier series

$$\frac{\phi(t_0+\theta)+\phi(t_0-\theta)}{2}\sim\sum a_n\cos nt_0\cos n\theta,$$

$$\frac{\phi(t_0+\theta)-\phi(t_0-\theta)}{2}\sim\sum a_n\sin nt_0\sin n\theta$$

satisfy the assumptions of Theorems IV and V respectively, hence are uniformly convergent at $\theta=0$. This proves Theorem 1. The proof of Theorem 2 follows on quite similar lines, since it has been proved [2] that

$$\lim_{\lambda\downarrow 1}\limsup_{n\to\infty}\sum_{n}^{\lambda n}|b_\nu| = 0. \tag{2.2}$$

It is clear from our proof that the assumptions of our theorems can be replaced by the sole assumptions (2.1) and (2.2) respectively.

We remark that in Theorems IV and V the assumptions (1.6) and (1.7) cannot be replaced by

$$\sum_{n}^{2n}|a_\nu| = O(1) \quad\text{and}\quad \sum a_n \text{ converges},$$

$$\sum_{n}^{2n}|b_\nu| = O(1) \quad\text{and}\quad \sum_{1}^{n}\nu b_\nu = o(n),$$

respectively. We give an example, suggested by a construction due to Fejér [1].

Let $P_n(z)=\sum_{\nu=0}^{n-1}z^\nu/(n-\nu)-\sum_{\nu=0}^{n-1}z^{n+\nu}/(\nu+1)$, then $|P_n(z)|<6$ for $|z|\leqq 1$. Let $\mu_n=2^{n^2}$, $\kappa_n=2^{n(n+1)}$, $n=1, 2, 3, \cdots$, and consider the polynomial series $\sum_1^\infty n^{-2}z^{\mu_n}P_{\kappa_n}(ze^{i/n})$. This series is clearly uniformly convergent for $|z|\leqq 1$, the degree of the nth term is $2\kappa_n+\mu_n-1<\mu_{n+1}$, hence writing out the polynomials successively we get a power series, convergent for $|z|<1:\sum_1^\infty c_nz^n=F(z)$, and $F(e^{it})$ is the Fourier power series of a continuous function. The structure of P_n easily yields $\sum_n^{2n}|c_\nu|=O(1)$. It can be proved, as in Fejér's example, that the series $\sum c_ne^{int}$ converges for each t, uniformly in $\epsilon\leqq t\leqq 2\pi-\epsilon$, $\epsilon>0$; but neither component converges uniformly at $t=0$. The same is true for the series $\sum a_n \cos nt$, $\sum a_n \sin nt$, where $a_n = R(c_n)$; $\sum a_n$ converges, so that $\sum_1^n \nu a_\nu = o(n)$. Again, using Fejér's device, and replacing $e^{i/n}$ by e^{it_n}, where the sequence $\{t_n\}$ is everywhere dense in $(0, 2\pi)$, we get a continuous function with a Fourier series and its conjugate nonuniformly convergent everywhere, while $|c_n|$ is the same as before.

3. **Proof of Theorem 3.** It follows from (1.8) that $\lim b_n$ exists, and now from (1.9) that $\lim b_n = 0$. Furthermore

$$\sum_1^{2^\kappa} |b_\nu - b_{\nu+1}| \leqq \sum_{n=0}^{\kappa-1} \sum_{2^n}^{2^{n+1}} |b_\nu - b_{\nu+1}| = \sum \frac{1}{2^n} O(1) = O(1),$$

hence

$$\sum_1^\infty |b_\nu - b_{\nu+1}| < \infty. \tag{3.1}$$

Moreover

$$\begin{aligned} \sum_n^\infty |b_\nu - b_{\nu+1}| &\leqq \sum_{\kappa=0}^\infty \sum_{n\cdot 2^\kappa}^{n\cdot 2^{\kappa+1}} |b_\nu - b_{\nu+1}| \\ &= O\left(\frac{1}{n} \sum \frac{1}{2^\kappa}\right) = O\left(\frac{1}{n}\right), \end{aligned} \tag{3.2}$$

hence

$$nb_n = n \sum_n^\infty (b_\nu - b_{\nu+1}) = O(1). \tag{3.3}$$

It was proved by Littlewood that boundedness of a sequence and Abel summability imply $(C, 1)$ summability; if we apply this to the sequence $\{nb_n\}$ it follows from (1.9) and (3.3) that

$$\sum_1^n \nu b_\nu = o(n). \tag{3.4}$$

Next, from Abel's formula

$$\sum_n^m b_\nu \sin \nu t = \sum_n^{m-1} (b_\nu - b_{\nu+1}) T_\nu(t) + b_m T_m(t) - b_n T_{n-1}(t), \tag{3.5}$$

where

$$T_n(t) = \frac{\cos t/2 - \cos (n + 1/2)t}{2 \sin t/2},$$

hence in any interval $\epsilon \leqq t \leqq 2\pi - \epsilon$

$$\left|\sum_{n}^{m} b_\nu \sin \nu t\right| < \epsilon^{-1}\pi \sum_{n}^{\infty} | b_\nu - b_{\nu+1} | + 2\epsilon^{-1}\pi(| b_n | + | b_m |).$$

Thus the series $\sum b_n \sin nt$ is uniformly convergent in $\epsilon \leqq t \leqq 2\pi - \epsilon$, $\epsilon > 0$. Let

$$\sum_{1}^{\infty} b_n \sin nt = f(t);$$

we shall prove next that $f(t) \to 0$ as $t \downarrow 0$. We write

$$f(t) = \left(\sum_{1}^{n} + \sum_{n+1}^{\infty}\right) b_\nu \sin \nu t = U_1(t) + U_2(t),$$

say, where $n = [\epsilon^{-1}t^{-1}]$. Now, employing (3.2), (3.3) and (3.5)

$$\begin{aligned} | U_2(t) | &< t^{-1}\pi \left(\sum_{n+1}^{\infty} | b_\nu - b_{\nu+1} | + | b_{n+1} | \right) \\ &= t^{-1}O(n^{-1}) = \epsilon O(1). \end{aligned} \tag{3.6}$$

As to $U_1(t)$, we have

$$U_1 = \sum_{1}^{n} \nu b_\nu \frac{\sin \nu t}{\nu} = \sum_{1}^{n-1} v_\nu \Delta_\nu + v_n \frac{\sin nt}{n},$$

where

$$v_n = \sum_{1}^{n} \nu b_\nu, \qquad \Delta_n = \Delta \frac{\sin nt}{n} = \frac{\sin nt}{n} - \frac{\sin (n+1)t}{n+1}.$$

We have

$$\Delta_n = \int_0^t (\Delta \cos nx) dx = R \int_0^t z^n(1 - z) dx, \qquad z = e^{ix},$$

hence

$$| \Delta_n | < \int_0^t | 1 - z | \, dx < t^2,$$

and

$$\begin{aligned} | U_1(t) | &< t^2 \sum_{1}^{n} | v_\nu | + n^{-1} | v_n | \\ &< \epsilon^{-2} n^{-1} \sum_{1}^{n} \nu^{-1} | v_\nu | + n^{-1} | v_n | \to 0 \end{aligned} \tag{3.7}$$

as $t \downarrow 0$, by (3.4).

Now (3.6) and (3.7) yield

$$\limsup_{t\to 0} |f(t)| \leqq \epsilon;$$

ϵ being arbitrary, we get $f(t)\to 0$ as $t\to 0$. In view of (3.3) uniform convergence now follows from Theorem II.

We remark that under the assumptions of Theorem 3 the sequence $\{nb_n\}$ need not have a limit. This is seen from the example

$$nb_n = 1 \quad \text{for} \quad n = 2^\nu, \ \nu = 0, 1, 2, \cdots, \quad b_n = 0 \text{ otherwise.}$$

Moreover in this case $b_n \geqq 0$ and $\sum b_n$ is convergent.

On the other hand for the example $\sum_2^\infty(-1)^n \sin (2n-1)t/n \log n$, $nb_n\to 0$, $\sum b_n$ converges, yet the series is divergent for $t=\pi/2$. Of course (1.8) is not satisfied, but $\sum_n^{2n}|b_\nu| = O(1/\log n)$.

4. Proof of Theorem 4. We shall employ the following lemma.

LEMMA 1. *Suppose that* $B_n\geqq 0$, *that for some* $q\geqq 0$

$$B_{n+1} \leqq (1 + q/n)B_n, \qquad n = 1, 2, 3, \cdots, \tag{4.1}$$

and that the sequence $\{B_n\}$ *is Abel summable to* B; *then* $B_n\to B$.

This is Lemma 5 of my paper [2]. Note that the inequalities $B_n\geqq 0$ and (4.1) need only be satisfied for all large n, $n\geqq n_0$, say. For the sequence $B_n' = B_{n_0}$, $n=1, 2, \cdots, n_0$, $B_n' = B_n$, $n>n_0$, satisfies the assumptions of the lemma, hence $\lim B_n = \lim B_n'$ exists.

Now for $nb_n+p=B_n$, from (1.9)

$$(1 - r)\sum B_n r^n \to p \quad \text{as} \quad r\uparrow 1; \tag{4.2}$$

from (1.10) and (1.11)

$$0 \leqq B_{n+1} \leqq (1 + q/n)B_n, \quad \text{for all large } n. \tag{4.3}$$

Lemma 1 now yields

$$B_n \to p, \quad \text{that is} \quad nb_n \to 0. \tag{4.4}$$

From (4.3)

$$(B_{n+1} - B_n) \leqq n^{-1}qB_n, \quad \text{for } n \geqq n_0, \text{ say.} \tag{4.5}$$

Write $\sum_n^{2n}(B_{\nu+1}-B_\nu) = \sum' + \sum''$, where $\sum'$ is the sum of the positive terms, and $\sum''$ the rest. From (4.4) and (4.5), $\sum' = O(1)$; furthermore

$$B_{2n+1} - B_n = \sum' + \sum'' = \sum' - \left|\sum''\right|,$$

hence $|\sum''| = B_n - B_{2n+1} + \sum' = O(1)$. It now follows that

$$\sum_{n}^{2n} | B_{\nu+1} - B_\nu | = \sum{}' + \left| \sum{}'' \right| = O(1);$$

this and (4.4) yield (1.8). Our theorem now follows from Theorem 3.

If we replace (1.9) by the assumption (A) lim $nb_n = \rho$, then the trigonometric series

$$\sum (b_n - \rho n^{-1}) \sin nt = \sum \beta_n \sin nt$$

satisfies the assumptions of Theorem 4, hence it is uniformly convergent, and we get $nb_n \to \rho$, and

$$(4.6) \qquad \sum b_n \sin nt \to \pi\rho/2 \quad \text{as} \quad t \downarrow 0.$$

Combined with Theorem 3 of our paper [2] we get the theorem.

THEOREM 5. *If* (4.2) *holds then a necessary and sufficient condition that* (4.6) *holds is* $nb_n \to \rho$.

For b_n positive and decreasing, $\rho = 0$, the result is due to Chaundy and Jolliffe, for $\rho \neq 0$ to Hardy. For references see [2].

5. **The cosine series.** We shall next prove the theorem:

THEOREM 6. *Suppose that*

$$(5.1) \qquad \sum_{n}^{2n} | a_\nu - a_{\nu+1} | = O(n^{-1}),$$

and that $\sum a_n$ *is Abel summable, then* $\sum a_n \cos nt$ *is uniformly convergent.*

Using Abel's formula

$$(5.2) \quad \sum_{n}^{m} a_\nu \cos \nu t = \sum_{n}^{m-1} (a_\nu - a_{\nu+1}) \gamma_\nu(t) + a_m \gamma_m(t) - a_n \gamma_{n-1}(t),$$

where

$$\gamma_n(t) = \frac{\sin (n + 1/2)t}{2 \sin (t/2)}.$$

As in §3 it follows from (5.1) that lim a_n exists, and now Abel summability of $\sum a_n$ implies that $a_n \to 0$. Furthermore

$$(5.3) \quad \sum_{1}^{\infty} | a_n - a_{n+1} | < \infty, \quad \sum_{n}^{\infty} | a_\nu - a_{\nu+1} | = O(n^{-1}), \quad na_n = O(1).$$

Hence, by a theorem of Littlewood, $\sum a_n$ converges.

Now (5.2) yields uniform convergence of $\sum a_n \cos nt$ in $\epsilon \leqq t \leqq \pi$, $\epsilon > 0$. Let

$$\sum_1^\infty a_n \cos nt = \phi(t), \qquad 0 < t \leqq \pi.$$

We write

$$\sum_1^\infty a_\nu \cos \nu t = \sum_1^n + \sum_{n+1}^\infty = V_1(t) + V_2(t),$$

say, where $n = [\epsilon^{-1}t^{-1}]$. Now from (5.2)

$$V_2(t) = \sum_{n+1}^\infty (a_\nu - a_{\nu+1})\gamma_\nu(t) - a_{n+1}\gamma_n(t),$$

hence

$$\begin{aligned} |V_2(t)| &< t^{-1}\pi\left(|a_{n+1}| + \sum_{n+1}^{\infty}|a_\nu - a_{\nu+1}|\right) \\ &= t^{-1}O(n^{-1}) = \epsilon O(1). \end{aligned} \tag{5.4}$$

To estimate V_1 put $\sum_n^\infty a_\nu = r_n$, then $r_1 = s$, and

$$V_1 = s \cos t - r_{n+1} \cos nt + 2\sum_2^n r_\nu \sin (t/2) \sin (\nu + 1/2)t,$$

hence

$$\begin{aligned} |V_1(t) - s \cos t| &\leqq |r_{n+1}| + t\sum_2^n |r_\nu| \\ &\leqq |r_{n+1}| + \epsilon^{-1}n^{-1}\sum_2^n |r_\nu| = \epsilon^{-1}o(1) \end{aligned} \tag{5.5}$$

as $t \to 0$. From (5.4) and (5.5) $\limsup_{t\to 0} |\phi(t) - s| \leqq \epsilon$, ϵ being arbitrary, we get $\phi(t) \to s$, as $t \to 0$. Our theorem now follows from Theorem I. The example $\sum 2^{-n} \cos 2^n t$ shows that na_n need not have a limit. Here is an alternative proof for the continuity of $\phi(t)$ at $t = 0$: From (5.2)

$$\begin{aligned} \phi(t) = &- a_1/2 + 2^{-1}\sum_1^\infty (a_n - a_{n+1}) \cos nt \\ &+ 2^{-1} \cos (t/2) \sum_1^\infty (a_n - a_{n+1}) \frac{\sin nt}{\sin (t/2)}; \end{aligned}$$

clearly $\sum (a_n - a_{n+1}) \cos nt$ is uniformly convergent. Furthermore

$$\sum_1^\infty (a_n - a_{n+1}) \sin nt = \sum_1^\infty n(a_n - a_{n+1}) \frac{\sin nt}{n} = \sum_1^\infty a_n' \frac{\sin nt}{n},$$

where

$$a_n' = n(a_n - a_{n+1}), \qquad \sum_n^{2n} | a_\nu' | = O(1), \quad \text{by } (5.1).$$

Now $\sum_1^n a_\nu' = \sum_1^{n+1} a_\nu - (n+1)a_{n+1}$; $\sum a_n$ being convergent, it follows that $n^{-1}\sum_1^n \nu a_\nu \to 0$, and $\sum a_n'$ is $(C, 1)$ summable to s, hence by Theorem 4 of our paper [3]

$$\sum_1^\infty a_n' \frac{\sin nt}{nt} \to \sum_1^\infty a_n = s, \quad \text{as} \quad t \to 0.$$

Thus $\phi(t)$ is continuous at $t=0$.

Theorems 3 and 6 combined yield the theorem:

THEOREM 7. *Suppose that*

$$\sum_n^{2n} | c_\nu - c_{\nu+1} | = O(n^{-1}) \quad as \quad n \to \infty,$$

and that $\sum c_n$ *is Abel summable; then the power series* $\sum c_n z^n$ *is uniformly convergent in the circle* $|z| \leqq 1$.

It suffices to consider the circle $|z| = 1$; suppose first that the c_n are real. The uniform convergence of $\sum c_n \cos nt$ follows from Theorem 6; it also follows that $n^{-1}\sum_1^n \nu c_\nu \to 0$, and Theorem 3 now yields the uniform convergence of $\sum c_n \sin nt$. If the c_n are complex, $c_n = a_n + ib_n$, then apply the result just obtained to $\sum a_n z^n$, $\sum b_n z^n$. This proves Theorem 7.

6. Further theorems on cosine series.

Our next theorem is:

THEOREM 8. *Suppose that for some constants* $p \geqq 0$ *and* $q \geqq 0$

$$(6.1) \quad 0 \leqq (n+1)s_{n+1} - ns_n + p \leqq (1 + q/n)[ns_n - (n-1)s_{n-1} + p],$$

$s_n = \sum_1^n a_\nu$, *and that* $\sum a_n$ *is Abel summable; then* $na_n \to 0$, *and* $\sum a_n \cos nt$ *is uniformly convergent.*

Put $ns_n - (n-1)s_{n-1} + p = \delta_n = s_n + (n-1)a_n + p$, $s_0 = 0$, then $\sum_1^n \delta_\nu = n(s_n + p) \geqq 0$, $s_n \geqq -p$, hence by a well known theorem of Tauberian type $\sum a_n$ is $(C, 1)$ summable, thus the sequence $\{\delta_n\}$ is $(C, 2)$ summable. This and $0 \leqq \delta_{n+1} \leqq (1+q/n)\delta_n$ imply by Lemma 1 that lim δ_n exists, $\delta_n \to \delta$, say. It follows that $n^{-1}\sum_1^n \delta_\nu = s_n + p \to \delta$, or $s_n \to \delta - p = s$, and now

$$(6.2) \qquad na_n = \delta_n - s_n + a_n - p \to 0.$$

Next from (6.1)

$$\delta_{n+1} - \delta_n \leqq (q/n)\delta_n, \tag{6.3}$$

furthermore

$$D_n = \sum_n^{2n} (\delta_{\nu+1} - \delta_\nu) = \delta_{2n+1} - \delta_n = O(1). \tag{6.4}$$

Write $D_n = D' + D''$, where D' denotes the sum of positive terms, $D'' = D_n - D'$. From (6.3)

$$0 \leqq D' \leqq q \sum_n^{2n} \nu^{-1}\delta_\nu = O(1),$$

and now from (6.4), $|D''| = O(1)$, hence

$$\sum_n^{2n} |\delta_{\nu+1} - \delta_\nu| = O(1).$$

Also $\delta_{\nu+1} - \delta_\nu = (\nu+1)(a_{\nu+1} - a_\nu) + 2a_\nu$, thus

$$\sum_n^{2n} \nu |a_{\nu+1} - a_\nu| \leqq O(1) + 2\sum_n^{2n} |a_\nu|.$$

But from (6.2), $\sum_n^{2n} |a_\nu| = O(1)$, hence

$$\sum_n^{2n} |a_{\nu+1} - a_\nu| = O(n^{-1}).$$

Our theorem now follows from Theorem 6.

We next prove the following analogue to Lemma 1:

LEMMA 2. *Suppose that* $B_n \geqq 0$ *for* $n > n_0$, *that for some* $q > 0$

$$B_{n+1} \geqq (1 - q/n)B_n, \qquad n > n_0, \tag{6.5}$$

and that (A) $\lim B_n = B$; *then* $B_n \to B$.

We may assume that $q/n < 1$ for $n > n_0$; then from (6.5)

$$\sum_n^{n+\kappa} B_\nu \geqq B_n \sum_{\nu=0}^{\kappa} (1 - q/n)^\nu = \frac{nB_n}{q}\left\{1 - (1 - q/n)^{\kappa+1}\right\}, \qquad n > n_0,$$

hence

$$\begin{aligned} &B_n \leqq \frac{q}{n}(B'_{n+\kappa} - B'_{n-1})\left\{1 - (1 - q/n)^{\kappa+1}\right\}^{-1}, \quad \text{where} \\ &B'_n = \sum_1^n B_\nu. \end{aligned} \tag{6.6}$$

Choose $\kappa = [\delta n]$, where $\delta > 0$; from Abel summability and from $B_n \geqq 0$, it follows that $n^{-1}B_n' \to B$. Now from (6.6)

$$\limsup B_n \leqq \frac{q\delta B}{1 - \exp(-q\delta)};$$

letting $\delta \downarrow 0$, we get

$$\limsup B_n \leqq B.$$

Similarly for $n - \kappa > n_0$

$$\sum_{\nu=0}^{\kappa} B_{n-\nu} \leqq B_n \sum_{\nu=0}^{\kappa} \left(1 - \frac{q}{n-\kappa}\right)^{-\nu}$$
$$= \frac{n-\kappa}{q} B_n \left(1 - \frac{q}{n-\kappa}\right)\left\{\left(1 - \frac{q}{n-\kappa}\right)^{-(\kappa+1)} - 1\right\},$$

hence

$$B_n \geqq (q/(n - \kappa - q))(B_n' - B_{n-\kappa-1}')\{(1 - q/(n-\kappa))^{-(\kappa+1)} - 1\}^{-1}.$$

Let now $\kappa = [n\delta]$, $0 < \delta < 1$, then

$$\liminf B_n \geqq \frac{q\delta}{1-\delta} B \left(\exp \frac{q\delta}{1-\delta} - 1\right)^{-1},$$

and $\delta \downarrow 0$ yields $\liminf B_n \geqq B$. This proves the lemma.

THEOREM 9. *Suppose that for some constants* $p \geqq 0$, $q \geqq 0$,

$$(6.7) \qquad \begin{aligned} &(n+1)s_{n+1} - ns_n + p \\ &\qquad \geqq (1 - q/n)[ns_n - (n-1)s_{n-1} + p] \geqq 0, \end{aligned}$$

and that (A) $\lim s_n = s$ *exists. Then* $na_n \to 0$ *and* $\sum a_n \cos nt$ *is uniformly convergent.*

As in the proof of Theorem 8, $s_n \geqq -p$, hence $\sum a_n$ is $(C, 1)$ summable; then by Lemma 2, $\delta_n \to \delta$, $na_n \to 0$. Next from (6.7)

$$\delta_{n+1} - \delta_n \geqq -qn^{-1}\delta_n, \quad \text{and}$$

$$(6.8) \qquad D_n = \sum_{n}^{2n} (\delta_{\nu+1} - \delta_\nu) = \delta_{2n+1} - \delta_n = O(1).$$

Write $D_n = D' + D''$, where D' denotes the sum of negative terms, $D'' = D_n - D'$. From (6.8), $0 \geqq D' \geqq -q\sum_n^{2n} \nu^{-1}\delta_\nu$, hence $D' = O(1)$, and $D'' = O(1)$; hence $\sum_n^{2n} |\delta_{\nu+1} - \delta_\nu| = O(1)$. The remaining part is the same as in the proof of Theorem 8.

7. Closing remarks. The assumption of Lemma 1 can be written as $0 \leqq B_{n+1} \leqq (n+q)/nB_n$, or $0 \leqq (\Gamma(n+q)/\Gamma(n))B_{n+1} \leqq (\Gamma(n+q+1)/\Gamma(n+1))B_n$, that is, $\Gamma(n)B_n/\Gamma(n+q)$ is decreasing. A similar lemma was proved by Hardy; for reference see [2]. Again in Lemma 2 the assumption is $B_{n+1} \geqq (n-q)/nB_n \geqq 0$, or

$$(\Gamma(n-q)/\Gamma(n))B_{n+1} \geqq (\Gamma(n-q+1)/\Gamma(n+1))B_n \geqq 0,$$

that is, $\Gamma(n)B_n/\Gamma(n-q)$ is increasing. The larger q the more general is the condition.

The differences $(n+1)s_{n+1}-ns_n=\tau_{n+1}$ are the $(C, -1)$ means of the series $\sum a_n$, that is, $s_n=n^{-1}\sum_1^n \tau_\nu$ $(\tau_1=s_1)$. The condition (6.1) may be written as

$$-(\tau_n+p) \leqq \tau_{n+1}-\tau_n \leqq (q/n)(\tau_n+p).$$

If it holds for some p, then it clearly holds for any $p'>p$. Similarly (6.7) becomes

$$\tau_{n+1}-\tau_n \geqq -(q/n)(\tau_n+p) \geqq -(\tau_n+p),$$

and here too p may be replaced by any $p'>p$. Clearly summability $(C, -1)$ of the series $\sum a_n$ is equivalent to convergence together with $na_n \to 0$.

We have seen that the first inequality of (6.1) and Abel summability of $\sum a_n$ imply $(C, 2)$ summability of the sequence $\{\tau_n\}$; it follows from a theorem of Tauberian type that $\sum a_n$ converges. It is an open question whether this and $\tau_n \geqq -p$, $n=1, 2, 3, \cdots$, imply uniform convergence of $\sum a_n \cos nt$ at $t=0$. Theorem IV asserts that this is the case if $\sum a_n \cos nt$ is the Fourier series of a function continuous at $t=0$. However it is doubtful whether even $(C, -1)$ summability of $\sum a_n$ itself implies uniform convergence of $\sum a_n \cos nt$, or continuity of the corresponding function at $t=0$.

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A GENERALIZATION OF CONTINUED FRACTIONS[1]

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1. **Introduction.**[2] The generalizations and analogues of regular continued fractions due to Pierce [8], Lehmer [5], and Leighton [6] concern the iteration of rational functions to obtain rational approximations to a real number. The present generalization proceeds from the fact that the continued fraction

$$\frac{1}{a_1 + \cfrac{1}{a_2 + \cdots}} \tag{1.1}$$

can be written in the form

$$f(a_1 + f(a_2 + \cdots \tag{1.2}$$

where $f(t) = 1/t$. This suggests the possibility of using functions other than $1/t$ to obtain generalizations of (1.1). In §2 a class F of functions which includes $1/t$ is defined and in §3 meaning is given to (1.2) for each $f \in F$ and each sequence $a_1, a_2, a_3, \cdots$ of positive integers. An algorithm is given for obtaining for a fixed $f \in F$ an expression of the form (1.2) corresponding to each number x in the interval $0 < x < 1$; this expression is then called the *f-expansion of x*. The analogue of the nth convergent of a simple continued fraction is defined, and its behavior with respect to x is noted. In §4 the form (1.2) is called an *f-expansion* when $f \in F$ and $a_1, a_2, a_3, \cdots$ is a sequence of positive integers. The convergence and some idea of the rapidity of convergence of an f-expansion are established. The one-to-one correspondence between f-expansions and f-expansions of numbers x, $0 < x < 1$, is given in §5 by Theorem 5. In §6 statistical independence of the a_i of an f-expansion is defined in the customary way and a subclass F_p of F for which the a_i are statistically independent is considered. Various sets of numbers x whose f-expansions are restricted by conditions on the a_i are considered and the linear Lebesgue measures of these sets are given. In §7, when $f \in F_p$, certain sets of numbers x which have been studied for $f(t) = 1/t$ by Borel [2] and F. Bernstein [1] are shown to be of measure zero.

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[2] Numbers in brackets refer to the bibliography.

2. The class F. Let F denote the class of real functions $f(t)$ defined for $t\geqq 1$ and having the following properties:

$$f(1) = 1; \tag{2.1}$$

$$f(t_1) > f(t_2) > 0, \qquad 1 \leqq t_1 < t_2; \tag{2.2}$$

$$\lim_{t\to\infty} f(t) = 0; \tag{2.3}$$

$$|f(t_2) - f(t_1)| < |t_2 - t_1|, \qquad 1 \leqq t_1 < t_2; \tag{2.4}$$

there is a constant λ such that $0<\lambda<1$ and

$$|f(t_2) - f(t_1)| < \lambda^2 |t_2 - t_1|, \qquad 1 + f(2) < t_1 < t_2. \tag{2.5}$$

3. The f-expansions of numbers. Let $f(t)\in F$ and x be a fixed number, $0<x<1$. Let z_0 be defined by $x=f(z_0)$ and let the sequences $z_1, z_2, \cdots, \theta_1, \theta_2, \cdots$, and $a_1, a_2, \cdots$ be defined by the relations

$$a_n = [z_{n-1}], \qquad \theta_n = z_{n-1} - a_n, \qquad \theta_n = f(z_n), \tag{3.1}$$

for $n=1, 2, \cdots$. If $\theta_n\neq 0$ for $n<k$ while $\theta_k=0$, we shall say the expansion terminates and that the *f-expansion of x* is[3]

$$f(a_1 + f(a_2 + \cdots + f(a_k). \tag{3.2}$$

In this case it is easy to see that $a_k\geqq 2$ and that the f-expansion of x is equal to x. If $\theta_n\neq 0$ for $n=1, 2, \cdots$, then the expansion will not terminate and we shall call

$$f(a_1 + f(a_2 + \cdots \tag{3.3}$$

the *f-expansion of x.*

By analogy with simple continued fractions we define

$$x_n = f(a_1 + f(a_2 + \cdots + f(a_n) \tag{3.4}$$

and call the elements of the sequence $x_1, x_2, \cdots$ the *convergents* of x. The integers $a_1, a_2, \cdots$ and the convergents $x_1, x_2, \cdots$ are uniquely determined by x for almost all x, $0<x<1$. When we wish to emphasize this functional dependence we shall write them in the form $a_1(x), a_2(x), \cdots$ and $x_1(x), x_2(x), \cdots$.

To facilitate notation we introduce the function $\phi_n(t)$ defined when $f\in F$ and $a_1, a_2, \cdots$ is a sequence of positive integers by

$$\phi_n(t) = f(a_1 + f(a_2 + \cdots + f(a_n + t), \qquad t \geqq 0. \tag{3.5}$$

A simple induction proves the following lemma.

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[3] In (3.2) and similar expressions we shall use a single parenthesis on the right.

LEMMA 1. *The function* $\phi_n(t)$ *is a decreasing (increasing) function of* t *when* n *is odd (even).*

THEOREM 1. *If* $f \in F$ *and* $0 < x < 1$, *then the odd (even) convergents of the* f*-expansion of* x *form a decreasing (increasing) sequence bounded below (above) by* x; *thus*

$$(3.6) \qquad 0 < x_2 < x_4 < \cdots \leqq x \leqq \cdots < x_3 < x_1 \leqq 1.$$

When $\phi_n(t)$ is defined by (3.5), we have $x_n = \phi_n(0)$, $x = \phi_n(\theta_n)$, and $x_{n+1} = \phi_n(f(a_{n+1}))$. Since $f(a_{n+1}) \geqq \theta_n > 0$, we can apply Lemma 1 to obtain $x_n > x \geqq x_{n+1}$ when n is odd and $x_n < x \leqq x_{n+1}$ when n is even. Since $f(a_{n+1} + f(a_{n+2})) > 0$ and $x_{n+2} = \phi_n(f(a_{n+1} + f(a_{n+2})))$, we similarly have $x_n > x_{n+2}$ when n is odd and $x_n < x_{n+2}$ when n is even. These inequalities establish Theorem 1.

COROLLARY. *If* $\lim_{n\to\infty} x_n$ *exists, then* $\lim_{n\to\infty} x_n = x$.

4. Convergence of f-expansions.

If $f \in F$ we shall mean by an *f-expansion* either a finite expansion $f(a_1 + f(a_2 + \cdots + f(a_k)$ in which the a_i are positive integers and $a_k \geqq 2$, or an infinite expansion $f(a_1 + f(a_2 + \cdots$ in which the a_i are positive integers. It is to be proved later that each f-expansion is generated by a unique x; meanwhile this is not assumed.

THEOREM 2. *Let* $f \in F$. *If sequences* x_n *and* y_n *are defined in terms of an* f*-expansion by the formulas*

$$(4.1) \qquad x_n = f(a_1 + f(a_2 + \cdots + f(a_n),$$

$$(4.2) \qquad y_n = f(a_1 + f(a_2 + \cdots + f(a_n + 1),$$

then

$$(4.3) \qquad 0 < x_2 < x_4 < \cdots < x_3 < x_1 \leqq 1$$

and

$$(4.4) \qquad x_{n+1} \in I(a_1, a_2, \cdots, a_n),$$

where $I(a_1, a_2, \cdots, a_n)$ *is the closed interval with end points at* x_n *and* y_n.

Proof of (4.3) is identical with a part of the proof of (3.6). The conclusion (4.4) follows from Lemma 1 since $x_n = \phi_n(0)$,

$$x_{n+1} = \phi_n(f(a_{n+1})), \qquad y_n = \phi_n(1), \quad \text{and} \quad 0 < f(a_{n+1}) \leqq 1.$$

LEMMA 2.[4] *Let* $f \in F$. *For a fixed positive integer* n, *the least upper*

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[4] We use the symbol $|E|$ to denote the linear Lebesgue measure of a set E.

bound of $|I(a_1, a_2, \cdots, a_n)|$ *for all sequences of positive integers* a_i *is less than* λ^{n-2} *where* λ *is the constant in* (2.5); *that is, if* $f \in F$ *and*

$$(4.5)\quad A_n = \underset{a_1, \cdots, a_n \geqq 1}{\text{l.u.b.}} \left| f(a_1 + \cdots + f(a_n + 1) - f(a_1 + \cdots + f(a_n) \right|,$$

where $a_1, a_2, \cdots, a_n$ *assume independently all positive integral values, then*

$$(4.6)\qquad A_n \leqq \lambda^{n-2}, \qquad n = 1, 2, \cdots.$$

For $n \geqq 1$, we can write

$$\begin{aligned} A_{n+2} &= \underset{a_1, \cdots, a_{n+2} \geqq 1}{\text{l.u.b.}} \frac{|I(a_1, \cdots, a_{n+2})|}{|I(a_3, \cdots, a_{n+2})|} \cdot |I(a_3, \cdots, a_{n+2})| \\ &\leqq A_n \cdot \underset{a_1, a_2 \geqq 1;\, 0 < u < v \leqq 1}{\text{l.u.b.}} \left| \frac{f(a_1 + f(a_2 + u) - f(a_1 + f(a_2 + v)}{u - v} \right|, \end{aligned}$$

from which we obtain

$$(4.7)\quad \begin{aligned} A_{n+2} \leqq A_n \cdot \underset{a_1, a_2 \geqq 1;\, 0 < u < v \leqq 1}{\text{l.u.b.}} & \left| \frac{f(a_1 + f(a_2 + u) - f(a_1 + f(a_2 + v)}{[a_1 + f(a_2 + u)] - [a_1 + f(a_2 + v)]} \right| \\ & \cdot \left| \frac{f(a_2 + u) - f(a_2 + v)}{u - v} \right|. \end{aligned}$$

If $a_2 = 1$, then $a_1 + f(a_2 + u) > a_1 + f(a_2 + v) \geqq 1 + f(2)$ when a_1 is a positive integer and $0 < u < v \leqq 1$, so that by (2.5) and (2.4) the first and second factors of the product of which the least upper bound is taken in (4.7) are less than λ^2 and 1, respectively. If $a_2 \geqq 2 > 1 + f(2)$, then the first and second factors are less than 1 and λ^2, respectively. So we have $A_{n+2} \leqq \lambda^2 A_n$, $n = 1, 2, \cdots$. Since $A_2 \leqq A_1 < 1$, the statement (4.6) follows easily by mathematical induction.

THEOREM 3. *If* $f \in F$, *then each infinite f-expansion converges to a number* x *in the interval* $0 < x < 1$; *moreover*

$$(4.8)\qquad |x_n - x| \leqq \lambda^{n-2}, \qquad n = 1, 2, \cdots,$$

where λ *is the constant in* (2.5).

From Theorem 2 and Lemma 2 we conclude that $|x_{n+1} - x_n| \leqq \lambda^{n-2}$ for $n = 1, 2, \cdots$ and since $0 < \lambda < 1$, x_n converges to a number x which by (4.3) lies in each of the intervals from x_n to x_{n+1}. This proves (4.8).

THEOREM 4. *If* $f \in F$ *and* $0 < x < 1$, *the f-expansion of* x *converges to* x.

In the terminating case the f-expansion of x obviously equals x

and in this sense converges to x. In the non-terminating case the conclusion follows directly from Theorem 3 and the corollary to Theorem 1.

Henceforth we shall use the notation $x = f(a_1 + f(a_2 + \cdots$ to mean that the f-expansion on the right side converges to x.

When $f(t) = 1/t$, the least upper bound of $|f(x) - f(y)|/|x-y|$ for $3/2 < x < y$ is $(2/3)^2$, and so we may take $\lambda = 2/3$. It follows from (4.8) that

$$|x_n(x) - x| \leq (2/3)^{n-2}, \qquad n = 1, 2, \cdots. \tag{4.9}$$

From the theory of simple continued fractions we know [7, 4] that

$$|x_n(x) - x| \leq z^{n-1}, \qquad n = 1, 2, \cdots, \tag{4.10}$$

where $z = (3 - 5^{1/2})/2$. Comparison of (4.9) and (4.10) shows that our method of obtaining estimates of the rapidity of uniform convergence of f-expansions gives, when applied to $f(t) = 1/t$, an estimate which is similar in form to the stronger estimate of (4.10).

5. Uniqueness. In this section we establish a one-to-one correspondence between f-expansions and f-expansions of numbers x, $0 < x < 1$. We note, as in simple continued fractions [7, p. 22], the following lemma.

LEMMA 3. *If $f \in F$, then any two of the three equations*

$$x = f(a_1 + f(a_2 + \cdots, \tag{5.1}$$

$$y = f(a_n + f(a_{n+1} + \cdots, \tag{5.2}$$

$$x = f(a_1 + f(a_2 + \cdots + f(a_{n-1} + y) \tag{5.3}$$

implies the third, the f-expansions in (5.1) *and* (5.2) *being infinite.*

The proof of Lemma 3 is straightforward.

THEOREM 5. *If $f \in F$ and $0 < x < 1$, then an f-expansion which converges to x and the f-expansion of x are identical.*

If the two infinite f-expansions $f(a_1 + f(a_2 + \cdots$ and $f(b_1 + f(b_2 + \cdots$ converge to the same x, then by successively applying Lemma 3 we obtain $a_n = b_n$, $n = 1, 2, \cdots$. A similar argument proves that an infinite f-expansion and a finite f-expansion or two different finite f-expansions do not converge to the same x. Theorem 4 completes the proof.

6. Statistical independence. From (3.6) and (4.4) we see that $I(c_1, c_2, \cdots, c_i)$ except for at most its end points is identical with the set of x, $0<x<1$, for which $a_j(x)=c_j$, $j=1, 2, \cdots, i$. More exactly we have[5]

$$\begin{aligned} E[a_j(x) &= c_j; j = 1, 2, \cdots, i] \\ &= I(c_1, c_2, \cdots, c_i) - \{f(c_1 + f(c_2 + \cdots + f(c_i + 1)\} \end{aligned} \tag{6.1}$$

unless $i=1$ and $c_1=1$ in which case

$$E[a_1(x) = 1] = I(1) - \{f(1)\} - \{f(2)\}. \tag{6.2}$$

LEMMA 4. *If $f\in F$ and $c_1, c_2, \cdots, c_n$ and $c_1', c_2', \cdots, c_n'$ are two sets of positive integers such that for at least one j, $1\leq j\leq n$, $c_j\neq c_j'$, then the intervals $I(c_1, c_2, \cdots, c_n)$ and $I(c_1', c_2', \cdots, c_n')$ have at most an end point in common.*

The proof of this lemma follows from (6.1) and (6.2) and from the fact that the sets $E[a_j(x)=c_j;\ j=1, 2, \cdots, n]$ and $E[a_j(x)=c_j';\ j=1, 2, \cdots, n]$ are mutually exclusive by Theorem 5.

COROLLARY. *If $f\in F$, then*

$$I(c_1, c_2, \cdots, c_n) = \{f(c_1 + \cdots + f(c_n)\} + \sum_{j=1}^{\infty} I(c_1, c_2, \cdots, c_n, j).$$

If $y_1, y_2, \cdots$ is a decreasing sequence of positive numbers such that $y_1=1$ and $y_n\to 0$ and $f(t)$ is the function whose graph is the polygon joining in order the points (n, y_n), $n=1, 2, \cdots$, then $f(t)\in F$. Let F_p be the class of all such polygonal functions.

THEOREM 6. *If $f\in F_p$, then for any positive integers i and k*

$$| E[a_i(x) = k] | = f(k) - f(k + 1).$$

By (6.1) and (6.2) we have $|E[a_1(x)=k]| = |I(k)| = f(k)-f(k+1)$. For any positive integer m, it follows from (6.1) and Lemma 4 that $|E[a_{m+1}(x)=k]| = \sum |I(b_1, b_2, \cdots, b_m, k)|$ where $\sum$ is to be taken independently over all positive integral values of $b_1, b_2, \cdots, b_m$. By the mean value theorem we have

$$\begin{aligned} &| E[a_{m+1}(x) = k] | \\ &= \sum | f(b_1 + \cdots + f(k + 1) - f(b_1 + \cdots + f(k) | \\ &= \sum | f(b_1) - f(b_1 + 1) | \, | f(b_2 + \cdots + f(k + 1) - f(b_2 + \cdots + f(k) | \\ &= (\sum | f(b_1) - f(b_1 + 1) |) \cdot (\sum | I(b_2, \cdots, b_m, k) |) \\ &= \sum | I(b_2, \cdots, b_m, k) | = | E[a_m(x) = k] |. \end{aligned}$$

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[5] The symbol $E[\cdots]$ shall denote the set of x satisfying the proposition in brackets.

An induction completes the proof.

The functions $a_i(x)$, $i=1, 2, \cdots$, are said to be *statistically independent* [4] if for each set of positive integers $n_1<n_2<\cdots<n_m$ and each set of positive integers $c_1, c_2, \cdots, c_m$

$$(6.3)\quad \left| E[a_{n_j}(x) = c_j; j = 1, 2, \cdots, m]\right| = \prod_{j=1}^{m} \left| E[a_{n_j}(x) = c_j]\right|.$$

THEOREM 7. *If $f\in F_p$, then the functions $a_i(x)$, $i=1, 2, \cdots$, are statistically independent.*

The equation (6.3) is trivial for $m=1$. By (6.1) and Lemma 4 we have

$$\begin{aligned}\left| E[a_{n_j}(x) = c_j; j = 1, 2, \cdots, m]\right| \\ = \sum{}' \left| I(b_1, \cdots, b_{n_1-1}, c_1 b_{n_1+1}, \cdots, c_2, \cdots, c_m)\right|\end{aligned}$$

where $\sum'$ is to be taken independently over all positive integral values of b_i for all indices i from one to n_m excepting $i=n_1, n_2, \cdots, n_m$. By an argument similar to that used in the proof of Theorem 6 we obtain

$$\begin{aligned}&\left| E[a_{n_j}(x) = c_j; j = 1, 2, \cdots, m]\right| \\ &\quad = \sum{}' \left| f(b_1) - f(b_1+1)\right| \cdot \left| I(b_2, \cdots, b_{n_1-1}, c_1, \cdots, c_m)\right| \\ &\quad = \sum{}' \left| I(b_2, \cdots, b_{n_1-1}, c_1, \cdots, c_m)\right| = \cdots \\ &\quad = \sum{}' \left| I(c_1, b_{n_1+1}, \cdots, c_m)\right| \\ &\quad = \left| f(c_1) - f(c_1+1)\right| \cdot \left(\sum{}' \left| I(b_{n_1+1}, \cdots, c_m)\right|\right) \\ &\quad = \left| E[a_{n_1}(x) = c_1]\right| \cdot \left| E[a_{n_j}(x) = c_j; j = 2, \cdots, m]\right|\end{aligned}$$

and again an induction completes the proof.

COROLLARY. *If $f\in F_p$, then for each set of positive integers $n_1<n_2<\cdots<n_m$ and each set of positive integers $c_1, c_2, \cdots, c_m$, $d_1, d_2, \cdots, d_m$ such that $c_j\leqq d_j$, $j=1, 2, \cdots, m$, we have*

$$\begin{aligned}\left| E[c_j \leqq a_{n_j}(x) \leqq d_j; j = 1, 2, \cdots, m]\right| &= \prod_{j=1}^{m} \left| E[c_j \leqq a_{n_j}(x) \leqq d_j]\right| \\ &= \prod_{j=1}^{m} \left| f(c_j) - f(d_j+1)\right|.\end{aligned}$$

7. **Sets of measure zero.**[6] The results of §6 will now be used in order to prove a few measuretheoretical facts concerning f-expansions under the assumption that $f\in F_p$.

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[6] Theorems, similar to those in this section, applying to the simple continued fraction have been proved by Borel [2] and Bernstein [1]; for expositions see [3].

THEOREM 8. *If $f \in F_p$, then the set of x, $0<x<1$, for which the sequence $a_1(x), a_2(x), \cdots$ is bounded, has measure zero.*

Let the set $E[a_i(x) \leqq k; i=1, 2, \cdots, m]$ be denoted by G_k^m. In the corollary to Theorem 7 we set $n_j=j$, $c_j=1$, $d_j=k$ and obtain

$$|G_k^m| = \prod_{j=1}^{m} \{1 - f(k+1)\} = \{1 - f(k+1)\}^m.$$

If we let $G_k = E[a_i(x) \leqq k; i=1, 2, \cdots]$, then $G_k \in G_k^m$, $m=1, 2, \cdots$, and so $|G_k|=0$. The set of x, $0<x<1$, for which the sequence $a_1(x), a_2(x), \cdots$ is bounded is $G=\sum_{i=1}^{\infty} G_i$ and consequently $|G|=0$.

Similarly the set of x, $0<x<1$, for which $a_i(x)>k$, $i=1, 2, \cdots, m$, has measure $\{f(k+1)\}^m$. An argument similar to that used in the proof of Theorem 8 proves the following theorem.

THEOREM 9. *If $f \in F_p$, then the set of x, $0<x<1$, for which $a_i(x)>1$, $i=1, 2, \cdots$, has measure zero.*

THEOREM 10. *If $f \in F_p$ and $\phi(1), \phi(2), \cdots$ is a sequence of positive integers for which*

$$\sum_{n=1}^{\infty} f(\phi(n) + 1) \tag{7.1}$$

is divergent, then the set of x, $0<x<1$, for which $a_n(x) \leqq \phi(n)$, $n=1, 2, \cdots$, has measure zero.

Let $H_m = E[a_i(x) \leqq \phi(i); i=1, 2, \cdots, m]$. By an argument similar to the one used in proving Theorem 8 we have

$$|H_m| = \prod_{i=1}^{m} \{1 - f(\phi(i) + 1)\}. \tag{7.2}$$

Since $0<f(\phi(i)+1)<1$ for $i=1, 2, \cdots$, the divergence of the series (7.1) is equivalent to the limit as $m \to \infty$ of the product in (7.2) being zero. If we let $H=E[a_i(x) \leqq \phi(i); i=1, 2, \cdots]$, then since $H \in H_m$ for every positive integer m, it follows that $|H|=0$.

The last three theorems can be generalized to infinite subsequences of the sequence $a_1(x), a_2(x), \cdots$.

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CORNELL UNIVERSITY

COMMENT ON THE REAL LINE

PAUL R. HALMOS

Is it possible to topologize the additive group of the real line so that it becomes a *compact* topological group? It is the purpose of this note to establish the affirmative answer to this question. While the fact is not deep and the proof not difficult, they are presented here mainly to call attention to the almost completely unsolved problem of algebraic characterization of those abstract abelian groups which can be compact.

(1) If G is any additively written abelian group and n is any positive integer, consider the subgroups G_n' and G_n'' defined as follows. G_n' is the set of all x's in G for which $nx=0$; G_n'' is the set of all x's which have the form $x=ny$ for some y in G. These two subgroups are dual in the sense that if G is a locally compact abelian group and Γ is its character group, then the annihilator of G_n' is Γ_n'' and (consequently) the annihilator of G_n'' is Γ_n'. (The first statement implies that the annihilator of Γ_n'' is G_n'; the second statement follows from this by interchanging the roles of G and Γ.) To prove this let $\xi=\xi(x)$ be any element of Γ, and consider the identity $\xi(nx)=n\xi(x)$. Since $\xi(nx)$ vanishes identically if and only if ξ belongs to the annihilator of G_n'', and $n\xi(x)$ vanishes identically if and only if $\xi\in\Gamma_n'$, the asserted duality follows. It follows also that a locally compact abelian group G is divisible if and only if its character group Γ is torsion free, and therefore that G has both properties if and only if Γ has.

(2) In a torsion free abelian group the division of an element by an integer is unique (if at all possible). Therefore a torsion free group in which division is always possible admits the additive group of rationals as operators. Consequently such a group is a vector space over the rationals and its algebraic structure is uniquely characterized by its vector dimension, that is by the cardinal number α of elements in a Hamel basis. For the additive group of the real line α is the power of the continuum. The assertion of this note can be proved therefore by exhibiting a compact group G which is both torsion free and divisible and in which there is a Hamel basis having the power of the continuum.

(3) Let Γ be the discrete group of rationals and G its compact character group. It follows from (1) that G is divisible and torsion free. The elements $x\in G$ may be thought of as functions $x(\xi)$ from

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Γ to the unit interval modulo 1. If the elements $x_1, \cdots, x_n$ are rationally dependent then there exist also *integers* $k_1, \cdots, k_n$ such that $k_1x_1(\xi)+\cdots+k_nx_n(\xi)\equiv 0 \pmod 1$ for all $\xi\in\Gamma$ and therefore in particular $k_1x_1(1)+\cdots+k_nx_n(1)\equiv 0 \pmod 1$. Since the mapping $x\rightarrow x(1)$ is an *onto* homomorphism from G to the unit interval modulo 1, the existence in G of a Hamel basis with power smaller than that of the continuum leads to a contradiction. For then every subset of G with the power of the continuum would contain a rationally dependent finite set and consequently every subset of the unit interval modulo 1 with the power of the continuum would contain a finite set which is integrally dependent modulo 1. Since this last possibility is absurd, and since the power of G itself is that of the continuum, it follows that a Hamel basis in G has exactly the power of the continuum. The affirmative answer to the question of this note is thus completely established.

SYRACUSE UNIVERSITY

ON CERTAIN SYSTEMS WHICH ARE ALMOST GROUPS

HENRY B. MANN

We consider systems $\mathfrak{S}$ of elements $A, B, \cdots$ satisfying the following postulates.

I. To every pair of elements A, B in $\mathfrak{S}$ there exists a uniquely determined product $A \cdot B$ in $\mathfrak{S}$.

II. $(AB)C = A(BC)$.

III. There exists in $\mathfrak{S}$ *an* element E such that $EA = A$ for every A.

IV. To every A in $\mathfrak{S}$ there exists *an* A' in $\mathfrak{S}$ such that $AA' = E$.

The system $\mathfrak{S}$ differs from a group only in that it contains a left unit and right inverse instead of a right unit and right inverse. We shall call such a system a left right system, abbreviated (l, r) system.

Definition 1. *An element F is called an idempotent if $F^2 = F$.*

Proposition 1. *If F is idempotent then $FE = E$.*

Proof. From $FF = F$ we have $FFF^{-1} = FF^{-1}$, hence $FE = E$.

Proposition 2. *If $FE = E$ then F is idempotent.*

For $FF = FEF = EF = F$.

Proposition 3. *The idempotents of a (l, r) system form a (l, r) system.*

Proof. If F and F' are idempotent then by Proposition 1

$$FF'E = FE = E,$$

hence FF' is idempotent and postulate I is fulfilled. Postulate II is fulfilled since it is fulfilled in $\mathfrak{S}$. E itself is idempotent and is by Proposition 1 a right inverse to every idempotent. Hence all the postulates of a (l, r) system are fulfilled.

Proposition 4. *If F is idempotent and A is any element then $FA = A$.*

For $FA = FEA = EA = A$.

Definition 2. *A (l, r) system which contains only idempotents will be called an idempotent (l, r) system.*

From Proposition 4 we see that in every idempotent (l, r) system every product is equal to its last factor. On the other hand it is easy to verify that every system of elements in which every product is

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equal to its last factor is an idempotent (l, r) system since every element of such a system can be considered the unit element and is then inverse to all elements. From the foregoing we have the following theorem.

THEOREM 1. *There exists one and only one idempotent* (l, r) *system of order* n $(n=1, 2, \cdots$ ad inf.). *It consists of the elements* $A_1, A_2, \cdots, A_n$ *with the law of composition* $A_iA_j=A_j$.

Remark. Similarly one and only one idempotent (l, r) system exists for every cardinal number.

PROPOSITION 5. *The elements of any* (l, r) *system which are of the form* AF *where* F *is a fixed idempotent form a group.*

PROOF. $(AF)(BF)=A(FB)F=ABF$ by Proposition 4. The composition is by hypothesis associative, F is a right unit and $A^{-1}F$ a right inverse. Since the system forms a group we must have $A^{-1}FAF=F$.

Let $\mathfrak{J}=(E, F_2, \cdots)$ be an idempotent (l, r) system, $\mathfrak{G}=(\overline{E}, A_2, \cdots)$ a group. We form a system $\mathfrak{S}$ in the following way: $\mathfrak{S}$ consists of all pairs A_iF_j with the following rule of combination:

$$A_iF_jA_kF_l = A_iA_kF_l.$$

It is easy to verify that $\mathfrak{S}$ is a (l, r) system.

DEFINITION 3. *The system* $\mathfrak{S}$ *defined in the preceding paragraph is called the direct product of* $\mathfrak{J}$ *and* $\mathfrak{G}$.

PROPOSITION 6. *Every element of a* (l, r) *system may be written in the form* $(AE)F$ *where* F *is idempotent.*

PROOF. We have $(A^{-1}A)(A^{-1}A)=A^{-1}(AA^{-1})A=A^{-1}(EA)=A^{-1}A$. Hence $A^{-1}A$ is idempotent. But $A=AA^{-1}A=A(EA^{-1}A)=AE(A^{-1}A)$.

PROPOSITION 7. *If* $AEF=BEF'$ *where* F *and* F' *are idempotents then* $AE=BE$ *and* $F=F'$.

PROOF. From $AEF=BEF'$ we have $AEFE=BEF'E$. Hence on account of Proposition 1, $AE=BE$. Since the elements of the form AE form a group, $A^{-1}E$ is a left inverse of AE and $F=F'$ follows.

From Propositions 5, 6 and 7 we have the following theorem.

THEOREM 2. *Every* (l, r) *system is the direct product of an idempotent* (l, r) *system and a group.*

PROPOSITION 8. *Every idempotent* F *of a* (l, r) *system* $\mathfrak{S}$ *is a unit element of* $\mathfrak{S}$.

For by Proposition 4, $FA = A$ for every A and $A(A^{-1}F) = EF = F$.

Let $S_1, \cdots, S_n$ be n statements. Let A_i be the statement: "All the preceding statements are annulled but S_i is true." It is interesting to note that the statements A_i form an idempotent (l, r) system.

OHIO STATE UNIVERSITY

A CONJECTURE IN ELEMENTARY NUMBER THEORY

LEON ALAOGLU AND PAUL ERDÖS

A well known conjecture of Catalan states that *if* $f(n)$ *is the sum of all divisors of* n *except* n, *then the sequence of iterates of* $f(n)$ *is either eventually periodic or ends at* 1. It not only seems impossible to prove this, but it is also very difficult to verify.[1]

Another conjecture of Poulet,[2] which appears equally difficult to prove, has the doubtful merit that it is easy to verify. Let $\sigma(n)$ be the sum of all divisors of n, and let $\phi(n)$ be Euler's function. Then *for any integer* n *the sequence*

$$f_0(n) = n, \qquad f_{2k+1}(n) = \sigma(f_{2k}(n)), \qquad f_{2k}(n) = \phi(f_{2k-1}(n))$$

is eventually periodic.

We have verified this conjecture to $n = 10000$ (extending Poulet's verification) by using Glaisher's tables.[3] The checking was facilitated by the following observation: if the conjecture is to be checked for all $n < x$, it is enough to find a member of the sequence other than the first which is less than x.

The longest cycle found was in the sequence $f_k(9216)$. It starts with $f_6(9216)$, and is: 34560, 122640, 27648, 81800, 30976, 67963, 54432, 183456, 48384, 163520, 55296, 163800, 34560. However our method of checking does not show that this is the largest cycle up to 10000, and in fact Poulet found that $f_k(1800)$ has the same length 12.

As a rule $\phi(\sigma(n))$ is less than n. In fact, it can be shown that *for every* $\epsilon > 0$, $\phi(\sigma(n)) < \epsilon n$, *except for a set of density* 0. The proof follows from the following two observations:

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[1] L. E. Dickson, *Theorems and tables on the sum of the divisors of a number*, Quart. J. Math. vol. 44 (1913) pp. 264–296, and P. Poulet, *La Chasse aux Nombres*, vol. 1, pp. 68–72, and vol. 2, p. 188.

[2] P. Poulet, *Nouvelles suites arithmétiques*, Sphinx vol. 2 (1932) pp. 53–54.

[3] J. W. L. Glaisher, *Number-divisor tables*, British Association for the Advancement of Science, Mathematical Tables, vol. 8.

(1) *For a given prime p, the set of all n such that $\sigma(n) \equiv 0 \pmod{p}$ is of density* 1.

The set of all integers not divisible by any prime q of the form $px-1$ is of density zero, since $\sum_q 1/q$ diverges. Hence the set of all integers divisible by a prime $q > N$ of this type is of density 1. But the set of all integers divisible by q^2, $q > N$, is of density less than $\sum_{q>N} 1/q^2 = o(1)$. Therefore, if x is large, the number of n less than x such that $\sigma(n) \equiv 0 \pmod{p}$ exceeds $(1-\epsilon)x$.

(2) *Except for ϵx integers n less than x, $\sigma(n) < c(\epsilon)n$.*

This follows from the fact that $\sum_{n<x} \sigma(n) \sim \pi^2 n^2/12$.

Choose p so that $\prod_{q \leqq p} (1-1/q) < \delta/c(\epsilon)$. Then, if x is sufficiently large, all except $\eta x + \epsilon x$ integers n less than x have $\sigma(n) < c(\epsilon)n$, $\sigma(n) \equiv 0 \pmod{q}$ for all $q \leqq p$. But, with these exceptions, $\phi[\sigma(n)] < \delta n$, which completes the proof, since η and ϵ are arbitrary.

In much the same way it can be shown that *for every $c > 0$, $\sigma[\phi(n)] > cn$ except for a set of density zero.*

Actually, much more can be shown. *Except for a set of density zero, $e^{\gamma}\phi[\sigma(n)] \log \log \log n \sim \sigma(n)$, and $e^{-\gamma}\sigma[\phi(n)]/\log \log \log n \sim \phi(n)$,* where γ is Euler's constant. The proof is suppressed, but it might be noted that the reason for this result is that, for almost all n, $\phi(n)$ and $\sigma(n)$ are both divisible by all primes less than $(\log \log n)^{1-\epsilon}$, and by relatively few primes greater than $(\log \log n)^{1+\epsilon}$.

There exist numbers for which $\phi(\sigma(n)) = n$. Up to 2500 these numbers are 1, 2, 8, 12, 128, 240, 720; while two further solutions are 2^{15} and 2^{31}. Poulet gives many others; we do not know whether there are infinitely many solutions.

We state two further conjectures:

(a) Form the sequence $\sigma(n)$, $\sigma(\sigma(n))$, $\phi(\sigma(\sigma(n)))$, $\sigma(\phi(\sigma(\sigma(n))))$ in which the functions are successively applied in the order $\sigma, \sigma, \phi, \sigma, \sigma, \phi, \sigma, \sigma, \phi, \cdots$. This sequence seems to tend to infinity if n is large enough.

(b) On the other hand, the sequence $\phi(n), \phi(\phi(n)), \sigma(\phi(\phi(n))), \cdots$, in which the order is $\phi, \phi, \sigma, \phi, \phi, \sigma, \phi, \phi, \sigma, \cdots$, seems to converge to 1, for all n.

Obviously many more such conjectures can be formulated.

PURDUE UNIVERSITY

CONGRUENCES INVOLVING THE PARTITION FUNCTION $p(n)$

WILLIAM H. SIMONS

1. **Introduction.** The purpose of this note is to give certain congruence properties of $p(n)$, the partition function, for moduli 13 and 17, analogous to those obtained by Ramanujan for moduli 5, 7 and 11. The method and notation employed are essentially those of Ramanujan in [1].[1] Let

$$P = P(x) = 1 - 24\sum_{n=1}^{\infty} \frac{nx^n}{1 - x^n},$$

$$Q = Q(x) = 1 + 240\sum_{n=1}^{\infty} \frac{n^3x^n}{1 - x^n},$$

$$R = R(x) = 1 - 504\sum_{n=1}^{\infty} \frac{n^5x^n}{1 - x^n},$$

$$f(x) = \prod_{i=1}^{\infty} (1 - x^i).$$

Then it is known that

$$\begin{aligned} f(x) &= 1 - x - x^2 + x^5 + x^7 - \cdots \\ &= \sum_{n=-\infty}^{+\infty} (-1)^n x^{n(3n-1)/2}, \end{aligned} \tag{1}$$

$$Q^3 - R^2 = 1728x[f(x)]^{24}. \tag{2}$$

Furthermore, let $\Phi_{r,s}(x) = \sum_{n=1}^{\infty} n^r\sigma_{s-r}(n)x^n$, where $\sigma_k(n)$ denotes the sum of the kth powers of the divisors of n. In particular

$$\Phi_{0,s}(x) = \sum_{n=1}^{\infty} \frac{n^s x^n}{1 - x^n} = \sum_{n=1}^{\infty} \sigma_s(n)x^n,$$

so that

$$\begin{aligned} P &= 1 - 24\Phi_{0,1}(x), \\ Q &= 1 + 240\Phi_{0,3}(x), \\ R &= 1 - 504\Phi_{0,5}(x). \end{aligned}$$

Then in terms of the functions $\sum_{r,s}(n)$, defined by

$$\sum\nolimits_{r,s}(n) = \sum_{m=0}^{n} \sigma_r(m)\sigma_s(n - m),$$

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[1] Numbers in brackets refer to the Bibliography at the end of the paper.

the main result for modulus 13 is given by the congruence (10) of §2,

$$(10)\qquad p(13n-7)-p(13n-176)-\cdots\equiv -2\Sigma_{5,5}(13n)\pmod{13},$$

and for modulus 17 by the congruence (14) of §3,

$$(14)\qquad p(17n-12)-p(17n-301)-\cdots\equiv \Sigma_{5,9}(17n)\pmod{17}.$$

Another interesting congruence for modulus 13 is found in (12) of §2, namely

$$(12)\qquad p(13n-7)\equiv -2\sum_{m=1}^{n}\tau(m)p\left(\frac{n-m}{13}\right)\pmod{13},$$

where $\tau(m)$ is the Ramanujan function defined by

$$(3)\qquad \sum_{m=1}^{\infty}\tau(m)x^{m}=x[f(x)]^{24},$$

that is, $\tau(m)$ is the coefficient of x^m in the expansion of $x[f(x)]^{24}$.

2. **Congruence properties, modulus 13.** We let I denote a power series in x with integral coefficients. Now $\Phi_{r,s}(x)=\mathfrak{J}^r\Phi_{0,s-r}(x)$, where $\mathfrak{J}=xd/dx$. Also $\mathfrak{J}P=-24\Phi_{1,2}(x)$, $\mathfrak{J}Q=240\Phi_{1,4}(x)$, and $\mathfrak{J}R=-504\Phi_{1,6}(x)$. Then from Tables I, II and III of [2] we may form the following tables for expressing $\Phi_{r,s}(x)$ in terms of P, Q, and R. Each entry has been reduced modulo 13 so that to each must be added $13I$. Use has also been made of the relations

$$(4)\qquad Q^3=3R^2-2+13I,$$

$$(5)\qquad Q^2R=P+13I.$$

TABLE I

$$\begin{aligned}
\Phi_{0,1}(x)&=-6P+6\\
\Phi_{0,3}(x)&=-2Q+2\\
\Phi_{0,5}(x)&=-4R+4\\
\Phi_{0,7}(x)&=-Q^2+1\\
\Phi_{0,9}(x)&=3QR-3\\
\Phi_{1,2}(x)&=6P^2-6Q\\
\Phi_{1,4}(x)&=5R-5PQ\\
\Phi_{1,6}(x)&=-2PR+2Q^2\\
\Phi_{1,8}(x)&=-5PQ^2+5QR\\
\Phi_{1,10}(x)&=-4PQR+R^2+3
\end{aligned}$$

TABLE II

$$\Phi_{2,3}(x) = P^3 - 3PQ + 2R$$
$$\Phi_{2,5}(x) = -P^2Q + 2PR - Q^2$$
$$\Phi_{2,7}(x) = P^2R - 2PQ^2 + QR$$
$$\Phi_{2,9}(x) = 6P^2Q^2 + PQR + 4R^2 + 2$$
$$\Phi_{2,11}(x) = 5P^2QR + 4P + 4PR^2$$
$$\Phi_{3,4}(x) = -3P^4 + 5P^2Q + 2PR - 4Q^2$$
$$\Phi_{3,6}(x) = 6P^3Q - 5P^2R + 5PQ^2 - 6QR$$
$$\Phi_{3,8}(x) = 5P^3R - 2P^2Q^2 + 2PQR + 3R^2 + 5$$
$$\Phi_{3,10}(x) = 5P^3Q^2 - 2P^2QR - 3PR^2$$
$$\Phi_{3,12}(x) = 5P^3QR - 6P^2 + 6P^2R^2 + 4QR^2 + 4Q$$
$$\Phi_{4,5}(x) = -P^5 - 3P^3Q + 6P^2R + 2PQ^2 - 4QR$$
$$\Phi_{4,7}(x) = -3P^4Q - P^3R - 5P^2Q^2 - PQR - 3$$
$$\Phi_{4,9}(x) = -6P^4R - 2P^3Q^2 + 3P^2QR - 4PR^2 - 4P$$
$$\Phi_{4,11}(x) = -3P^4Q^2 - P^3QR + P^2R^2 + 6P^2 - 3QR^2$$
$$\Phi_{5,6}(x) = 5P^6 + 3P^4Q + 5P^3R - 4P^2Q^2 + 3PQR - 4R^2 + 5$$

Now let $\Delta = (Q^3 - R^2)$. Then by the use of (4) and (5) we may express Δ^7 in terms of P, Q, and R as follows:

$$\Delta^7 = -5P^6 - 2P^4Q + 6P^3R - 6P^2Q^2$$
$$-6PQR - 2R^2 + 2 + 13I.$$

Then, making use of Tables I and II,

$$\Delta^7 = -\Phi_{5,6}(x) + 4\Phi_{4,7}(x) + 3\Phi_{3,8}(x)$$
$$-6\Phi_{2,9}(x) + 3\Phi_{1,10}(x) - 3\Phi_{0,5}(x)$$
$$+2[\Phi_{0,5}(x)]^2 + 13I.$$

But since

$$\Phi_{0,5}(x) = \sum_{n=1}^{\infty} \sigma_5(n)x^n,$$

we have

$$[\Phi_{0,5}(x)]^2 = \left[\sum_{n=1}^{\infty} \sigma_5(n)x^n\right]\left[\sum_{m=1}^{\infty} \sigma_5(m)x^m\right]$$
$$= \sum_{n=2}^{\infty}\sum_{m=1}^{n-1} \sigma_5(m)\sigma_5(n-m)x^n,$$

and then

$$
\begin{aligned}
\Delta^7 = &- \Phi_{5,6}(x) + 4\Phi_{4,7}(x) + 3\Phi_{3,8}(x) \\
&- 6\Phi_{2,9}(x) + 3\Phi_{1,10}(x) - 3\Phi_{0,5}(x) \\
&+ 2\sum_{n=2}^{\infty} (\Sigma'_{5,5}(n))x^n + 13I,
\end{aligned}
\tag{6}
$$

where[2]

$$\Sigma'_{5,5}(n) = \sum_{m=1}^{n-1} \sigma_5(m)\sigma_5(n-m).$$

Instead of equation (6) we may obtain a slightly different form for Δ^7 by introducing Ramanujan's function $\tau(n)$, defined by equation (3). Then

$$Q^3 - R^2 = (3R^2 - 2 - R^2) + 13I = 2(R^2 - 1) + 13I$$

and

$$
\begin{aligned}
6(R^2 - 1) &= -3x[f(x)]^{24} + 13I \\
&= -3\sum_{n=1}^{\infty} \tau(n)x^n + 13I,
\end{aligned}
$$

so that

$$
\begin{aligned}
\Delta^7 = &- \Phi_{5,5}(x) + 4\Phi_{4,7}(x) + 3\Phi_{3,8}(x) \\
&- 6\Phi_{2,9}(x) + 3\Phi_{1,10}(x) \\
&- 3\sum_{n=1}^{\infty} \tau(n)x^n + 13I.
\end{aligned}
\tag{7}
$$

Now, by (2),

$$
\begin{aligned}
\Delta^7 = (Q^3 - R^2)^7 &= \{1728x[f(x)]^{24}\}^7 \\
&= -x^7f(x^{169})/f(x) + 13I.
\end{aligned}
$$

But by (1),

$$
\begin{aligned}
\frac{x^7f(x^{169})}{f(x)} &= x^7\left\{\sum_{n=-\infty}^{+\infty} (-1)^n x^{169n(3n-1)/2}\right\}\left\{\sum_{m=0}^{\infty} p(m)x^m\right\} \\
&= \sum_{n=7}^{\infty} \{p(n-7) - p(n-176) - p(n-345) + \cdots\}x^n,
\end{aligned}
$$

where the numbers 7, 176, 345, $\cdots$, are alternately of the form $(13m-2)(39m-7)/2$ and $(13m+2)(39m+7)/2$. We therefore have

---

[2] Ramanujan defines $\Sigma_{r,s}(n) = \sum_{m=0}^{n} \sigma_r(m)\sigma_s(n-m)$ so that $\Sigma_{5,5}(n) = \Sigma'_{5,5}(n) + 2\sigma_5(0)\sigma_5(n)$, $\sigma_5(0) = \zeta(-5)/2 = -1/504$.

$$\Delta^7 = -\sum_{n=7}^{\infty}\{p(n-7)-p(n-176)-p(n-345)+\cdots\}x^n+13I. \tag{8}$$

Comparing coefficients of x^n in (6) and (8) we obtain

$$\begin{aligned} &p(n-7)-p(n-176)-p(n-345)+\cdots \\ &\qquad \equiv + n^5\sigma_1(n) - 4n^4\sigma_3(n) - 3n^3\sigma_5(n) \\ &\qquad\quad + 6n^2\sigma_7(n) - 3n\sigma_9(n) + 3\sigma_5(n) \\ &\qquad\quad - 2\Sigma'_{5,5}(n) \pmod{13}. \end{aligned} \tag{9}$$

If we replace n by $13n$ we get

$$\begin{aligned} &p(13n-7)-p(13n-176)-\cdots \\ &\qquad \equiv 3\sigma_5(13n) - 2\Sigma'_{5,5}(13n) \pmod{13} \\ &\qquad \equiv -2\Sigma_{5,5}(13n) \pmod{13}. \end{aligned} \tag{10}$$

In a similar fashion, from (7) and (8) we obtain

$$p(13n-7)-p(13n-176)-\cdots \equiv 3\tau(13n) \pmod{13}. \tag{11}$$

We next wish to compare this result with that obtained by Zuckerman [3]. For this we rewrite (11) as follows

$$\sum_{n=1}^{\infty} p(13n-7)x^{13n}\sum_{n=-\infty}^{+\infty}(-1)^n x^{169n(3n+1)/2} = 3\sum_{n=1}^{\infty}\tau(13n)x^{13n} + 13I.$$

Therefore

$$\begin{aligned} \sum_{n=1}^{\infty} p(13n-7)x^{13n} &= 3\sum_{n=1}^{\infty}\tau(13n)x^{13n}\frac{1}{f(x^{169})} + 13I \\ &= 3\sum_{n=1}^{\infty}\tau(13n)x^{13n}\sum_{n=0}^{\infty} p\left(\frac{n}{13}\right)x^{13n} + 13I, \end{aligned}$$

$$p(13n-7) \equiv 3\sum_{m=1}^{n}\tau(13m)p\left(\frac{n-m}{13}\right) \pmod{13}.$$

But since

$$\tau(sp^\lambda) = \tau(p)\tau(sp^{\lambda-1}) - p^{11}\tau(sp^{\lambda-2}), \qquad (s,p)=1,$$

we have

$$\begin{aligned} \tau(13m) &= \tau(13^\alpha m'), \ (m',13)=1, \\ &\equiv \tau(13)\tau(13^{\alpha-1}m') \pmod{13} \\ &\equiv -5\tau(m) \pmod{13}, \end{aligned}$$

and so

$$(12)\qquad p(13n-7) \equiv -2\sum_{m=1}^{n} \tau(m)p\left(\frac{n-m}{13}\right) \pmod{13}.$$

Zuckerman's result gives

$$\sum_{n=0}^{\infty} p(13n+6)x^n = -2\frac{f(x^{13})}{f(x)} + 13I$$

from which equation (11) may also be obtained.

3. **Congruence properties, modulus 17.** Let I again denote a power series in x with integral coefficients. As before we express each $\Phi_{r,s}(x)$ in terms of P, Q and R. To each entry must be added $17I$.

TABLE III

$\Phi_{0,1}(x) = -5P + 5$

$\Phi_{0,3}(x) = -8Q + 8$

$\Phi_{0,5}(x) = 3R - 3$

$\Phi_{0,7}(x) = -4Q^2 + 4$

$\Phi_{0,9}(x) = -2QR + 2$

$\Phi_{0,11}(x) = 8Q^3 + 6R^2 + 3$

$\Phi_{0,13}(x) = -5Q^2R + 5$

$\Phi_{1,2}(x) = P^2 - Q$

$\Phi_{1,4}(x) = 3PQ - 3R$

$\Phi_{1,6}(x) = -7PR + 7Q^2$

$\Phi_{1,8}(x) = 3PQ^2 - 3QR$

$\Phi_{1,10}(x) = 4PQR + Q^3 - 5R^2$

$\Phi_{1,12}(x) = 8PQ^3 + 6PR^2 + 3Q^2R$

$\Phi_{1,14}(x) = -3PQ^2R + 8QR^2 - 5$

$\Phi_{2,3}(x) = 3P^3 + 8PQ + 6R$

$\Phi_{2,5}(x) = -3P^2Q + 6PR - 3Q^2$

$\Phi_{2,7}(x) = 3P^2R - 6PQ^2 + 3QR$

$\Phi_{2,9}(x) = -2P^2Q^2 + 4PQR - 3Q^3 + R^2$

$\Phi_{2,11}(x) = -2P^2QR - PQ^3 + 5PR^2 - 2Q^2R$

$\Phi_{2,13}(x) = 3P^2Q^3 - 2P^2R^2 - 2PQ^2R + 8QR^2 - 7$

$\Phi_{2,15}(x) = -8P^2Q^2R + 3PQR^2 + 5P$

$$\Phi_{3,4}(x) = 5P^4 + 4P^2Q + 6PR + 2Q^2$$
$$\Phi_{3,6}(x) = 7P^3Q - 4P^2R + 4PQ^2 - 7QR$$
$$\Phi_{3,8}(x) = 2P^3R - 6P^2Q^2 + 6PQR - Q^3 - R^2$$
$$\Phi_{3,10}(x) = 4P^3Q^2 + 5P^2QR + PQ^3 - 6PR^2 - 4Q^2R$$
$$\Phi_{3,12}(x) = -2P^3QR + 7P^2Q^3 - P^2R^2 - 6PQ^2R + 7QR^2 - 5$$
$$\Phi_{3,14}(x) = -5P^3Q^3 - 8P^3R^2 + P^2Q^2R + 7P - 6PQR^2 - 6Q^3R$$
$$\Phi_{3,16}(x) = -5P^3Q^2R + 6P^2QR^2 + 8P^2 + 4PQ^3R - PR^3 + Q + 4Q^2R^2$$

$$\Phi_{4,5}(x) = -4P^5 + 6P^3Q + 5P^2R - 8PQ^2 + QR$$
$$\Phi_{4,7}(x) = -3P^4Q - 5P^3R - P^2Q^2 - 5PQR + 6Q^3 + 8R^2$$
$$\Phi_{4,9}(x) = -7P^4R - 6P^3Q^2 - 8P^2QR - 3PQ^3 - 3PR^2 - 7Q^2R$$
$$\Phi_{4,11}(x) = -2P^4Q^2 + 8P^3QR - P^2Q^3 + 6P^2R^2 + 8PQ^2R - QR^2 - 1$$

$$\Phi_{5,6}(x) = 4P^6 + 8P^4Q + 7P^3R + 7P^2Q^2 - 6PQR + 3Q^3 - 6R^2$$
$$\Phi_{5,8}(x) = -2P^5Q - 7P^4R - 3P^3Q^2 + 3P^2QR + 3PQ^3 + 4PR^2 + 2Q^2R$$
$$\begin{aligned}\Phi_{5,10}(x) = {} & -3P^5R - 2P^4Q^2 + 4P^3QR - 2P^2Q^3 - 2P^2R^2 + 2PQ^2R \\ & + 2QR^2 + 1\end{aligned}$$

$$\begin{aligned}\Phi_{6,7}(x) = {} & 2P^7 - 8P^5Q + 4P^4R - 6P^3Q^2 - 2P^2QR + 2PQ^3 - 4PR^2 \\ & - 5Q^2R\end{aligned}$$
$$\begin{aligned}\Phi_{6,9}(x) = {} & 7P^6Q - 8P^5R + 3P^4Q^2 - 4P^3QR - 6P^2Q^3 - 8P^2R^2 \\ & - 8PQ^2R - 4QR^2 - 6\end{aligned}$$
$$\begin{aligned}\Phi_{7,8}(x) = {} & 4P^8 + 7P^6Q + 6P^5R - 7P^4Q^2 - 5P^3QR - P^2Q^3 + 2P^2R^2 \\ & + 5PQ^2R + 5QR^2 + 1\end{aligned}$$

In the reduction of the powers of Q and R in the expansion of $(Q^3-R^2)^{12}$ we have made use of the relations

$$Q^4 = 3QR^2 - 2 + 17I, \qquad R^3 = 4Q^3R - 3P + 17I,$$

together with combinations of these to form

$$\begin{aligned} Q^4R &= 7PQ - 6R + 17I, \\ QR^3 &= 8PQ - 7R + 17I, \\ Q^8R &= 4PQ^2R^2 - 5PQ + 2R + 17I, \\ Q^5R^4 &= 5P^2Q^2 + 5PQR + 8R^2 + 17I, \\ QR^6 &= 6Q^3R^2 - 4P^2Q - PR + 17I. \end{aligned}$$

We obtain

$$\begin{aligned}\Delta^{12} = (Q^3 - R^2)^{12} = &- P^8 - 4P^6Q + 6P^5R + 3P^4Q^2\\ &+ 4P^3QR + 6P^2Q^3 - 3P^2R^2 + 2PQ^2R\\ &+ 6QR^2 - 2 + 17I.\end{aligned}$$

Making use of the relations in Table III we have

$$\begin{aligned}\Delta^{12} = &4\Phi_{7,8}(x) - 7\Phi_{6,9}(x) + 2\Phi_{5,10}(x) + 6\Phi_{4,11}(x)\\ &- 4\Phi_{3,12}(x) + 2\Phi_{2,13}(x) - 8\Phi_{1,14}(x) + 3P^2R^2\\ &+ 2QR^2 - 5 + 17I.\end{aligned}$$

But

$$\begin{aligned}3P^2R^2 + 2QR^2 - 5 = &6\Phi_{2,7}(x)\Phi_{0,5}(x) + 6\Phi_{0,9}(x)\Phi_{0,5}(x) + \Phi_{2,7}(x)\\ &- 2\Phi_{1,14}(x) + \Phi_{0,9}(x) + 5\Phi_{0,5}(x) + 17I.\end{aligned}$$

Therefore

$$\begin{aligned}\Delta^{12} = &4\Phi_{7,8}(x) - 7\Phi_{6,9}(x) + 2\Phi_{5,10}(x) + 6\Phi_{4,11}(x) - 4\Phi_{3,12}(x)\\ &+ 2\Phi_{2,13}(x) + 7\Phi_{1,14}(x) + \Phi_{2,7}(x) + \Phi_{0,9}(x) + 5\Phi_{0,5}(x)\\ &+ 6\Phi_{2,7}(x)\Phi_{0,5}(x) + 6\Phi_{0,9}(x)\Phi_{0,5}(x) + 17I.\end{aligned}$$

Now we also know that

$$\Delta^{12} = - 4x^{12}[f(x)]^{289}/f(x) + 17I = - 4x^{12}f(x^{289})/f(x) + 17I,$$

and therefore

$$\begin{aligned}x^{12}f(x^{289})/f(x) = &- \Phi_{7,8}(x) + 6\Phi_{6,9}(x) + 8\Phi_{5,10}(x) + 7\Phi_{4,11}(x)\\ &+ \Phi_{3,12}(x) + 8\Phi_{2,13}(x) - 6\Phi_{1,14}(x) + 4\Phi_{2,7}(x)\\ &+ 4\Phi_{0,9}(x) + 3\Phi_{0,5}(x) + 7\Phi_{2,7}(x)\Phi_{0,5}(x)\\ &+ 7\Phi_{0,9}(x)\Phi_{0,5}(x) + 17I.\end{aligned} \tag{13}$$

Equating the coefficients of x^n on both sides of (13) we obtain

$$\begin{aligned}p(n - 12) - p(n - 301) - \cdots \equiv &- n^7\sigma_1(n) + 6n^6\sigma_3(n) + 8n^5\sigma_5(n)\\ &+ 7n^4\sigma_7(n) + n^3\sigma_9(n) + 8n^2\sigma_{11}(n) - 6n\sigma_{13}(n) + 4n^2\sigma_5(n)\\ &+ 4\sigma_9(n) + 3\sigma_5(n) + 7\sum_{m=1}^{n-1} m^2\sigma_5(m)\sigma_5(n - m)\\ &+ 7\sum_{m=1}^{n-1} \sigma_5(m)\sigma_9(n - m) \pmod{17},\end{aligned}$$

where the numbers 12, 301, $\cdots$ are alternately of the form

$(17n-3)(51n-8)/2$ and $(17n+3)(51n+8)/2$. Now replacing n by $17n$ we get

$$p(17n-12)-p(17n-301)-\cdots\equiv 4\sigma_9(17n)+3\sigma_5(17n)$$
$$+7\sum_{m=1}^{17n-1}m^2\sigma_5(m)\sigma_5(17n-m)+7\sum_{m=1}^{17n-1}\sigma_5(m)\sigma_9(17n-m)\pmod{17}.$$

In order to further simplify this result we note that

$$\Phi_{0,5}(x)\Phi_{2,7}(x)-4[\Phi_{1,6}(x)]^2=-3\Phi_{2,7}(x)+3\Phi_{0,5}(x)\Phi_{0,9}(x)-8\Phi_{0,9}(x)$$
$$-6\Phi_{0,5}(x)+17I$$

and therefore

$$\sum_{m=1}^{n-1}m^2\sigma_5(m)\sigma_5(n-m)\equiv 4\sum_{m=1}^{n-1}m(n-m)\sigma_5(m)\sigma_5(n-m)-3n^2\sigma_5(n)$$
$$+3\sum_{m=1}^{n-1}\sigma_5(m)\sigma_9(n-m)-8\sigma_9(n)-6\sigma_5(n)\pmod{17}.$$

But

$$\sum_{m=1}^{n-1}m(n-m)\sigma_5(m)\sigma_5(n-m)=n\sum_{m=1}^{n-1}m\sigma_5(m)\sigma_5(n-m)$$
$$-\sum_{m=1}^{n-1}m^2\sigma_5(m)\sigma_5(n-m)$$

and

$$\sum_{m=1}^{n-1}m\sigma_5(m)\sigma_5(n-m)=\frac{n}{2}\sum_{m=1}^{n-1}\sigma_5(m)\sigma_5(n-m)$$

so that finally

$$\sum_{m=1}^{17n-1}m^2\sigma_5(m)\sigma_5(17n-m)\equiv 4\sum_{m=1}^{17n-1}\sigma_5(m)\sigma_9(17n-m)$$
$$-5\sigma_9(17n)-8\sigma_5(17n)\pmod{17}.$$

Therefore

$$p(17n-12)-p(17n-301)-\cdots\equiv\sum_{m=1}^{17n-1}\sigma_5(m)\sigma_9(17n-m)$$
$$+3\sigma_9(17n)-2\sigma_5(17n)\pmod{17},$$

$$p(17n-12)-p(17n-301)-\cdots\equiv\Sigma_{5,9}(17n)\pmod{17}. \tag{14}$$

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AUTOMORPHISMS OF FIELDS OF FORMAL POWER SERIES

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We propose to discuss in this note on power series fields in one variable the special automorphisms which do not alter the fields of coefficients. It will be proved that the pseudo-ramification groups introduced by MacLane are universal ramification groups, in the sense that a special ramification group must always be a subgroup of a well determined pseudo-ramification group. Finally we interpret the automorphism group of the field as an automorphism group of an infinite Lie ring.

Let Ω be an arbitrary field of characteristic χ. In the sequel we shall consider the field F of all formal power series $a=\sum_{j>-\infty}^{\infty}\omega_j t^j$ where the ω_j are in Ω and t is a transcendental element over Ω.[1] The field F is complete with respect to the rank one valuation V defined by $Va=m$ where m is the smallest subscript j for which $\omega_j\neq 0$. Let $\mathfrak{O}$ be the valuation ring of all holomorphic series and $\mathfrak{P}=(t)$ the principal prime ideal of $\mathfrak{O}$.

Suppose that S is an automorphism of F. We show that $\mathfrak{O}^S$ is also a valuation ring of F. For the proof[2] let a, b be any two nonzero elements of F. We must show that at least one of the quotients a/b, b/a lies in $\mathfrak{O}^S$. By assumption on S there exist unique elements c, d with $c^S=a$, $d^S=b$. Now observe that at least one of the quotients c/d or d/c lies in $\mathfrak{O}$ for $\mathfrak{O}$ is a valuation ring. Therefore at least one of the

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[1] For the basic properties of valuations see [1, 4, 5, 10]. Numbers in brackets refer to the bibliography at the end of the paper.

[2] See [4, p. 165].

elements $(c/d)^S=a/b$ or $(d/c)^S=b/a$ lies in $\mathfrak{O}^S$. Furthermore we observe that the ideal theory of $\mathfrak{O}$ is carried over isomorphically to $\mathfrak{O}^S$. Thus $\mathfrak{O}^S$ defines a rank one valuation V^S on F.

DEFINITION 1.[3] *An automorphism S of F is called analytic if $Va<Vb$ is equivalent to $V^Sa^S\leqq V^Sb^S$ for each pair of elements a, b in F.*

LEMMA 1. *Each automorphism of F is analytic.*

PROOF. We shall prove that F is complete with respect to the valuation V^S. Let $a_n\equiv a_{n-1}$ (mod $(\mathfrak{P}^S)^n$) be an infinite consistent system of congruences with respect to V^S. There exist uniquely determined elements c_n with $c_n^S=a_n$ and we have $c_n\equiv c_{n-1}$ (mod $\mathfrak{P}^n$). Hence there exists an element x in F with $x\equiv c_n$ (mod $\mathfrak{P}^n$), for F is complete with respect to V. Applying S to x we have $x^S\equiv c_n^S\equiv a_n$ (mod $(\mathfrak{P}^S)^n$). Hence F is complete with respect to V^S. Consequently F would be multiply complete if V and V^S were distinct valuations. Thus it would follow[4] that F is algebraically closed, in contradiction to the construction of F. Hence V and V^S are equivalent valuations, that is, S is analytic.

Suppose that s is an automorphism of Ω. If we observe the rules for the addition and multiplication of elements in F, the correspondence $\sum_{j>-\infty}^{\infty}\omega_j t^j\to\sum_{j>-\infty}^{\infty}\omega_j^s t^j=(\sum_{j>-\infty}^{\infty}\omega_j t^j)^S$ defines an automorphism of F. These automorphisms of F determine a subgroup of the automorphism group A of F which is isomorphic with the automorphism group of Ω. A simple computation shows that this subgroup is not normal if and only if it is not the trivial group consisting of the identity. Now let T be an arbitrary automorphism of F. The element T either induces an automorphism on $\Omega\subset F$ or it maps Ω into an isomorphic subfield $\Omega^T\subset F$. *We shall consider only those automorphisms of F for which all the elements of Ω are invariant.* These automorphisms form a subgroup G of A. This group G corresponds to the inertial group considered in the theory of normal algebraic extensions of fields with valuations.

LEMMA 2. *A field F which is complete with respect to a rank one valuation has no immediate extensions.*[5]

PROOF. Suppose that K is an immediate extension of F. Let A be an element of K. We shall show that A must lie in F. By assumption the value of A is the value of an element a in F. Thus A/a is a unit and therefore $A/a \equiv \omega_0 \pmod{\mathfrak{P}_K}$, where $\mathfrak{P}_K$ is the prime ideal of K and ω_0 is in Ω. Next there is at least one element a_1 in F with $A/a - \omega_0 \equiv a_1 \pmod{\mathfrak{P}_K^{n_1}}$, where $n_1 = V(A/a - \omega_0)$. By complete induction there exists a sequence of elements $a\omega_0$, $a_0 + aa_1\omega_1, \cdots, a\omega_0 + \cdots + aa_1 \cdots a_\nu\omega_\nu, \cdots$ such that $A \equiv a\omega_0 + \cdots + aa_1 \cdots a_\nu\omega_\nu \pmod{\mathfrak{P}_K^{m}}$ with $m = n_1 + \cdots + n_\nu \to \infty$ for $\nu \to \infty$. This sequence has a limit b in F and $A - b$ has value ∞ in K. Hence $A = b$ in F.[6]

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[3] See [5, footnote on p. 373; 12].

[4] See [12].

[5] See [4, p. 191].

Suppose that $S \neq 1$ is an automorphism of F with $\omega = \omega^S$ for all ω in Ω. Then $t^S \neq t$, for $t = t^S$ implies $a = a^S$ for every a in F. Hence each $S \neq 1$ determines by $t^S = tu(S)$ an element $u(S)$ of the unit group U of F.

THEOREM 1. *The groups G and U are in one-to-one correspondence as sets.*

PROOF. As seen above each automorphism S determines relative to the prime element t a unit $u(S)$. For the converse we shall show that the mapping $t \to t^* = tu$ determines, for given u in U, an automorphism of F. We associate to an arbitrary element $a = \sum_{j > -\infty}^{\infty} \omega_j t^j$ the quantity $a^* = \sum_{j > -\infty}^{\infty} \omega_j (t^*)^j$. The elements a^* form by definition a subfield F^* of F which is isomorphic to F. By construction the valuation V of F induces a valuation V^* on the complete field F^* with $V^*t^* = Vt^* = Vt$ so that F and F^* have the same residue class field. Therefore F is an immediate extension of F^* and hence $F = F^*$ by Lemma 2. Consequently the mapping $a \to a^*$ is an automorphism of F. We remark that the identity elements of G and U correspond to each other.

We next give a definition and a set of formulas which can be used to compute explicitly the coefficients of a^* relative to the prime element t. We define the derivative $D_t a$ of $a = \sum \omega_j t^j$ as $\sum j\omega_j t^{j-1}$ and $D_t^i = (1/i!)(d/dt)^i$. Then[7]

$$\begin{aligned} D_t^i(a+b) &= D_t^i a + D_t^i b, \quad D_t^i t^n = C_{n,i} t^{n-i}, \qquad i \geqq 0, \\ D_t^i(\omega a) &= \omega D_t^i a, \quad \omega \text{ in } \Omega, \\ D_t^i(ab) &= \sum_i D_t^{i_1} a D_t^{i_2} b; \qquad i_1 \geqq 0,\ i_2 \geqq 0,\ i_1 + i_2 = i. \end{aligned}$$

As in the calculus the inversion formula of Lagrange holds, $t = \sum_{j=1}^{\infty} \lambda_j (t^*)^j$ with $\lambda_j = [D_t^j\{D_t t^* \cdot (t/t^*)^j\}]_{t=0}$, $\lambda_1 \neq 0$. Hence t lies in F^* and thus $F^* = F$. Using the Taylor developments of the elements $a^* = \sum \omega_j (t^*)^j = \sum D_t^j\{a^*\}_{t=0} t^j$ it follows that each unit u determines by $t \to t^* = tu$ an automorphism of F. Letting $b^* = \sum \rho_j (t^*)^j = \sum D_t^j (b^*)_{t=0} t^j$ we have, by the rules on the derivatives D_t^j,

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[6] For another proof see [5, pp. 379–380].

[7] See [2].

$$\begin{aligned}(a+b)^* &= \sum \{D_t^j(a^*+b^*)\}_{t=0}t^j\\ &= \sum \{D_t^j a^*\}_{t=0}t^j + \sum \{D_t^j b^*\}_{t=0}t^j = a^*+b^*,\\ (ab)^* &= \sum \{D_t^j(a^*b^*)\}_{t=0}t^j\\ &= \sum_j \sum \{D_t^{j_1}a^*\}_{t=0}\{D_t^{j_2}b^*\}_{t=0}t^{j_1+j_2} = a^*b^*,\end{aligned}$$

where $j_1\geqq 0$, $j_2\geqq 0$, $j_1+j_2=j$, and similarly $(a\omega)^*=a^*\omega$.

Now let S, T be two automorphisms of G. On applying ST to the prime element t we find $t^{ST}=(t^T)^S=(tu(T))^S=t^Su(T)^S=tu(S)u(T)^S=tu(ST)$ where the u's are the units corresponding to the automorphisms. Thus the mapping $S\rightarrow u(S)$ gives rise to a crossed representation[8] of the automorphism group G in the unit group U, for $u(ST)=u(S)u(T)^S$. The latter relation may be viewed as a new multiplication $\times$ between the elements of U. We define $u_1\times u_2$ to be $u_1u_2^W$ where $W=S(u_1)$ is the automorphism which corresponds to u_1 by virtue of Theorem 1. The associativity of the group G implies that U is a group with respect to the operation $\times$.

We now proceed to a different interpretation of this operation. Let Ω_x be the set of all formal power series $\sum_{i=0}^{\infty}\omega_i x^i$, where x is an indeterminate over Ω.

Definition 2. *If* $A=\sum\alpha_i x^i$ *and* $B=\beta_i x^i$ *then define* $A\otimes B$ *as* $\sum_i\beta_i(\sum_j\alpha_j x^j)^i$ *if and only if* $\alpha_0=0$.

From the definition of Ω_x we observe that Ω_x may be considered as an infinite vector space over Ω with restricted multiplication.[9] The element x plays the role of a left and right unit on the subset $\mathbf{U}$ of all elements without constant terms. The set $\mathbf{U}$ is a group, as may be verified by using Taylor expansions with respect to x. Obviously the product of elements corresponds to the operation of taking the function of a function. It is now easy to set up a relation between U with $\times$ as operation and $\mathbf{U}$. If $u=\sum_{j=0}^{\infty}\omega_j t^j$, $\omega_0\neq 0$, is given then we take for the corresponding element $\mathbf{u}=\sum_{j=0}^{\infty}\omega_j x^{i+1}$. The product $u_1\times u_2$ of two elements u_1, u_2 in U is then determined as $\{(\mathbf{u}_1\otimes\mathbf{u}_2)x^{-1}\}_{x=t}$ where the factor x^{-1} is to indicate a scalar division of $\mathbf{u}_1\otimes\mathbf{u}_2$ by x. In the sequel it will be useful to use the representation of the elements S in $\mathbf{G}$ by means of the elements in $\mathbf{U}$; $S\rightleftarrows u(S)\rightleftarrows \mathbf{u}(S)$. We shall use the symbol S ambiguously for the automorphism S and the representation $\mathbf{u}(S)$.

Suppose now that μ_1, μ_2 are two nonzero elements of Ω. These elements determine, by $t\rightarrow t\mu_i$, automorphisms $S(\mu_i)$ in $\mathbf{G}$, $i=1, 2$. The multiplication rule of U shows $S(\mu_1)S(\mu_2)=(x\mu_1\otimes x\mu_2)=S(\mu_1\mu_2)$ $=S(\mu_2\mu_1)=(x\mu_2\otimes x\mu_1)=S(\mu_2)S(\mu_1)$. Hence the multiplicative group Ω^* of Ω has an isomorphic image in G.

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[8] See [7, p. 313].

[9] See [8, 9].

The group G contains an infinite series of subsets G_i defined by the elements $S_i=x+\sum_{\nu=i+1}^{\infty}\omega_\nu x^\nu$.

THEOREM 2. *The sets G_i are invariant subgroups of G, their intersection $\bigcap_i G_i$ is the identity and each factor group G_i/G_{i+1} is isomorphic to the additive group Ω. The factor group G/G_1 is isomorphic to the multiplicative group Ω^*.*

PROOF. Let $S=x+\omega_1x^i+\cdots$, $T=x+\omega_2x^i+\cdots$ be two elements of G_i. Then, by definition of the product in U,

$$\begin{aligned} S\otimes T &= x+(\omega_1+\omega_2)x^i+\cdots,\\ (*)\qquad T\otimes S &= x+(\omega_2+\omega_1)x^i+\cdots,\\ S^{-1} &= x+(-\omega_1)x^i+\cdots. \end{aligned}$$

Thus G_i is a group. To show that G_i is an invariant subgroup of G let $W=\alpha_0x+\alpha_1x^2+\cdots$ be an arbitrary element of G. The inversion formula for power series implies $W^{-1}=\alpha_0^{-1}x+(-\alpha_1/\alpha_0^3)x^2+\cdots$. Consequently, by direct computation, $W\otimes S\otimes W^{-1}=x+\alpha_0^{i-1}\omega_1x^i$ $+\cdots$ lies also in G_i. Definition of the G_i implies $G\supset G_1\supset\cdots$ and $\bigcap_i G_i=x$, the identity of G. The relations (*) imply that G_i/G_{i+1} is isomorphic to the additive group Ω. Now let $S(\alpha_0)$ be determined by the element $W=\alpha_0x+\alpha_1x^2+\cdots$. Then $S(\alpha_0)^{-1}\otimes W=\alpha_0(\alpha_0^{-1})x$ $+\alpha_1(\alpha_0^{-1})x^2+\cdots=x+(\alpha_1/\alpha_0)x^2+\cdots$ lies in G_1. Hence $G=\sum_{\alpha\in\Omega^*}S(\alpha)G_1$ and therefore $G/G_1\cong\Omega^*$.

COROLLARY 1. *The group G is a group extension of G_1 by Ω^* with factor set unity.*

PROOF. As seen before the elements $S(\mu)$, μ in Ω^*, form a subgroup of G. The elements $S(\mu)=\mu x$ induce automorphisms on the group $G_1=\{x+\omega_1x^2+\cdots+\omega_nx^{n+1}+\cdots\}$. The associated combinations are determined by

$$\begin{aligned} x\mu\otimes(x+\omega_1x^2+\cdots+\omega_nx^{n+1}+\cdots) &= x\mu+x^2\omega_1\mu^2+\cdots\\ &\qquad+x^{n+1}\omega_n\mu^{n+1}+\cdots,\\ (x+\omega_1x^2+\cdots+\omega_nx^{n+1}+\cdots)\otimes x\mu &= x\mu+x^2\omega_1\mu+\cdots\\ &\qquad+x^{n+1}\omega_n\mu+\cdots, \end{aligned}$$

whence

$$\begin{aligned} &x\mu\otimes(x+\omega_1x^2+\cdots+\omega_nx^{n+1}+\cdots)\otimes(x\mu)^{-1}\\ &\quad=x+x^2\omega_1\mu+\cdots+x^{n+1}\omega_n\mu^n+\cdots. \end{aligned}$$

Since $\{S(\mu),\ \mu\in\Omega^*\}$ is a subgroup the factor set of Ω^* with respect to G is the unit factor set.

COROLLARY 2. *The group G_1 is a complete metric group; if the field Ω is finite then G_1 is compact.*

PROOF. The system of normal subgroups G_i defines a topology in G_1. The group G_1 is complete with respect to this topology for G_1 contains all elements $x+\omega_1x^2+\cdots+\omega_nx^{n+1}+\cdots$ with arbitrary coefficients ω_n. We next define a metric in G_1 which is consistent with the system of defining neighborhoods G_i, $i=1, 2, \cdots$. Suppose $S = x + \alpha_1x^2+ \cdots + \alpha_{n-1}x^n + \alpha_nx^{n+1}+ \cdots$, $T = x + \beta_1x^2+ \cdots +\beta_{n-1}x^n+\beta_nx^{n+1}+ \cdots$ are two arbitrary elements of G_1. We define $\delta(S, T)$ to be e^{-n} if and only if $\alpha_i=\beta_i$ for all $i<n$ and $\alpha_n\neq\beta_n$. Then $\delta(S, T)=\delta(T, S)$, $\delta(S, T)>0$ for $S\neq T$, $\delta(S, T)\leqq\max[\delta(S, R), \delta(T, R)]$. Moreover G_n consists of all elements S in G_1 which have distance $\delta(1, S)\leqq e^{-n}$. Thus G_1 is a 0-dimensional metric group. The factor groups G_1/G_n contain nontrivial elements of finite order if and only if $\chi<\infty$ holds for the characteristic of Ω. If $\chi<\infty$ the elements of G_1/G_n have at most order χ^{n-1}. Hence $\lim_{n\to\infty} g^{\chi^n}=1$ for each element g of G_1, thus G_1 is a generalized χ-adic group.[10] In particular, G_1 is compact if Ω is finite. In this case $G_n/G_{n+1}\cong\Omega$, and thus G_1 is isomorphic to the inverse limit $\lim_{n\to\infty} G_1/G_n$ where the factor groups G_1/G_n are all finite.

DEFINITION 3. *The totality of elements S in G_1 with $a^S\equiv a \pmod{\mathfrak{P}^n}$ for each a in $\mathfrak{O}$ is called the nth pseudo-ramification group R_n of F.*[11]

THEOREM 3. *The groups G_n and R_n coincide.*

PROOF. We first remark that the first pseudo-ramification group R_1 coincides with G_1, for the general element $x+\omega_1x^2+\cdots$ of G_1 induces the identity mapping on the residue class field Ω of F. Suppose now that $g=x+\sum_{\nu=n+1}^{\infty}\omega_\nu x^\nu$ is an arbitrary element of G_n. Let $a=\alpha_0+\alpha_1t+\cdots+\alpha_{n-1}t^{n-1}+\alpha_nt^n+\cdots$. Then

$$a^g = \alpha_0 + \alpha_1\left(t + \sum_{\nu=n}^{\infty}\omega_\nu t^\nu\right) + \cdots + \alpha_{n-1}\left(t + \sum_{\nu=n}^{\infty}\omega_\nu t^\nu\right)^{n-1}$$
$$+ \alpha_n\left(t + \sum_{\nu=n}^{\infty}\omega_\nu t^\nu\right)^{n} + \cdots$$

---

[10] A topological group G is called χ-adic, if $\lim_{n\to\infty}g^{\chi^n}=1$ for each g in G, $\chi<\infty$, in a sense of the topology of G. Examples of χ-adic groups are the additive subgroup of the χ-adic number field and multiplicative groups in χ-adic algebras.

[11] See [6, p. 438].

$$
\begin{aligned}
&= \alpha_0 + \alpha_1 t + \alpha_1 \sum_{\nu=n}^{\infty} \omega_\nu t^\nu + \cdots + \alpha_{n-1} t^{n-1} + \alpha_{n-1}(n-1) t^n \omega_n \\
&\quad + \cdots + \alpha_n t^n + \cdots + \alpha_n n t^{n+\nu} \omega_n + \cdots \\
&\equiv \alpha_0 + \alpha_1 t + \cdots + \alpha_{n-1} t^{n-1} \pmod{\mathfrak{P}^n},
\end{aligned}
$$

whence $G_n \subseteq R_n$. Conversely, let $g \in R_n$ and $t^g = t + \beta_1 t^2 + \cdots + \beta_{n-1} t^n + \beta_n t^{n+1} + \cdots$. We compute

$$
\begin{aligned}
a^g &= \alpha_0 + \alpha_1(t + \beta_1 t^2 + \cdots + \beta_{n-1} t^n + \beta_n t^{n+1} + \cdots) \\
&\quad + \alpha_2(t + \beta_1 t^2 + \cdots)^2 + \cdots + \alpha_{n-1}(t + \beta_1 t^2 \cdots)^{n-1} + \cdots \\
&\equiv \alpha_0 + \alpha_1 t + \alpha_1 \beta_1 t^2 + \cdots + \alpha_2 t^2 + 2\alpha_2 \beta_1 t^3 + \cdots + \cdots \\
&\quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
&\quad + \alpha_{n-1} t^{n-1} + \cdots \\
&\equiv \alpha_0 + \alpha_1 t + \alpha_2 t^2 + \cdots + \alpha_{n-1} t^{n-1} \pmod{\mathfrak{P}^n}.
\end{aligned}
$$

Reducing this congruence successively mod $\mathfrak{P}^2, \mathfrak{P}^3, \cdots, \mathfrak{P}^{n-1}$ we obtain $\beta_1 = \cdots = \beta_{n-2} = 0$, whence $G_n \supseteq R_n$.

REMARK. As in the ordinary Hilbert theory it follows that R_n is an invariant subgroup of G_1. This furnishes a new proof for the invariance of the groups G_n.

THEOREM 4. *Suppose that H is a finite subgroup of G_1 and let K be the associated subfield of F. The ramification groups H_i of F/K are equal to $G_i \cap H$.*

PROOF. Let V_K be the valuation which is induced by V on the subfield K. Suppose that $\Gamma(F)$ and $\Gamma(K)$ are the value groups associated to V and V_K, respectively. We first observe that Ω is contained in K for $H \subset G_1$. Let $\bar{K}$ be the completion of K with respect to V_K, then $K \subseteq \bar{K} \subseteq F$ for the fundamental sequences in K are special fundamental sequences in F by the definition of V_K. Next observe that V_K has only V for its prolongation to F. Remark that all prolongations of V_K are given as V^h, h in H; thus, by Lemma 1, $V^h = V$. Combining these facts, we find $[F:K] = [\Gamma(F):\Gamma(K)] = [\Gamma(F):\Gamma(\bar{K})] = [F:\bar{K}]$. Consequently $K = \bar{K}$, that is, K is a complete field and H is the decomposition and inertial group for F/K. Now let H_i be the ith ramification group of F/K. Then $H_i \subset R_i = G_i$ by the proof of Theorem 3. Thus $H_i \subseteq R_i \cap H$. Conversely, let S be an element of $R_i \cap H$. Then $a^S \equiv a \pmod{\mathfrak{P}^i}$ by definition of R_i. Hence S lies in H_i. Therefore $H_i = R_i \cap H$ as asserted.

Let K be an arbitrary subfield of F such that $\Omega \subset K$ and $[F:K] < \infty$. The field K is then complete with respect to the valuation V_K

induced by V. For the proof we observe that the valuation V has a unique prolongation to the normal closure of F over K, for the latter is complete since it is a finite extension of F. The ramification theory implies, as at the beginning of the proof for Theorem 4, that K is complete. Now let $\pi(F)$ be any prime element of F, that is $V\pi(F)=1$, $F=\Omega\{\pi(F)\}$. Then $N_{F/K}\pi(F)$ is a prime element $\pi(K)$ for K because $V\pi(K)=[F:K]$ and $\Omega\{\pi(K)\}=K$ by Lemma 2 applied to the chain $\Omega\{\pi(K)\}\subseteq K\subset F$. We have $\pi(K)=\sum_{\nu=0}^{\infty}\omega_\nu\pi(F)^{n+\nu}$, where $[F:K]=n$ and $\omega_0\neq 0$. Applying an automorphism S of G to $\pi(F)$ we obtain a field $K^S=\Omega\{\pi(K)^S\}$ which is isomorphic to K and $K=K^S$ if and only if S is an automorphism of F/K. The group SHS^{-1} is the full automorphism group of F/K^S if H is the automorphism group of F/K. We apply these results to prove the following theorem.

THEOREM 5. *The automorphism group G of F contains elements of finite order.*

PROOF. We first observe that F always has normal completely ramified extensions M. The existence of such extensions depends on the nature of the residue class field Ω.[12] In case that n is an integer prime to the characteristic χ and Ω contains a primitive nth root of unity the field F surely has at least one cyclic completely ramified extension of degree n. For example, there is surely such a quadratic extension if $(\chi, 2)=1$ or $\chi=\infty$. If χ is finite then Ω contains some field of (χ^m-1)th roots of unity, $m\geqq 1$, and any $n\mid(\chi^m-1)$ may be used. For finite χ there are infinitely many cyclic completely ramified extensions of degree χ over F.[13] Hence we know that completely ramified normal extensions M/F exist. Since M is completely ramified, say of degree n, we have $\pi(F)=\sum_{\nu=0}^{\infty}\omega_\nu\pi(M)^{\nu+n}$, $\omega_0\neq 0$, for a prime element $\pi(M)$ of M. By the structure theory of fields of power series the fields M and F are isomorphic.[14] Suppose that ϕ is a realization of the isomorphism $M\cong F$, that is, $M^\phi=F$. We apply ϕ to the equation for $\pi(F)$ and obtain $\pi(F)^\phi=\sum_{\nu=0}^{\infty}\omega_\nu[\pi(M)^\phi]^{n+\nu}$. Then $\pi(F)^\phi$ determines by $\Omega\{\pi(F)^\phi\}$ a subfield K of F so that the Galois groups of M/F and $F/\Omega\{\pi(F)^\phi\}$ are isomorphic. Observing that the Galois group of M/F is isomorphic to a finite subgroup of G, the assertion of the theorem follows.[15]

As a special case we consider cyclic subgroups H of order n in G for which $(n, \chi)=1$. Let K be the field of invariants for H. We have

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[12] See [11, 14].

[13] See [14].

[14] See [1, 6, 13, 14].

[15] See [3, p. 890].

$K=\Omega\{ut^n\}$ where u is a unit of F. Then $u=\omega v$ where $\omega\neq 0$ lies in Ω and $v\equiv 1 \pmod{P}$. We have $K=\Omega\{vt^n\}$, for K is an immediate extension of $\Omega\{vt^n\}$. Since $(n, \chi)=1$ there exists[16] a unit y of F with $y\equiv 1 \pmod{P}$, $y^n=v$. Therefore $K=\Omega\{(yt)^n\}$. Since $\Omega\{yt\}=\Omega\{t\}=F$ we obtain $H=SH_0S^{-1}$ where S is the automorphism of G_1 with $t^S=yt$ and H_0 is the finite subgroup of G generated by the element ζx, ζ a primitive nth root of unity. Finally $K=\Omega\{t^n\}$ if and only if $SH_0S^{-1}=H_0$, that is, S induces an automorphism on H_0. Expressing $K=\Omega\{t^n\}$ in terms of the prime elements we have $(yt)^n/t^n=y^n$ lies in $\Omega\{t^n\}$. Consequently y^n must be a power series $1+\sum_{j=1}^{\infty}\alpha_j(t^n)^j$. Conversely each unit y with y^n in $\Omega\{t^n\}$ gives rise to $K=\Omega\{t^n\}$. Since there are always elements y for which y^n does not lie in $\Omega\{t^n\}$ and $y^n=1+\omega t+\cdots$, $\omega\neq 0$, we have the existence of automorphisms S with $SH_0S^{-1}\neq H_0$ and $\Omega\{t^n\}\cap\Omega\{(yt)^n\}=\Omega$, as can be shown by comparing the series of the fields. This shows that the set of normal subfields K with $[F:K]<\infty$, $([F:K], \chi)=1$ is not a lattice. We remark that conditions on y may be derived to describe $\Omega\{(yt)^s\}=\Omega\{t^s\}$ for $s\mid n$.

The group G can be interpreted as a group of automorphisms of an infinite Lie ring $\mathfrak{D}_L$ over Ω. We define $\mathfrak{D}_L$ as the ring $\mathfrak{D}$, considered as an infinite vector space over Ω, in which a product $[f, g]$ is defined as follows. We set $[f, g]=(dg/dt)f-(df/dt)g$. The rules of differentiation imply that $[f, g]$ obeys the rules of the Jacobi bracket. For a basis of $\mathfrak{D}_L$ we may take the elements $e_i=t^{i+1}$, $i=-1, 0, 1, \cdots$ and $[e_i, e_j]=(j-i)e_{i+j}$. By actual computation it can be shown that the mapping

$$(**)\qquad f(t)\rightarrow f(t^S)/(dt^S/dt)$$

is an automorphism of $\mathfrak{D}_L$.[17] Moreover, distinct elements of G give rise to distinct automorphisms in the automorphism group $A(\mathfrak{D}_L)$ of $\mathfrak{D}_L$.

THEOREM 6. *The groups G and $A(\mathfrak{D}_L)$ coincide if $\chi=\infty$.*

PROOF. Suppose that Σ is an automorphism of $\mathfrak{D}_L$. Let $t^\Sigma=\Phi(t)$. We shall show that there exists an automorphism S in G so that Σ is determined by the formula (**) applied to t, that is, $t^\Sigma=t^S/(dt^S/dt)$. Such an element S determines then the automorphism Σ on all of $\mathfrak{D}_L$ because the elements t^{i+1} form a basis of $\mathfrak{D}_L$. To determine S it suffices to find a unique element $\phi(t)=\alpha_1t+\alpha_2t^2+\cdots$, $\alpha_1\neq 0$, for which $t^\Sigma=\Phi(t)=f[\phi(t)]/(d\phi/dt)$. By formal integration we find

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[16] See [10, p. 561; 11, p. 441].

[17] Compare [15, pp. 37–47].

$\phi(t)=\exp[\int\Phi^{-1}(t)dt]$ where $\int\Phi^{-1}(t)dt$ is the formal indefinite integral without a constant of integration. We observe that all formal operations involved can be carried out because they are determined in $\mathfrak{O}$. The automorphism S is given by $\phi(x)$ using the representation of G by U. In concluding we remark that $G\subset A(\mathfrak{O}_L)$ for $\chi<\infty$. The inequality may be explained by the fact that t^n is never the derivative of an element in F if $n\equiv 0 \pmod{\chi}$.

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THE EQUATION $x' \equiv xd - dx = b$

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Let $\mathfrak{A}$ be an associative algebra with a possibly infinite basis over a field Φ. Then if d is a fixed element in $\mathfrak{A}$, it is well known that the mapping $x \rightarrow x' \equiv [x, d] = xd - dx$ is a derivation[1] in $\mathfrak{A}$; that is,

$$(x + y)' = x' + y', \qquad (x\alpha)' = x'\alpha, \qquad (xy)' = x'y + xy'$$

for all x, y in $\mathfrak{A}$ and all α in Φ. The constants relative to such a derivation are the elements of $\mathfrak{A}$ that commute with d. We shall call an element b a *d-integral* if $b = a'$ for some element a in $\mathfrak{A}$, that is, if the equation $x' = xd - dx = b$ has a solution in $\mathfrak{A}$. Clearly if a is a solution of this equation then the totality of solutions is the set $\{a + c\}$ where c ranges over the set of d-constants. In a recent paper appearing in this Bulletin, R. E. Johnson obtained a necessary and sufficient condition that an element b be a d-integral under the assumption that $\mathfrak{A}$ is a separable algebraic division ring.[2] In this note we allow $\mathfrak{A}$ to be an arbitrary algebra but we make the assumption that d is an algebraic element in the sense that it satisfies a polynomial equation with coefficients in Φ. We obtain a necessary condition, which is equivalent to Johnson's condition when $\mathfrak{A}$ is a division ring, that b be a d-integral. If the minimum polynomial $\mu(\lambda)$ of d is relatively prime to its derivative $\mu'(\lambda)$, then it is easy to see that the condition is also sufficient and one may give an explicit formula for a solution of the equation $x' = b$. If we assume that $\mathfrak{A}$ is a simple algebra satisfying the descending chain condition for left ideals then we can show that our condition is also sufficient when $\mu(\lambda)$ is a product of distinct irreducible factors in $\Phi[\lambda]$ and in certain other cases. Here, however, we do not display a solution but merely prove its existence. Our results include, of course, Johnson's result for algebraic division rings, since the minimum polynomial of an element in such a ring is irreducible. No assumption about separability is required.

In order to obtain a condition for the solvability of the equation $x' = b$ we consider the matrices

$$(1) \qquad u = \begin{pmatrix} d & 0 \\ 0 & d \end{pmatrix}, \qquad v = \begin{pmatrix} d & b \\ 0 & d \end{pmatrix}$$

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[1] Cf. the author's paper *Abstract derivation and Lie algebras*, Trans. Amer. Math. Soc. vol. 42 (1937) pp. 206–224.

[2] *On the equation $\chi\alpha = \gamma\chi + \beta$ over an algebraic division ring*, Bull. Amer. Math. Soc. vol. 50 (1944) pp. 202–208.

in the matrix algebra $\mathfrak{A}_2$ of two-rowed matrices with elements in $\mathfrak{A}$. If x is any element in $\mathfrak{A}$

$$\begin{pmatrix}1 & -x\\ 0 & 1\end{pmatrix} = \begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}^{-1}$$

and the equation $x'=xd-dx=b$ is equivalent to the matrix equation

$$\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\begin{pmatrix}d & 0\\ 0 & d\end{pmatrix}\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}^{-1} = \begin{pmatrix}d & b\\ 0 & d\end{pmatrix}. \tag{2}$$

Thus if b is a d-integral the matrices (1) are similar in $\mathfrak{A}_2$. We suppose now that d is an algebraic element and let

$$\phi(\lambda) = \lambda^m + \alpha_1\lambda^{m-1} + \cdots \tag{3}$$

be the minimum polynomial of d over Φ. Then it is clear that $\phi(u)=0$. Hence a necessary condition that b be a d-integral is that $\phi(v)=0$. Now

$$v^r = \begin{pmatrix}d^r & B_r\\ 0 & d^r\end{pmatrix}$$

where $B_r=\sum_{k=0}^{r-1}d^kbd^{r-k-1}$. Hence the condition that $\phi(v)=0$ is that

$$B_m + \alpha_1B_{m-1} + \cdots + \alpha_{m-1}B_1 = 0. \tag{4}$$

As we have shown elsewhere[3]

$$B_r = C_{r,1}d^{r-1}b + C_{r,2}d^{r-2}b' + \cdots + b^{(r-1)}$$

where $b^{(k)}=(b^{(k-1)})'$. Hence if we define

$$\phi_k(\lambda) = C_{m,k}\lambda^{m-k} + C_{m-1,k}\alpha_1\lambda^{m-k-1} + \cdots \equiv \phi^{(k)}(\lambda)/k!$$

we may write (4) in the more useful form

$$\phi_1(d)b + \phi_2(d)b' + \cdots + \phi_m(d)b^{(m-1)} = 0. \tag{5}$$

We suppose now that $\phi_1(\lambda)\equiv\phi'(\lambda)$ is relatively prime to $\phi(\lambda)$.[4] Then $\phi_1(d)$ is a regular element in $\mathfrak{A}$. Hence if b is an element such that (5) holds,

$$b = -\phi_1(d)^{-1}\phi_2(d)b' - \cdots - \phi_1(d)^{-1}\phi_m(d)b^{(m-1)} = x'$$

where

$$x = -\phi_1(d)^{-1}\phi_2(d)b - \cdots - \phi_1(d)^{-1}\phi_m(d)b^{(m-2)}. \tag{6}$$

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[3] Loc. cit. footnote 1, p. 209.

[4] This condition will be satisfied if d generates a separable algebraic field over Φ.

This proves the following theorem.

THEOREM 1. *Let $\mathfrak{A}$ be an arbitrary algebra and let d be an algebraic element of $\mathfrak{A}$ having a minimum polynomial $\phi(\lambda)$ relatively prime to its derivative. Then* (5) *is a necessary and sufficient condition in order that the element b be a d-integral. When the condition holds, $b=x'$ where x is given by* (6).

We suppose now that $\mathfrak{A}$ is a simple algebra with an identity satisfying the descending chain condition for left (right) ideals. Then $\mathfrak{A}=\mathfrak{D}_h$, a matrix algebra of h rows over the (not necessarily finite) division algebra $\mathfrak{D}$ and conversely any algebra of this form satisfies our condition. As before let d be an algebraic element of $\mathfrak{A}$ and let $\phi(\lambda)$ be its minimum polynomial. Let b be an element of $\mathfrak{A}$ such that (5) holds. Then (4) holds and hence the minimum polynomial of v as well as of u is $\phi(\lambda)$. Since $d, b\in\mathfrak{A}=\mathfrak{D}_h$ the matrices u and $v\in\mathfrak{D}_{2h}$ and these may be regarded as the matrices of linear transformations in a $2h$-dimensional vector space $\mathfrak{R}$ over $\mathfrak{D}$. Let T be the linear transformation corresponding to v. Then according to the form of v we have an h-dimensional subspace $\mathfrak{S}$ of $\mathfrak{R}$ invariant under T such that the matrix of T in $\mathfrak{S}$ is d and the matrix of T in the difference space $\mathfrak{R}-\mathfrak{S}$ is also d.

We suppose now that $\phi(\lambda)$ is a product of irreducible factors in $\Phi[\lambda]$. In this case the linear transformation is completely reducible.[5] Hence there exists a subspace $\mathfrak{S}'$ invariant under T such that $\mathfrak{R}=\mathfrak{S}+\mathfrak{S}'$, $\mathfrak{S}\cap\mathfrak{S}'=0$ and such that the matrix of T in $\mathfrak{S}'$ is also d. Let $x_1, \cdots, x_h, x_{h+1}, \cdots, x_{2h}$ be the original basis of $\mathfrak{R}$ relative to which T has the matrix v so that $x_1, \cdots, x_h$ is a basis for $\mathfrak{S}$. Corresponding to the decomposition $\mathfrak{R}=\mathfrak{S}+\mathfrak{S}'$ we have the basis $x_1, \cdots, x_h, x'_{h+1}, \cdots, x'_{2h}$. The matrix relating this basis to the original one has the form

$$\begin{pmatrix} 1 & p \\ 0 & q \end{pmatrix},$$

where $p, q\in\mathfrak{D}_h$ and the matrix of T relative to the basis $x_1, \cdots, x_h, x'_{h+1}, \cdots, x'_{2h}$ is u. Hence we have the equation

$$\begin{pmatrix} d & 0 \\ 0 & d \end{pmatrix} = \begin{pmatrix} 1 & p \\ 0 & q \end{pmatrix}^{-1} \begin{pmatrix} d & b \\ 0 & d \end{pmatrix} \begin{pmatrix} 1 & p \\ 0 & q \end{pmatrix}.$$

This implies that $dq=qd$ and that $bq=pd-dp$. Since the matrix

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[5] See the author's paper *Pseudo-linear transformations*, Ann. of Math. vol. 38 (1937) p. 498.

$$\begin{pmatrix} 1 & p \\ 0 & q \end{pmatrix}$$

is regular, q is regular and hence we have the relation $b = xd - dx$ where $x = pq^{-1}$.

THEOREM 2. *Let $\mathfrak{A} = \mathfrak{D}_k$ where $\mathfrak{D}$ is a division algebra over Φ and let d be an algebraic element of $\mathfrak{A}$. Then if the minimum polynomial $\phi(\lambda)$ of d is a product of distinct irreducible factors in $\Phi[\lambda]$, the condition* (5) *is necessary and sufficient in order that the element b of $\mathfrak{A}$ be a d-integral.*

We next let $\mathfrak{A} = \Phi_k$, the matrix algebra of h rows over Φ. Let d be a non-derogatory matrix in Φ_k. Thus d has only one invariant factor $\phi(\lambda) \neq 1$ and $\phi(\lambda)$ is the minimum polynomial of d. Let b be a matrix such that (5) holds and consider the matrix v as before. The minimum polynomial of v is $\phi(\lambda)$. If T is the linear transformation in the $2h$-dimensional space over Φ associated with the matrix v then $\mathfrak{R}$ contains an invariant subspace $\mathfrak{S}$ whose matrix is d. Since d is non-derogatory, $\mathfrak{S}$ is a cyclic subspace and its order is the minimum polynomial of T in $\mathfrak{R}$. Now it is known that this implies that $\mathfrak{R} = \mathfrak{S} + \mathfrak{S}'$, $\mathfrak{S} \cap \mathfrak{S}' = 0$ where $\mathfrak{S}'$ is also invariant relative to T.[6] A repetition of the argument used to prove Theorem 2 will now yield the following theorem.

THEOREM 3. *Let d be a non-derogatory matrix in the matrix algebra Φ_k and let $\phi(\lambda)$ be its minimum polynomial. Then the condition* (5) *is necessary and sufficient that the matrix b be a d-integral.*

We give finally an example in which the condition (5) is not sufficient to insure that an element be a d-integral. Let

$$d = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad b = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then the minimum polynomial of d is $\phi(\lambda) = \lambda^2$. Since $bd = db = 0$, b satisfies (5). On the other hand, the invariant factors of the matrices u and v here are respectively $\lambda^2, \lambda^2, \lambda, \lambda$ and $\lambda^2, \lambda^2, \lambda^2$. It follows that these matrices are not similar and hence b is not a d-integral.

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[6] See van der Waerden's *Moderne Algebra*, vol. 2, pp. 129–130. The proof given there of this theorem for ordinary finite groups is also valid for vector spaces relative to a single linear transformation.

SYMBOLIC SOLUTION OF CERTAIN PROBLEMS IN PERMUTATIONS

IRVING KAPLANSKY

1. Introduction. It is our purpose in this paper to show how algebraic symbolism can be applied to the problem of enumerating certain types of restricted permutations. The method rests on a symbolic interpretation of the method of inclusion and exclusion, or rather its probability analogue, the formula of Poincaré; the use of the terminology of probability is a matter of convenience only, the considerations being purely combinatorial. The problems considered are of two kinds: those where the restriction of position of elements is absolute (§3, "card-matching" problems), and those where elements are restricted relative to others (§§5, 6). In §4 approximations to the symbolic formulas are obtained. Though these problems are rather special in character, the foundations of the method (§2) are quite general and should find other applications.

2. Symbolic expressions. Consider n events $A_1, \cdots, A_n$ and let $p(A_{i_1} \cdots A_{i_k})$ denote the probability of the joint occurrence of $A_{i_1} \cdots A_{i_k}$. The probability that none of $A_1, \cdots, A_n$ occurs, which we shall denote by P_0, is given by Poincaré's formula:

$$P_0 = 1 - \sum p(A_i) + \sum p(A_iA_j) \cdots . \tag{1}$$

In the case of complete symmetry, that is, where each $P(A_{i_1} \cdots A_{i_k})$ is a function ϕ_k of k alone,

$$P_0 = 1 - n\phi_1 + {}_nC_2\phi_2 - \cdots .$$

By using the displacement operator E defined by $E^k\phi_0 = \phi_k$, we may write more compactly $P_0 = (1-E)^n\phi_0$.

In the cases which we shall consider, however, this complete symmetry will be lacking and instead we shall have the following partial substitute, which we shall call *quasi-symmetry*: $p(A_{i_1} \cdots A_{i_k})$ is either equal to zero or to a function ϕ_k of k alone. To evaluate P_0 for such cases a symbolic device of Broderick [2][1] is helpful; we may write (1) in the form

$$P_0 = (1 - pA_1)(1 - pA_2) \cdots (1 - pA_n), \tag{2}$$

where the multiplication is symbolic in the sense that $p(A_i)p(A_j) \cdots$

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[1] Numbers in brackets refer to the Bibliography at the end of the paper.

is interpreted as $p(A_iA_j \cdots)$.[2] The following rule can then be stated.

To evaluate (2) *in the case of quasi-symmetry, first suppress all products that vanish; from the remaining expression form a polynomial* $f(E)$ *by replacing each term* $p(A_i)$ *by* E. *Then* P_0 *will equal* $f(E)\phi_0$.

The advantage of this point of view lies in the fact that in the computation of $f(E)$ we may resort to all the devices of formal algebra. A second advantage, perhaps more important, was pointed out by Fréchet [3]. Once $f(E)$ has been found, we have simultaneously solved the problem of finding P_r, the probability that exactly r of the events $A_1, \cdots, A_n$ occur. For Poincaré's formula gives

$$P_r = \sum_{k=r}^{n} (-1)^{k-r} {}_kC_{k-r} \sum p(A_{i_1} \cdots A_{i_k})$$

which can be rewritten

$$P_r = \sum_{k=0}^{n} (-1)^{r} {}_kC_r \sum (-1)^k p(A_{i_1} \cdots A_{i_k}). \tag{3}$$

Comparison of (1) and (3) shows that we may use the same polynomial $f(E)$ to evaluate P_r provided we replace ϕ_k by $(-1)^r {}_kC_r\phi_k$; that is,

$$P_r = f(E)\psi_0 \quad \text{where} \quad \psi_k = (-1)^r {}_kC_r\phi_k.$$

Moreover, since

$${}_kC_0 - {}_kC_1 + {}_kC_2 - \cdots + (-1)^r {}_kC_r = (-1)^r {}_{k-1}C_r,$$

the use of $(-1)^r {}_{k-1}C_r\phi_k$ will give the probability that *at most* r of the n events happen.

3. **Card-matching problems.**[3] A general version of the so-called "card-matching" problem can be formulated as follows: in a permutation of $1, \cdots, n$ what is the probability that the integer i is not in the jth place, also k is not lth, and so on? More generally we can ask for the probability that these conditions are violated precisely r times. For a convenient notation let p_{ij} denote the probability that i is jth; then we are to evaluate a product of the form

$$(1 - p_{ij})(1 - p_{kl}) \cdots .$$

We must first observe that the condition of quasi-symmetry is fulfilled; in fact the (symbolic) product of k of the p's is either 0 or

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[2] We refrain from introducing a special notation for this symbolic multiplication, as it will always be clear from the context when it is intended.

[3] Cf. [1] for an extensive bibliography.

$(n-k)!/n!$. In accordance with our previous notation we shall therefore write

$$(4) \qquad \phi_k = (n-k)!/n!, \qquad \psi_k = (-1)^r {}_kC_r(n-k)!/n!$$

in the remainder of the paper. The question of determining when a product of p_{ij}'s vanishes is readily answered; $p_{ij}\,p_{kl}=0$ if $i=k$ or $j=l$ and a product of several p's is 0 if and only if it contains such a pair. This observation makes it comparatively easy to compute $f(E)$ in some famous problems of the card-matching type.

In each of the problems that follows, the solution is provided by finding the polynomial $f(E)$ *defined in* §2; *the probability that a permutation violates the conditions* r *times is* $f(E)\psi_0$, *with* $\psi_k=E^k\psi_0$ *given in* (4).

(i) The classical "problème des rencontres" requires merely that i shall not be ith. We have to compute $(1-p_{11})\cdots(1-p_{nn})$. No products vanish so this is $(1-E)^n$.

(ii) In a widely known generalization of the preceding problem, the n integers are divided into n/a subsets of a each, and every integer is forbidden to appear in each of the a places originally occupied by the members of its subset. Thus, with $a=2$, we can suppose the elements so numbered that the subsets are $(1, 2)$, $(3, 4)$, $\cdots$, and the conditions read: neither 1 nor 2 is 1st or 2nd, neither 3 nor 4 is 3rd or 4th, and so on. We find $n/2$ groups of 4 factors, the first of which is

$$\begin{aligned}(1-p_{11})(1-p_{12})(1-p_{21})(1-p_{22}) \\ = 1 - p_{11} - p_{12} - p_{21} - p_{22} + p_{11}p_{22} + p_{12}p_{21}.\end{aligned}$$

All vanishing products having now been eliminated, we have the answer $(1-4E+2E^2)^{n/2}$. For the usual deck of 52 cards a similar calculation yields

$$(1 - 16E + 72E^2 - 96E^3 + 24E^4)^{13},$$

and for the 5×5 deck used in telepathic experiments:

$$(1 - 25E + 200E^2 - 600E^3 + 600E^4 - 120E^5)^5.$$

We can generalize somewhat further by supposing that each of the a members of a subset are forbidden to appear in b, rather than a, places. We have then to compute

$$(5) \qquad \begin{aligned} F(a, b) &= \prod (1 - p_{ij}) \qquad (i = 1, \cdots, a; j = 1, \cdots, b) \\ &= \sum_{t=0}^{a} {}_aC_t\, {}_bC_t\, t!(-E)^t, \end{aligned}$$

as follows from a simple combinatorial argument. We can even suppose that the subsets are of different sizes $a_1, a_2, \cdots$ with the members forbidden $b_1, b_2, \cdots$ places respectively; this generalization leads to the polynomial $F(a_1, b_1)F(a_2, b_2)\cdots$. We remark again that by applying this operator to ψ_0, ψ defined as in (4), we obtain the probability that a permutation will violate the stated conditions precisely r times. This result was published without proof in [4].

(iii) In the "problème des ménages" the conditions read: 1 shall not be 1st or 2nd, 2 shall not be 2nd or 3rd, $\cdots$, n not nth or 1st. The product

$$(1 - p_{11})(1 - p_{12}) \cdots (1 - p_{nn})(1 - p_{n1})$$

was evaluated in [5] and found to be

$$U_n = \sum_{k=0}^{n} {}_{2n-k}C_k 2n(-E)^k/(2n - k). \tag{6}$$

But we are able further to solve Lucas' more difficult problem [6, p. 491] of finding the number of permutations "discordant" with any two given ones.[4] (The problème des ménages is the case where the given permutations are 1, 2, $\cdots$, n and 2, 3, $\cdots$, n, 1.) The solution is as follows: considering one of the given permutations to be a permutation of the other, represent it as a product of disjoint cycles of length $a, b, c, \cdots$. Then the polynomial giving the probability of a permutation discordant with both is $U_aU_bU_c \cdots$, where U_n is given by (6) for $n \neq 1$ and $U_1 = 1 - E$.

(iv) As a final somewhat more intricate example, suppose 1, $\cdots$, n to be divided again into subsets of a each, and let every integer be forbidden to appear in the places occupied by all the *other* members of its subset—its own original position is permitted. The answer, which we state without proof, is

$$[F(a, a) + aEF(a - 1, a - 1) + {}_aC_2E^2F(a - 2, a - 2) + \cdots + E^a]^{n/a},$$

F being defined as in (5).

4. Asymptotic approximations. The symbolic formulas obtained in §3 provide the exact answers to the problems in question and are readily calculated for small values of n. For large values of n their exact evaluation is out of the question, but we can readily find approximations. The method is sufficiently well illustrated by a consideration of the symbolic expression $(1-aE)^{zn}\phi_0$, where ϕ_k is defined as in (4) and zn is a positive integer not exceeding n.

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[4] In an article submitted to the American Mathematical Monthly, John Riordan gives a detailed investigation of this solution, identifying it with one of Touchard [8], and applying it to the enumeration of n by 3 Latin rectangles.

$$(1 - aE)^{zn}\phi_0 = \sum_{k=0}^{n} (- a)^k(zn)_k/k!n_k, \tag{7}$$

where $n_k=n(n-1)\cdots(n-k+1)$. Since

$$(zn)_k/n_k = z^k - (k^2 - k)(z^{k-1} - z^k)n^{-1}/2 + O(n^{-2}), \tag{8}$$

it is natural to conclude from (7) and (8) that

$$(1 - aE)^{zn}\phi_0 = e^{-az}[1 - a^2(z - z^2)n^{-1}/2 + O(n^{-2})]. \tag{9}$$

However, since the error term of (8) involves k, a more detailed investigation is necessary.

Let $m=[n^{1/3}]$. We split the sum (7) into two parts: S_1, the sum from $k=0$ to $m-1$, and S_2, the sum from m to n. In all the applications $z\leqq 1$ and we shall treat only that case. Then $(zn)_k\leqq n_k$ and

$$|S_2| \leqq \sum_{m}^{\infty} |a|^k/k! = O(m^{-6}) = O(n^{-2}). \tag{10}$$

To estimate S_1 the following lemma is convenient.

LEMMA. *Suppose $k\leqq m$ and let R denote the sum of the absolute values of the terms in n_k which are of degree not greater than $k-2$ in n. Then $R<k^5n^{k-2}$.*

PROOF. The term in n^i is less than

$$n^i(1 + 2 + \cdots + k - 1)^{k-i} < n^ik^{2k-2i}.$$

Now $i\leqq k-2$ implies

$$i + (2k - 2i - 4)/3 \leqq k - 2.$$

Hence $n^ik^{2k-2i}\leqq k^4n^{k-2}$. Since R contains $k-1$ terms all less than k^4n^{k-2}, we have $R<k^5n^{k-2}$.

Next we consider the expression

$$Q = (zn)_k - n_kz^k + n_k(k^2 - k)(z^{k-1} - z^k)n^{-1}/2. \tag{11}$$

Since the terms in n^k and n^{k-1} cancel out, we may apply our lemma to estimate each of the three terms of Q. Remembering $z\leqq 1$, we find $|Q|<Ck^7n^{k-2}$ where C depends only on z. We next note that $n^m(n-m)^{-m}\to 1$ as $n\to\infty$, so that $n^k(n-k)^{-k}$ has an upper bound independent of n and k. Hence

$$|Q/n_k| < Ck^7n^{-2}n^k(n - k)^{-k} < C_1k^7n^{-2},$$

and finally

$$\sum_{0}^{m-1} (-a)^k Q/k!n_k = O(n^{-2}). \tag{12}$$

If we substitute in (12) the expression for Q given by (11) we verify that S_1 equals the right member of (9). Together with (10) this proves (9).

The same method, with more elaborate calculation, gives the general result (here again ψ_k is defined in (4), and zn is a positive integer not exceeding n)

$$\begin{aligned}(1 - aE + bE^2 \cdots)^{zn}\psi_0 &= e^{-az}(az)^r(r!)^{-1} \\ &\cdot [1 - (a^2 + 2b - a^2z)(r^2 - r - 2arz + a^2z^2)(2a^2zn)^{-1} + O(n^{-2})],\end{aligned}$$

showing that the distribution of the number of "hits" obtained in the card-matching problems approaches the Poisson distribution with parameter az. One may of course compute as many more terms of the asymptotic expansion as desired. (The author has computed the next two terms, but as the result is complicated and unverified, it will not be stated.)

For example, with a deck of n cards divided into 4 suits, we found the probability of exactly r matches to be

$$\begin{aligned}(1 - 16E + 72E^2 \cdots)^{n/4}\psi_0 \\ = e^{-4}4^r(r!)^{-1}[1 - 21(r^2 - 9r + 16)/8n] + O(n^{-2}).\end{aligned}$$

The probability of no matches ($r = 0$) is

$$(1 - 16E + 72E^2 \cdots)^{n/4}\phi_0 = e^{-4}(1 - 42n^{-1}) + O(n^{-2}).$$

However the approximation is still too crude for n as small as 52.

5. **The "n-kings problem."** There is a second type of restricted permutations to which our method can be applied. Let it be required to find the probability that, in a permutation of $1, \cdots, n$, the integer i does not immediately precede j, k does not immediately precede l, and so on. We shall denote the probability that i immediately precedes j by q_{ij}. The expression to be evaluated is then

$$(1 - q_{ij})(1 - q_{kl}) \cdots .$$

The fundamental requirement of quasi-symmetry (§2) is fulfilled; in fact, the product of k of the q's is either 0 or the same expression as in (4): $\phi_k = (n-k)!/n!$. Moreover it is again true that $q_{ij}q_{kl} = 0$ if $i = k$ or $j = l$; this time however we have the further complication $q_{ij}q_{ji} = 0$,

$q_{ij}q_{jk}q_{ki}=0$, and so on. With this modification, the calculations proceed as before.

The simplest problem of this kind requires merely that i shall never precede $i+1$. We find

$$(1 - q_{12})(1 - q_{23}) \cdots (1 - q_{n-1,n}) = (1 - E)^{n-1}.$$

A modification that suggests itself is to require further that $i+1$ shall not precede i. Or we can reformulate the question as a chess-board problem: in how many ways can n kings be placed on an n by n board, one on each row and column, so that no two attack each other? We recognize a first cousin (albeit a shabby relative) of the famous "n-queens problem."[5]

We have to evaluate

$$A_n = (1 - q_{12})(1 - q_{21})(1 - q_{23})(1 - q_{32}) \cdots (1 - q_{n-1,n})(1 - q_{n,n-1}).$$

Denote by B_n the result of deleting from A_n the last factor. Then

$$A_n = B_n(1 - q_{n,n-1}), \tag{13}$$

$$B_n = A_{n-1}(1 - q_{n-1,n}). \tag{14}$$

Further we find

$$A_{n-1}q_{n-1,n} = B_{n-1}(1 - q_{n-1,n-2})q_{n-1,n} = B_{n-1}q_{n-1,n}. \tag{15}$$

From (14) and (15):

$$B_n = A_{n-1} - B_{n-1}q_{n-1,n}. \tag{16}$$

On substituting (16) in (13) we get

$$A_n = B_n - A_{n-1}q_{n,n-1}. \tag{17}$$

Now in the product $B_{n-1}q_{n-1,n}$ it is readily checked that the latter factor does not enter into any vanishing products. Its effect therefore is simply to multiply the polynomial B_{n-1} by E. That is to say, $B_{n-1}q_{n-1,n}=EB_{n-1}$, whence we can rewrite (16) as

$$B_n = A_{n-1} - EB_{n-1}. \tag{18}$$

Moreover it is clear from symmetry that $A_{n-1}q_{n-1,n}=A_{n-1}q_{n,n-1}$. From (15) and (17) we therefore obtain

$$A_n = B_n - EB_{n-1}. \tag{19}$$

Elimination of A from (18) and (19) yields

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[5] If there were a chess piece "*" which combined the moves of rook and king, the name "n-* problem" would be more suitable than n-kings. It might be added that the author has failed in several efforts to solve the n-queens problem by these methods.

$$B_n = (1 - E)B_{n-1} - EB_{n-2}. \tag{20}$$

The solution of the difference equation (20), with initial values $B_1=1$, $B_2=1-E$, is

$$B_n = (u^n - v^n)/(u - v)$$

where u, $v=(1-E\pm(1-6E+E^2)^{1/2})/2$. A_n can be found from (19), and $A_n\phi_0$ is the solution of the n-kings problem. The first 3 terms are:

$$A_n = 1 - (2n - 2)E + (2n^2 - 8n + 8)E^2 \cdots .$$

Check for $n=4$:

$$\begin{aligned} B_4 &= u^3 + u^2v + uv^2 + v^3 = 1 - 5E + 5E^2 - E^3, \\ B_3 &= u^2 + uv + v^2 = 1 - 3E + E^2, \\ A_4 &= B_4 - EB_3 = 1 - 6E + 8E^2 - 2E^3, \\ A_4\phi_0 &= (4! - 6\cdot 3! + 8\cdot 2! - 2\cdot 1!)/4! = 2/4!, \end{aligned}$$

corresponding to the two permissible arrangements 3142 and 2413.

6. **A problem of Netto.** In [7, p. 84], the following problem is proposed. Let the integers $1, \cdots, n$ be divided into n/a sets of a each. What is the probability that in a permutation there are nowhere two consecutive elements from the same subset? Thus for a deck of cards ($a=4$), we require the probability of going through the deck and never encountering two consecutive cards of the same denomination (aces, deuces, and so on).

If the integers $1, \cdots, a$ are taken to form the first subset, we have to compute

$$V_a = \prod (1 - q_{ij}),$$

the product taken over all $i, j \leq a$ with $i \neq j$; then the answer to Netto's problem will be $V_a^{n/a}$. Let W_a denote the result of deleting from V_a the $a-1$ factors $(1-q_{ia})$ $(i=1, \cdots, a)$. Then

$$V_a = W_a(1 - q_{1a}) \cdots (1 - q_{a-1,a}) = W_a - (a - 1)W_aq_{a-1,a}.$$

Now in the product $W_aq_{a-1,a}$ the effect of multiplication by $q_{a-1,a}$ will be to delete from W_a the remaining terms of the form $q_{a-1,i}$ and also the term $q_{a,a-1}$. In the product that remains it is now merely a harmless change of notation to replace a by $a-1$, after which precisely V_{a-1} is left. Hence

$$\begin{aligned} W_aq_{a-1,a} &= EV_{a-1}, \\ V_a &= W_a - (a - 1)EV_{a-1}. \end{aligned} \tag{21}$$

By a repetition of virtually the same argument we can show that

$$W_a = V_{a-1} - (a-1)EW_{a-1}. \tag{22}$$

By eliminating W from (21) and (22), we obtain

$$V_a = (1 + 2E - 2aE)V_{a-1} - (a-1)(a-2)E^2V_{a-2}.$$

The solution of this difference equation with initial values $V_1 = 1$, $V_2 = 1 - 2E$ is $F(a, a-1)$ in the notation of (5). Our final answer is thus $[F(a, a-1)]^{n/a}\phi_0$.

With $a = 4$ the result becomes

$$(1 - 12E + 36E^2 - 24E^3)^{n/4}\phi_0 = e^{-3} + O(n^{-1}).$$

The odds are about 19 to 1, although to the author's knowledge this game has been played at even money!

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ORBIT-CLOSURE DECOMPOSITIONS AND ALMOST PERIODIC PROPERTIES

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Let X be a metric space with metric ρ, let $f(X) \subset X$ be a continuous mapping, and let $h(X) = X$ be a homeomorphism. For $x \in X$, the set $\sum_{n=0}^{+\infty} f^n(x)$ is called the *semi-orbit* of x under f and the set $\sum_{n=-\infty}^{+\infty} h^n(x)$ is called the *orbit* of x under h. For $x \in X$, the closure of the semi-orbit of x under f is called the *semi-orbit-closure* of x under f and the closure of the orbit of x under h is called the *orbit-closure* of x under h.

A nonvacuous subset Y of X is said to be *semi-minimal* (*minimal*) under $f(h)$ provided that the semi-orbit-closure (orbit-closure) of each point of Y is Y. Clearly, any two semi-minimal (minimal) sets are either coincident or disjoint. It is easily proved that a subset Y of X is semi-minimal (minimal) under $f(h)$ if and only if Y is nonvacuous, closed, $f(Y) \subset Y(h(Y) = Y)$, and furthermore Y contains no proper subset with these properties. We follow Birkhoff [2, p. 198][1] in the terminology of "minimal set."

A *decomposition* of X is defined to be a collection of nonvacuous closed pairwise disjoint subsets of X which fill up X. We say that the mapping f *gives a semi-orbit-closure* (*a semi-minimal set*) *decomposition* provided that the collection of semi-orbit-closures (semi-minimal sets) is a decomposition of X. Also, it is said that the homeomorphism h *gives an orbit-closure* (*a minimal-set*) *decomposition* provided that the collection of orbit-closures (minimal sets) is a decomposition of X.

A point x of X is said to be *almost periodic* under f provided that to each $\epsilon > 0$ there corresponds a positive integer N with the property that in every set of N consecutive positive integers appears an integer n such that $\rho(x, f^n(x)) < \epsilon$. The mapping f is said to be *pointwise almost periodic* provided that each point of X is almost periodic under f. It is to be noted that various writers use the above terms in different senses and employ other terminologies for these notions.

LEMMA 1. *The mapping f (homeomorphism h) gives a semi-orbit-closure (an orbit-closure) decomposition if and only if $f(h)$ gives a semi-minimal-set (a minimal-set) decomposition; and in either event, the two decompositions coincide.*

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[1] Numbers in brackets refer to the bibliography at the end of the paper.

The proof is easy and will be omitted.

LEMMA 2. *In order that the homeomorphism h give an orbit-closure decomposition it is sufficient that h give a semi-orbit-closure decomposition; and in case X is compact, this condition is also necessary. In either event, the two decompositions coincide.*

PROOF. The proof of the sufficiency is easy and will be omitted. We establish the necessity. Let C be an orbit-closure. By Lemma 1, it is enough to show that C is a semi-minimal set. Let Y be a nonvacuous closed subset of C such that $h(Y)\subset Y$. The proof will be completed if we show $Y=C$. Define $Z=\prod_{n=0}^{+\infty}h^n(Y)$. Now Z is a nonvacuous closed subset of C such that $h(Z)=Z$. Since C is a minimal set by Lemma 1, $Z=C$ and, hence, $Y=C$.

LEMMA 3. *If $x\in X$ is almost periodic under f, then the semi-orbit-closure C of x is semi-minimal.*

PROOF. Suppose C is not semi-minimal. Then there exists a point y of C such that the semi-orbit-closure Y of y is not C. Now $Y\subset C$. Also $x\notin Y$, since otherwise $C\subset Y$ and thus $Y=C$. Let 2ϵ be the distance from x to Y. There exists a positive integer N such that in every set of N consecutive positive integers appears an integer n so that $\rho(x, f^n(x))<\epsilon$. Choose $\delta>0$ so small that $z\in X$ with $\rho(y, z)<\delta$ implies $\rho(f^i(y), f^i(z))<\epsilon$ $(i=1, 2, \cdots, N)$. Now there exists a nonnegative integer p such that $\rho(y, f^p(x))<\delta$. Also it is possible to find an integer q, $1\leqq q\leqq N$, so that $\rho(x, f^{p+q}(x))<\epsilon$. Hence, $\rho(x, f^q(y))<2\epsilon$ which is impossible.

LEMMA 4. *If X is locally compact and if the subset Y of X is semi-minimal under f, then each point of Y is almost periodic.*

PROOF. Assume some point x of Y is not almost periodic. There exists a neighborhood U of x such that $\overline{U}$ is compact and such that for each positive integer n there exists a point x_n of $U\cdot Y$ with the property that $f^m(x_n)\notin U$ $(m=1, 2, \cdots, n)$. Some subsequence of $x_1, x_2, \cdots$ converges to some point, say y, of $\overline{U}\cdot Y$. There exists a positive integer M such that $f^M(y)\in U$ and, hence, there also exists a neighborhood V of y such that $f^M(V)\subset U$. Choose an integer N so that $N>M$ and $x_N\in V$. Then, $f^M(x_N)\in U$ which is a contradiction.

THEOREM 1. *In order that the mapping f give a semi-orbit-closure decomposition, it is sufficient that f be pointwise almost periodic; and in case X is locally compact, this condition is also necessary.*

The proof follows easily from Lemmas 1, 3 and 4.

Theorem 2. *In order that the homeomorphism h give an orbit-closure decomposition it is sufficient that h be pointwise almost periodic*; *and in case X is compact, this condition is also necessary.*

The proof proceeds easily from Theorem 1 and Lemma 2.

The preceding results are related to certain results of Birkhoff [2, p. 199] and Hall and Kelley [4]. Gottschalk [3] contains a direct proof of Theorem 2.

We pause in our main development to comment on the rôle of local compactness in the second part of Theorem 1. In this case the semi-orbit-closures are actually compact, as the following indicates.

Theorem A. *If $x\in X$ is almost periodic under f and if there exists a neighborhood U of x whose closure is compact, then the semi-orbit-closure of x is itself compact.*

Proof. There exists an integer N and a sequence $\{n_i \mid i=0, 1, \cdots\}$ of integers such that $0=n_0<n_1<\cdots$, $n_{i+1}-n_i\leqq N$ and $f^{n_i}(x)\in U$ for all i. Define $K=\sum_{r=0}^{N} f^r(\overline{U})$. Now K is compact. We show the semi-orbit of x is contained in K, which completes the proof. Let n be any non-negative integer. There exists a non-negative integer i such that $n_i\leqq n\leqq n_{i+1}$. Hence, $f^n(x)=f^{n-n_i}f^{n_i}(x)\in f^{n-n_i}(U)\subset K$.

Theorem B. *If $Y\subset X$ is semi-minimal under f and if Y intersects a neighborhood U whose closure is compact* (*in particular, if X is locally compact*), *then Y is itself compact.*

Proof. Let $y\in Y\cdot U$. By the argument used in the proof of Lemma 4, it can be shown that y is almost periodic. The conclusion now follows from Theorem A. (Theorem B is not valid for a minimal set as the example of a discrete infinite orbit shows.)

Besicovitch [1] has constructed an interesting example of a homeomorphism of the plane onto itself which possesses *some* semi-orbits dense in the plane and which leaves the origin fixed. He seems to remark at the end of his paper (p. 65) that *every* semi-orbit, excluding the origin, is also dense in the plane. Theorem B would indicate that either this remark or our interpretation of it is in error. For, take $X=Y$ to be the plane with the origin deleted. If every semi-orbit is dense in the punctured plane, then the punctured plane would be compact. *Question*: *In Besicovitch's example* [1], *is the orbit of every point of the plane, excepting the origin, dense in the plane?* We now continue with the main sequence of theorems.

Let $\{X_n \mid n=1, 2, \cdots\}$ be a sequence of subsets of X. The set of all points x of X such that each neighborhood of x intersects X_n for almost all (infinitely many) positive integers n is denoted by $\liminf \{X_n\}$ ($\limsup \{X_n\}$). In case $\liminf \{X_n\} = \limsup \{X_n\}$, we denote this set by $\lim \{X_n\}$. Of course, for any sequence $\{X_n\}$, $\liminf \{X_n\} \subset \limsup \{X_n\}$.

Let D be a decomposition of X. For $x \in X$, let $D(x)$ denote the element of D containing x. The decomposition D is said to be *continuous* provided that $x_0, x_n \in X$ ($n=1, 2, \cdots$) with $x_n \rightarrow x_0$ implies $D(x_0) \subset \liminf \{D(x_n)\} \subset \limsup \{D(x_n)\} \subset D(x_0)$.

Following G. A. Hedlund, we say the mapping f is *uniformly pointwise almost periodic* provided that to each $\epsilon > 0$ there corresponds a positive integer N such that if $x \in X$, then in every set of N consecutive positive integers appears an integer n so that $\rho(x, f^n(x)) < \epsilon$. Clearly, if f is uniformly pointwise almost periodic, then f is pointwise almost periodic.

LEMMA 5. *For $x \in X$, let $C(x)$ denote the semi-orbit-closure of x under f. If $x_0, x_n \in X$ ($n=1, 2, \cdots$) with $x_n \rightarrow x_0$, then $C(x_0) \subset \liminf \{C(x_n)\}$.*

PROOF. Let $x \in C(x_0)$ and let U be any neighborhood of x. For some non-negative integer k, $f^k(x_0) \in U$. By the continuity of f^k, $f^k(x_n) \in U$ for almost all positive integers n, that is, U intersects $C(x_n)$ for almost all positive integers n. The conclusion follows.

THEOREM 3. *In order that the mapping f give a continuous semi-orbit-closure decomposition it is sufficient that f be uniformly pointwise almost periodic; and in case X is compact, this condition is also necessary.*

PROOF. We establish the sufficiency. Let D denote the collection of semi-orbit-closures. By Theorem 1, D is a decomposition of X. By Lemma 5, it is enough to prove that $\limsup \{C(x_n)\} \subset C(x_0)$ for $x_0, x_n \in X$ ($n=1, 2, \cdots$) with $x_n \rightarrow x_0$, where $C(x)$ denotes the semi-orbit-closure of the point x. Assume this is false. Then there exist points $x_0, x_n \in X$ ($n=1, 2, \cdots$) such that $x_n \rightarrow x_0$ and $\limsup \{C(x_n)\} \not\subset C(x_0)$. Thus there exist a point x of X, a monotone increasing sequence $n_1, n_2, \cdots$ of positive integers, and a sequence $m_1, m_2, \cdots$ of non-negative integers such that $f^{m_i}(x_{n_i}) \rightarrow x$ and $x \notin C(x_0)$. Hence, $C(x) \cdot C(x_0) = \Lambda$ and $x_0 \notin C(x)$. Let 2ϵ denote the distance from x_0 to $C(x)$. Since f is uniformly pointwise almost periodic, there exists a positive integer k such that for each positive integer i it is possible to find an integer k_i with the properties that $1 \leqq k_i \leqq k$ and $\rho(x_{n_i}, f^{m_i+k_i}(x_{n_i})) < \epsilon$. There exists an integer k_0 such that $k_i = k_0$ for infinitely many positive integers i. Since also $x_{n_i} \rightarrow x_0$ and $f^{m_i+k_0}(x_{n_i}) \rightarrow f^{k_0}(x)$, we have $\rho(x_0, f^{k_0}(x)) \leqq \epsilon < 2\epsilon$. Hence, the distance from x_0 to $C(x)$ is less than 2ϵ. This is a contradiction.

We establish the necessity. Suppose f is not uniformly pointwise almost periodic. Then there exists a positive number ϵ such that for each positive integer n it is possible to find a positive integer m_n and a point x_n of X such that $N_\epsilon(x_n)\cdot\sum_{i=0}^{n} f^{m_n+i}(x_n)=\Lambda$, where $N_\delta(y)$ denotes the δ-neighborhood of the point y. We may suppose that $x_n\rightarrow x_0$ and $f^{m_n}(x_n)\rightarrow x$ for some points x_0, x of X. For all sufficiently large positive integers n, $N_{\epsilon/2}(x_0)\cdot\sum_{i=0}^{n} f^{m_n+i}(x_n)=\Lambda$. It follows that $N_{\epsilon/2}(x_0)\cdot C(x)=\Lambda$ and $x_0\notin C(x)$, where $C(y)$ denotes the semi-orbit-closure of the point y. Now $\lim \{C(x_n)\}=C(x_0)$ and $C(x) = \lim \{C(f^{m_n}(x_n))\} = \lim \{C(x_n)\}$. Hence, $C(x_0)=C(x)$ and $x_0\in C(x)$. This is a contradiction.

THEOREM 4. *In order that the homeomorphism h give a continuous orbit-closure decomposition it is sufficient that h be uniformly pointwise almost periodic; and in case X is compact, this condition is also necessary.*

The proof follows readily from Theorem 3 and Lemma 2.

It is worthy of note that if X is compact, then there exists a subset Y of X such that $f(Y)=Y$ is uniformly pointwise almost periodic. The proof is short. The property P of being a nonvacuous closed subset Z of X such that $f(Z)\subset Z$ is easily shown to be inducible. By the Brouwer reduction theorem, there exists a subset Y which has property P irreducibly. Then Y is semi-minimal and $f(Y)=Y$, since $f(Y)$ has property P. The conclusion now follows from Theorem 3.

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ON CONTINUA WHOSE LINKS ARE NON-INTERSECTING

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There are many examples known of compact continua no two of whose simple links intersect, even though uncountably many of the links are nondegenerate. On page 346 of his *Foundations of point set theory* [1],[1] R. L. Moore has given an example of a compact continuum which is the sum of an uncountable collection of nondegenerate simple links, only a countable number of which intersect. In view of these results it seems natural to inquire whether there is a compact continuum which is the sum of an uncountable collection of mutually exclusive nondegenerate simple links. Theorem 1 gives a negative answer. If we do not require that the continuum be compact, but do require that it be a separable locally connected Moore space,[2] then Theorem 7 shows that the answer is again negative.

References to theorem and page numbers refer to Moore's book, to which the reader is directed for definitions not given here.

THEOREM 1.[3] *If M is a compact metric continuum which is not a simple link of itself and no two of its simple links intersect, then uncountably many are degenerate, and the set which is the sum of the nondegenerate simple links of M is the sum of a countable number of closed point sets.*[4]

LEMMA. *Under the hypothesis of Theorem 1, the simple links of M form an upper semi-continuous collection, G, filling up M, such that G is an acyclic continuous curve every interval of which contains a degenerate element of G.*

PROOF. By Theorem 93, p. 67, each simple link of M is a continuum. Let P be the sequential limiting point of a sequence of points of distinct links L_1, L_2, L_3, $\cdots$ of M. Let Q be any point of the limiting set of L_1, L_2, L_3, $\cdots$. Clearly no point separates P from Q

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[1] Numbers in brackets refer to the Bibliography at the end of the paper.

[2] A space is said to be a Moore space if it satisfies Axiom 0 and the first three parts of Axiom 1 of Moore's book.

[3] In the terminology of transformations, Theorem 1 is equivalent to the following: Let f be a monotone mapping of the compact continuum M into a dendrite N. If for each point X of N no point of M separates any two points of $f^{-1}(X)$ and for each two points X and Y of N some point of M separates $f^{-1}(X)$ from $f^{-1}(Y)$, then for uncountably many points X of N the set $f^{-1}(X)$ is degenerate.

[4] The proof of this theorem was altered after refereeing.

in M, and since by Theorem 103, p. 71, no point of M is an improper point of M, it follows that the collection G is upper semi-continuous. By Theorem 23(c), p. 344, G is a dendrite with respect to its elements. Suppose that the interval AB of G contains no degenerate element. Then the set, K, of all points each of which separates A from B in M is countable, for no link can contain more than two points of K, and by Theorem 24(a), p. 350, no more than a countable number of nondegenerate links can contain cut points of M. However, the collection, T, of all simple links of M in AB that intersect K is dense in AB. From Theorem 24, p. 348, we have that G contains an uncountable continuous subcollection, C, dense in AB.[5] Let P be a point of K contained in a link L, and let M' denote the subcontinuum of M consisting of the sum of the elements of AB. Then $M'-P$ is the sum of two mutually separated sets, H_1 and H_2, containing A and B respectively; and since L separates AB into two connected sets of elements of G, $M'-P=L_1+L_2+(L-P)$, where L_1 and L_2 are mutually separated connected sets not intersecting L. If $L-P$ contains limit points of both L_1 and L_2, then it intersects both K_1 and K_2, which contradicts Theorem 98, p. 70. Hence $L-P$ and H_1, say, are mutually separated. But since C is dense in AB it follows that there is a sequence $L_1, L_2, L_3, \cdots$ of elements of C whose limiting set (in M) is P. Since G is upper semi-continuous, as $n \to \infty$, diam $L_n \to 0$. Let H be an element of C, and let d be its diameter. Every open segment of AB containing H also contains elements of T, and therefore contains elements of C of diameter less than $d/2$, which contradicts the continuity of C. Hence AB contains a degenerate link of M.

PROOF OF THEOREM 1. By the lemma and Theorem 5.1, p. 131, of [4], the collection G' of all nondegenerate elements of G is semi-closed, and each element of G' is a component of G'^*. By Theorem 5.41, p. 132, of [4], G'^* is the sum of a countable number of closed point sets. If the degenerate elements of G were countable in number, the decomposition space G would be the sum of a countable number of closed sets, no one of which contains a domain, which would contradict Theorem 15, p. 11.

A well known example due to Anna M. Mullikin [2], and described on p. 366 of Moore's book, of a connected set in the plane which is the sum of a countable number of mutually exclusive arcs, can be modified by replacing each arc by a simple closed curve to show that Theorem 1 is not true if the hypothesis of compactness is replaced by separability, even if each link is a compact continuum.

Another argument for Theorem 1 could be given by proving that under the hypothesis of Theorem 1 every simple link of M is a set $M(P)$.[6] Then Theorem 1 would follow from Theorem 2, which has some interest, due to the connection between dendratomic subsets, simple links, and sets $M(P)$ shown by Moore in chap. 5 of his book.

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[5] Theorem 24 does not state that C is dense in AB, but this readily follows here.

THEOREM 2. *If the compact metric continuum M is not an $M(P)$, then uncountably many sets $M(P)$ of M are degenerate.*

PROOF. By Theorem 23(a), p. 343, the collection, G, of all sets $M(P)$ of M is upper semi-continuous. By Theorem 24(a), p. 350, no more than a countable number of nondegenerate elements of G contain cut points of M. By hypothesis M has uncountably many cut points. If the theorem were false some element of G would contain uncountably many cut points of M, which contradicts Theorem 23(b), p. 344.

Since every simple link is a subset of some set $M(P)$, this theorem says that if the simple links are allowed to intersect in certain groups so that no two points in any group are separated by more than a countable number of points, but so that any two points in different groups are separated by uncountably many points, then the conclusion of Theorem 1 still holds.

Locally connected continua. If to the hypothesis of Theorem 1 we add the condition that M be locally connected, our theorem becomes trivial, since the nondegenerate simple links of a compact continuous curve are its true cyclic elements, and there are only a countable number of these. If we also remove the condition of compactness, then the problem becomes more interesting, and indeed the theorem would not be true without the assumption of separability. However, if M were assumed to be locally connected, separable, and complete, the theory of almost cyclic elements developed by F. B. Jones [3] would apply and our theorem could readily be proved, in virtue of Theorem 19 of his paper. Failure to assume completeness creates more difficulties, since arcwise connectivity would not be available. The theorem remains true, however, and the proof is not difficult. We prove first several theorems concerning simple links.

In the next four theorems, M will denote a locally connected, connected Moore space.

THEOREM 3.[7] *If L is a simple link of M, and R is a connected domain intersecting L, then $L \cdot R$ is connected. Hence L is connected and locally connected.*

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[6] If P is a point of M, by $M(P)$ is meant the set of all points of M which are not separated from P by each of an uncountable set of points.

[7] Cf. Theorem 66, p. 147, of Moore, and Theorems 8 and 20 of Jones' paper.

PROOF. Suppose that $L \cdot R$ is the sum of two mutually separated sets, H and K. Some component, C, of $R - R \cdot L$ has a limit point, P, in H, and a limit point, Q, in K, as may be seen with the use of Theorem 2(b), p. 91. Let X be any point of C. No point of L separates P or Q from X in R. But by Theorem 12, p. 97,[8] some point Y separates X from $P+Q$ in R and is not itself separated from $P+Q$ by any point. Let Z be a proper point of M belonging to L. Some point, W, separates Y from Z in M, and W can be neither P nor Q. But then W cannot belong to L, and hence separates $P+Q$ from Y, which is impossible.

If R is taken to be M, it follows that L is connected.

THEOREM 4.[9] *If M is the sum of its simple links, and no two simple links of M intersect, then every improper point belongs to a nondegenerate continuum of improper points, no two lying in the same simple link.*

PROOF. Let X be an improper point of M. Let H be a point set containing X such that no two points of H belong to the same simple link of M or are separated by any point, and which is maximal with respect to this property. Since the simple links of M are mutually exclusive, H is nondegenerate. Also, every point of H is an improper point. Further, H is closed, for if the point Y of $M-H$ is a limit point of H, then no point separates Y from any point of H. Hence Y belongs to the same simple link as some point Z of H. But then Z separates Y from $H-Z$, which is impossible.

Suppose that H is the sum of two mutually exclusive closed sets, H_1 and H_2. Paralleling the argument of Theorem 3, we show the existence of a point Q of $M-H$ and two points, P_1 and P_2, belonging to H_1 and H_2 respectively, such that no point separates Q from both P_1 and P_2. Hence no point separates Q from any point of H. Further, Q is not in the same simple link as any point of H. Therefore Q belongs to H, which is a contradiction, and completes our proof.

THEOREM 5. *If M is the sum of its simple links and is not a simple link of itself, and the simple links of M are non-intersecting, they form an uncountable collection.*

PROOF. Suppose that the theorem is false. Then every point of M is a proper point of M, for if P were an improper point of M, by Theorem 4, P would lie in a nondegenerate continuum H of points, no two of which lie in the same simple link of M. But H must be countable, and no nondegenerate countable set is connected.[10] Hence, if A and B belong to distinct simple links of M, the set $I(A, B)$ of all points separating A from B is non-empty, and indeed any two points of $I(A, B)+A+B$ belonging to distinct links are separated by some point. Since no link can contain more than two points separating A from B, $I(A, B)$ is countable. It is then easy to construct two sequences of points, $P_1, P_2, P_3, \cdots$ and $Q_1, Q_2, Q_3, \cdots$ such that (1) for each positive integer i, P_{i+1} precedes P_i, Q_{i+1} follows Q_i, and P_i follows Q_i in the order from A to B in $I(A, B)$; and (2) for any point X of $I(A, B)$ there is an i such that X is not between P_i and Q_i. But $P_1+P_2+P_3+\cdots$ cannot then have a limit point in $I(A, B)$, which contradicts Theorem 4.2, p. 51, of [4], the proof of which is valid in this space.

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[8] See remarks on p. 465 of Moore concerning this theorem.

[9] This theorem should be contrasted with Theorem 99, p. 71.

The example given following Theorem 1 shows that this theorem is not true without local connectivity.

The condition that M is the sum of its simple links in the hypothesis of this theorem is necessary, for there exist non-compact continuous curves which do not contain any simple links whatsoever. We give a rough description of one such: Jones has given on p. 781 of [3] an example of a space, H_1, containing a simple closed curve, C, each of whose points is an improper point, and each of whose simple links is a simple closed curve, $C_{1,\alpha}$, having only a point in common with C. By applying Jones' construction to each set $C_{1,\alpha}$, we can construct a space H_2 in which every point of H_1 is an improper point, and whose simple links are simple closed curves, $C_{2,\alpha}$, having only one point in common with H_1. Repeating Jones' construction again on the sets $C_{2,\alpha}$, we obtain a space H_3 in which every point of H_2 is an improper point. Continue this process indefinitely. The space H which is the limit of $H_1, H_2, H_3, \cdots$ is such that every point is an improper point, and hence H has no simple links at all.

THEOREM 6. *If M is separable, the nondegenerate simple links of M orm a countable collection, and each nondegenerate link is separable.*

PROOF. By hypothesis, M contains a countable dense subset, H. By Theorem 12, p. 97, each nondegenerate simple link contains a cut point of M. Suppose that the collection, G, of all nondegenerate simple links of M is uncountable. Either (1) there is a point, P, common to uncountably many elements of G, or (2) with Theorem 72, p. 50, and Theorem 68, p. 47, there can be shown to exist an uncountable subcollection, G', of G, two points, A and B, and a set, K, such that each element of G' contains one and only one point of K, and such that each point of K separates A from B, and belongs to an element of G'. In the last case, each element, L, of G' is intersected by a connected domain, D_L, whose closure does not intersect $I(A, B)+A+B$ (which by Theorem 4.2, p. 51, of [4] is closed and compact). There is a point, X, of H which is common to uncountably many of the domains D_L. Let G'' denote the collection of all links, L, of G' such that D_L contains X, and let K' denote the set of all points of K intersecting links of G''. Some point, Y, is a point of condensation of K', and there is a connected domain, D, containing Y, such that for uncountably many elements L of G'', $D_L \cdot D=0$. But then no point separates a point of an element of G'' from any point of another element of G'', which is impossible. A similar argument shows that the first case is likewise impossible, P taking the role of Y.

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[10] Since a countable Moore space is completely separable, and hence metric, it cannot be connected, if it is nondegenerate.

We show that each simple link is separable. Let L be any nondegenerate link of M, and let H be a countable dense subset of M. For each point X of $H-H\cdot M$, there is a point, P_X, of L which separates X from some point of L. The set, H', consisting of $H\cdot L$ and all points P_X is countable. If D is a connected domain intersecting L, and Y is a point of $H\cdot D$, either Y is in L or P_Y is in D. In either case, D intersects H'. Hence H' is a countable dense subset of L.

The last part of the theorem is false without local connectivity.

THEOREM 7. *If M is separable, and is not a simple link of itself, but is the sum of its simple links, and no two links of M intersect, then uncountably many are degenerate.*

PROOF. This follows immediately from Theorems 5 and 6.

The example of Jones referred to following Theorem 4 shows that the condition of separability in Theorem 7 cannot be removed. A simple modification of this example gives a similar example where in addition the almost cyclic elements are mutually exclusive.

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ON THE QUADRICS ASSOCIATED WITH A POINT OF A SURFACE

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It is well known that among the three-parameter family of quadrics having a contact of the second order with an analytic surface at a point there are two systems of quadrics known as the quadrics of Darboux and the quadrics of Moutard. The particular quadric of the first family introduced in this note is of some interest. We propose to give a new geometrical construction for the cone of Segre by using Moutard quadrics and study the canonical pencil of lines associated with the surface.

Let $x^i(u, v)$ $(i=1, \cdots, 4)$ be the projective normal coordinates of a point on a surface S referred to its asymptotic net (u, v); then the functions x are solutions of a completely integrable system of differential equations in Fubini's canonical form

$$\begin{aligned} x_{uu} &= \theta_u x_u + \beta x_v + p_{11} x, \\ x_{vv} &= \gamma x_u + \theta_v x_v + p_{22} x, \\ \theta &= \log \beta\gamma. \end{aligned} \tag{1}$$

As usual we introduce nonhomogeneous local coordinates of a point with respect to the tetrahedron $(x x_u x_v x_{uv})$, so that the equation of any quadric of Darboux is found to be

$$z - xy + kz^2 = 0, \tag{2}$$

where k denotes a parameter, while the equation of the quadric of Moutard which belongs to the tangent $y - nx = 0$, $z = 0$ is

$$\begin{aligned} &[4(\beta + \gamma n^3)^2 + 3n(\beta_u + 4\beta_v n + 4\gamma_u n^3 + \gamma_v n^4)]z^2 \\ &\qquad + 36n^3\left[z - xy + \frac{z^2}{2}\,\frac{\partial^2 \log \beta\gamma}{\partial u \partial v}\right] \\ &\qquad - 12n(\beta - 2\gamma n^3)\left(y + \frac{1}{2}\,\frac{\partial \log \beta\gamma}{\partial u}\,z\right)z \\ &\qquad - 12n^2(\gamma n^3 - 2\beta)\left(x + \frac{1}{2}\,\frac{\partial \log \beta\gamma}{\partial v}\,z\right)z = 0. \end{aligned} \tag{3}$$

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These quadrics intersect each other in the asymptotic tangents and a conic in the plane

$$z\left[4(\beta+\gamma n^3)^2+3n(\beta_u+4\beta_v n+4\gamma_u n^3+\gamma_v n^4)+36n^3K\beta\gamma - 6n\frac{\partial\log\beta\gamma}{\partial u}(\beta-2\gamma n^3)-6n^2\frac{\partial\log\beta\gamma}{\partial v}(\gamma n^3-2\beta)\right] - 12n(\beta-2\gamma n^3)y-12n^2(\gamma n^3-2\beta)x=0,$$

where we have placed

$$k=\frac{1}{2}\frac{\partial^2\log\beta\gamma}{\partial u\partial v}-K\beta\gamma. \tag{4}$$

The plane coordinates of this plane are evidently

$$\begin{aligned} u_0 &= 0,\\ \rho u_1 &= -12n^2(\gamma n^3-2\beta), \qquad \rho u_2 = -12n(\beta-2\gamma n^3),\\ \rho u_3 &= 4\beta^2-3\beta\phi+12\beta n^2\psi+(8\beta\gamma+36K\beta\gamma)n^3+12\gamma\phi n^4\\ &\quad -3\gamma n^5\psi+4\gamma^2n^6, \end{aligned} \tag{5}$$

where $\phi=\partial\log\beta\gamma^2/\partial u$ and $\psi=\partial\log\beta^2\gamma/\partial v$, so that the cone enveloped by it may be obtained by eliminating n from the equations

$$\begin{aligned} \pi n^3+\omega n^2+tn+s&=0,\\ \sigma n^3+\tau n+\omega n+p&=0, \end{aligned} \tag{6}$$

where

$$\begin{aligned} \omega &= 7u_1\psi/2+7u_2\phi/2-6u_3,\\ \pi &= u_2\psi-2\gamma u_1, & p &= u_1\phi-2\beta u_2,\\ s &= (4+6K)\beta u_1, & \sigma &= (4+6K)\gamma u_2,\\ t &= u_1\phi+6(1+2K)\beta u_2, & \tau &= u_2\psi+6(1+2K)\gamma u_1. \end{aligned}$$

The result of carrying out the computation for a given value of K is

$$\begin{vmatrix} \pi & \omega & t & s\\ \sigma & \tau & \omega & p\\ 0 & p\pi-\sigma s & \omega p-\tau s & pt-\omega s\\ \sigma\omega-\pi\tau & \sigma t-\pi\omega & \sigma s-\pi p & 0 \end{vmatrix}=0. \tag{7}$$

It should be remarked that when and only when $K=-2/3$ the equation of the cone enveloped by the planes (5) is different from (7), since the latter in this case becomes an identity. From (2) and (4) we may infer naturally that *in the pencil of Darboux the quadric*

$$z - xy + \left(\frac{1}{2}\frac{\partial^2 \log \beta\gamma}{\partial u \partial v} + \frac{2}{3}\beta\gamma\right)z^2 = 0 \tag{8}$$

differs from other quadrics by a certain geometrical characterization.

The third polar of $z=0$ with respect to the cone (7) is

$$6u_1u_2u_3 - 3(u_1\psi + u_2\phi)u_1u_2 - (\beta u_2^3 + \gamma u_1^3) = 0, \tag{9}$$

which does not contain K. That is to say, *all the cones* (7) *with different values of K have the third polar of $z=0$ in common.*

As to the second polar of the same plane $z=0$ we have

$$[\cdots]u_1u_2 + u_1u_2\omega_2^2 + 6[(u_1\psi + u_2\phi)u_1u_2 - 2(\beta u_2^3 + \gamma u_1^3)] = 0 \tag{10}$$

where $[\cdots]$ does not contain ω. *Hence through each tangent of the surface S at P there are besides $z=0$ two tangent planes p_1 and p_2 of the cone* (10) *and the harmonic conjugate plane of $z=0$ with respect to p_1 and p_2 just envelopes the cone* (9). Thus we obtain a correspondence between a tangent of the surface and a plane through it. Especially, *the three corresponding planes of the tangents of Darboux are concurrent in the first directrix of Wilczynski.*

The equation of the cone (9) in point coordinates is found to be

$$\begin{aligned} -3\left(y+\frac{\phi}{2}z\right)^2\left(x+\frac{\psi}{2}z\right)^2 &+ 2\gamma z\left(y+\frac{\phi}{2}z\right)^3 \\ &+ 2\beta z\left(x+\frac{\psi}{2}z\right)^3 - \frac{3}{2}\beta\gamma\left(x+\frac{\psi}{2}z\right)\left(y+\frac{\phi}{2}z\right)z^2 \\ &+ \frac{\beta^2\gamma^2}{16}z^4 = 0. \end{aligned} \tag{11}$$

It follows that $z=0$ *is the only double tangent plane along the two asymptotic tangents of S and therefore that there are three cuspidal generators, each passing through one of the points*

$$\left(0, \frac{\omega^2}{2}(\beta\gamma^2)^{1/3} - \frac{\psi}{2}, \frac{\omega}{2}(\gamma^2\beta)^{1/3} - \frac{\phi}{2}, 1\right), \qquad (\omega^3 = 1, \omega \neq 1).$$

The harmonic conjugate line of $z=0$ with respect to the trihedral of the three cuspidal generators is also the first directrix of Wilczynski. Furthermore the three cuspidal tangent planes of the cone (9) *are concurrent in the same directrix.*

It is worth noticing that the cones (10) with variable K have a common tangent plane which passes through each asymptotic tangent. The two planes thus obtained intersect in the canonical line $C(-7/12)$.[2]

Let us now attend to the particular quadric (8). Eliminating n from the equations

$$\pi n^2 + \omega n + p = 0,$$
$$\beta u_1(1 - 2(\gamma/\beta)n^3) = n(\gamma n^3 - 2\beta)u_2,$$

we obtain that the cone in this case is

$$(12) \qquad \begin{vmatrix} p\gamma u_2 & 2\gamma u_1 p & \pi\beta u_1 & \omega\beta u_1 - 2\beta u_2 p \\ -2\gamma\pi u_1 + \gamma u_2\omega & \gamma u_2 p & 2\beta\pi u_2 & \beta u_1\pi \\ \pi & \omega & p & 0 \\ 0 & \pi & \omega & p \end{vmatrix} = 0.$$

The second polar of the plane $z=0$ with respect to this cone is

$$(13) \qquad -u_1u_2\omega^2 + (u_1^2\pi + u_2^2 p)\omega = 0.$$

Therefore the quadric (8) *is distinguished from other quadrics in the pencil of Darboux by the fact that the corresponding cone* (12) *decomposes into a cone of the third class and a pencil of planes with* $C(-7/12)$ *as its axis*. Let p_1 and p_2 be the two tangent planes of the cone (12) through a given tangent of S at P but distinct from $z=0$, and t a plane through the same tangent such that

$$(z = 0, t; p_1, p_2) = \tau = \text{const.}$$

Then the equation of t is easily found to be

$$(14) \qquad \begin{aligned} (1-\tau)(u_1x + u_2y) + z\Big[\frac{7}{12}(\psi u_1 + \phi u_2) - 5\tau(\psi u_1 + \phi u_2) \\ - \frac{\tau}{3}\left(\frac{\gamma u_1^2}{u_2} + \frac{\beta u_2^2}{u_1}\right)\Big] = 0, \end{aligned}$$

u_1/u_2 being a parameter. *This plane envelopes a cone of the fourth order with three cuspidal tangent planes concurrent in the canonical line* $C(-(7-5K)/12(1-K))$, *while the latter is also the polar line of* $z=0$ *with respect to the trihedral of the three cuspidal generators.*

Finally, we shall give a geometrical interpretation to the cone of

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[2] This line appeared in Sannia's paper. See Rendiconti R. Accademia dei Lincei (6) vol. 9 (1929) pp. 1081–1085.

Segre. Consider the two quadrics of Moutard belonging to the tangents t and t' at a point of the surface; they intersect in the asymptotic tangents and a conic. The limiting position of the plane of this conic as $t'\to t$ is

$$z[4\beta^2 - 4\gamma^2 n^6 - 2\beta n\phi + 2\gamma n^5\psi + 4\beta\psi n^2 - 4\gamma\phi n^4] + 8(\gamma n^3 + \beta)n^2 x - 8(\beta + \gamma n^3)ny = 0, \tag{15}$$

which envelopes the cone of Segre.

NATIONAL UNIVERSITY OF CHEKIANG

ON A REPRESENTATION IN SPACE OF GROUPS OF CIRCLE AND TURBINE TRANSFORMATIONS IN THE PLANE

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1. Introduction. In a previous paper [6][1] the author showed that the oriented lineal elements in the euclidean plane can be mapped continuously and (1, 1) upon the points of *quasi-elliptic*[2] three-space Q_3 so that the whirl-similitude group of turbine[3] transformations in the euclidean plane is represented isomorphically upon the group of projective automorphisms of Q_3. By means of this representation *proper* turbines in the plane are mapped upon those real lines in Q_3 which do not intersect a line L, the real part of the quasi-elliptic absolute. It is the purpose of this note to investigate this representation analytically and to extend it so as to yield a (1, 1) continuous mapping of the turbines (proper and improper) in the *Moebius* plane upon all the lines in projective S_3. By such means we establish the isomorphism between certain groups of projective transformations in space on the one hand, and on the other of Kasner's 15-parameter group of turbine transformations [7] and some of its important subgroups, namely the Moebius, Laguerre, and Lie groups of circle transformations.

Other representations of turbines in space are due to Kasner and DeCicco [7, 8] and to A. Narasinga Rao [9]. The former use a

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[1] The numbers in brackets refer to the bibliography at the end of this paper.

[2] The term *quasi-elliptic space* is due to Blaschke [1, 2]. The absolute of this space is composed of a pair of conjugate imaginary planes $x_3^2+x_4^2=0$ and a pair of conjugate imaginary points $(1:\pm i:0:0)$.

[3] The geometry of turbines was initiated by Kasner [7]. An extensive bibliography on the subject is to be found in [5].

representation of lineal elements upon the points of S_3, similar to that used here; the latter maps turbines in the plane upon the points of S_4, and shows thereby that Kasner's group and its subgroups are isomorphic with certain groups of projectivities in S_4. Kasner and DeCicco do not indicate how their representation would map the groups of Moebius, Laguerre, and Lie in S_3.

2. The mapping $\mathfrak{M}$.

We shall need the following definitions.

DEFINITIONS. A *proper turbine* is a series of ∞^1 oriented lineal elements the points of which lie on a cycle (oriented circle) and the directions of which are inclined at the same angle to the direction of the cycle.

An *improper turbine* is a series of ∞^1 oriented parallel lineal elements the points of which lie on a straight line.

A *proper flat field* is a set of ∞^2 lineal elements which lie on the ∞^1 cycles having a given lineal element (the center of the flat field) in common.

An *improper flat field* is a set containing all of the ∞^2 lineal elements parallel to a given direction.

The proper turbines include the point turbines (=point cycles); the improper turbines include the spears (=oriented lines).

Let the oriented lineal element $\mathfrak{e}$ passing through the point x, y in the xy-plane have the direction θ. Let $z=x+iy$, $\bar{z}=x-iy$, $\zeta=e^{i\theta}$, $0\leqq\theta<2\pi$. Let the symbol (z, ζ) represents $\mathfrak{e}$; z and ζ shall be called the coordinates of $\mathfrak{e}$. In terms of these coordinates a proper turbine has an equation [6] of the form

$$z - l = r\zeta. \tag{2.1}$$

The point l is the center of the turbine and $|r|$ is the length of its radius. We shall represent the turbine (2.1) by the symbol $[l, r]$; l, r shall be called its coordinates. Turbines $[l, r]$ such that $r+\bar{r}=0$ are cycles, and those such that $r=0$ are point turbines.

A proper flat field having the lineal element (s, σ) for its center is given by the equation [6]

$$z - s = (\bar{z} - \bar{s})\sigma\zeta. \tag{2.2}$$

Let the symbol $\{s, \sigma\}$ represent this flat field. Two proper flat fields $\{s, \sigma\}$ and $\{s', \sigma'\}$ such that $\sigma\neq\sigma'$ determine a proper turbine $[l, r]$ where

$$l = (s\sigma' - s'\sigma)/(\sigma' - \sigma), \qquad r = (\bar{s} - \bar{s}')/(\bar{\sigma}' - \bar{\sigma}). \tag{2.3}$$

Let a point $\mathfrak{p}$ in S_3 have the homogeneous coordinates $x_1:x_2:x_3:x_4$.

The representation $\mathfrak{M}$ mapping $(z, \zeta) \rightleftarrows \mathfrak{p}$ continuously and (1, 1) is given by the equations

$$(2.4) \qquad z = 2(x_1 + ix_2)/(x_4 + ix_3), \qquad \zeta = (x_4 - ix_3)/(x_4 + ix_3)$$

and

$$(2.4^*) \qquad \begin{aligned} \rho x_1 &= i(z\bar\zeta + \bar z), & \rho x_2 &= z\bar\zeta - \bar z, \\ \rho x_3 &= 2(\bar\zeta - 1), & \rho x_4 &= 2i(1 + \bar\zeta). \end{aligned}$$

These equations define $\mathfrak{M}$ for all lineal elements in the euclidean plane and all points in S_3 except those on the line $L: x_3 = x_4 = 0$. The points of any plane $u_i x_i = 0$ such that $u_2 + iu_1 \neq 0$, that is, a plane that does not pass through L, are mapped upon the lineal elements of a proper flat field $\{s, \sigma\}$ where

$$(2.5) \qquad s = -2(u_3 + iu_4)/(u_2 + iu_1), \qquad \sigma = (u_2 - iu_1)/(u_2 + iu_1),$$

$$(2.5^*) \qquad \begin{aligned} \rho u_1 &= 2i(1 - \bar\sigma), & \rho u_2 &= 2(\bar\sigma + 1), \\ \rho u_3 &= -(s\bar\sigma + \bar s), & \rho u_4 &= i(s\bar\sigma - \bar s). \end{aligned}$$

Any plane u that does pass through L is mapped upon an improper flat field composed of ∞^2 parallel lineal elements (z, ζ) where

$$(2.6) \qquad \zeta = (u_3 + iu_4)/(u_3 - iu_4).$$

Let $\mathfrak{g}$ be any line in S_3 except L. Let the Plücker axial coordinates of $\mathfrak{g}$ be $q_{ij} = u_i v_j - u_j v_i$. Let us assume, as we may, that neither of the planes u and v passes through L. Evidently $\mathfrak{g}$ is mapped by $\mathfrak{M}$ upon a series of ∞^1 lineal elements common to the proper flat fields $\{s, \sigma\}$ and $\{s', \sigma'\}$ which correspond respectively to u and v. Such a series is a proper turbine when $\sigma \neq \sigma'$ and an improper turbine when $\sigma = \sigma'$.[4] Consequently, the proper turbines in the euclidean plane correspond (1, 1) to those real lines in S_3 which do not intersect L, and the improper turbines correspond (1, 1) to the real lines in S_3 that do intersect L; line L itself so far has no $\mathfrak{M}$-image. The correspondence between turbines and lines is given by the equations:

$$(2.7) \qquad \begin{aligned} \rho q_{12} &= 4i(\bar\sigma' - \bar\sigma), \\ \rho q_{13} &= -i[(s - s')\bar\sigma\bar\sigma' - (\bar s - \bar s') + (\bar s + s')\bar\sigma' - (s + \bar s')\bar\sigma], \\ \rho q_{14} &= -(s - s')\bar\sigma\bar\sigma' - (\bar s - \bar s') + (s - \bar s')\bar\sigma - (s' - \bar s)\bar\sigma', \\ \rho q_{34} &= -i(\bar s s'\bar\sigma' - s\bar s'\bar\sigma), \\ \rho q_{42} &= i[(s - s')\bar\sigma\bar\sigma' - (\bar s - \bar s') + (s + \bar s')\bar\sigma - (\bar s + s')\bar\sigma'], \\ \rho q_{23} &= (s - s')\bar\sigma\bar\sigma' + (\bar s - \bar s') + (s - \bar s')\bar\sigma + (\bar s - s')\bar\sigma'. \end{aligned}$$

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[4] $\sigma = \sigma'$ if and only if $\mathfrak{g}$ intersects L.

With the aid of equations (2.3) we find that the proper turbine determined by the flat fields $\{s, \sigma\}$ and $\{s', \sigma'\}$ such that $\sigma \neq \sigma'$ is a cycle when $(\bar{s}-\bar{s}')\sigma\sigma' = s - s'$, and a point turbine when $s = s'$. If $\sigma = \sigma'$ the pair of flat fields determines an improper turbine, which, in the event that $(\bar{s}-\bar{s}')\sigma^2 = s - s'$, becomes a spear. From (2.7) we find that to the cycles and spears in the euclidean plane correspond (1, 1) all but one of the lines of the linear complex

$$\mathcal{A}: \qquad q_{13} - q_{42} = 0;$$

the excepted line is L ($q_{12}:q_{13}:q_{14}:q_{34}:q_{42}:q_{23} = 0:0:0:1:0:0$). To the point turbines in the euclidean plane correspond (1, 1) all the lines but one, namely L, of the rotationally symmetric elliptic congruence

$$\mathcal{B}: \qquad q_{13} - q_{42} = 0, \qquad q_{14} - q_{23} = 0,$$

which has the lines $x_1 + ix_2 = x_3 - ix_4 = 0$ and $x_1 - ix_2 = x_3 + ix_4 = 0$ as directrices.

To make $\mathfrak{M}$ (1, 1) for all the lines of S_3, we therefore close the euclidean plane by adjoining one ideal point turbine at infinity t_∞, which shall serve as the $\mathfrak{M}$-image of L. To the point turbines of the closed (Moebius) plane there now correspond continuously and (1, 1) the lines of $\mathcal{B}$, and to the cycles and spears of the Moebius plane there correspond (1, 1) the lines of $\mathcal{A}$.

To the improper turbines correspond (1, 1) the lines of the special complex $q_{12} = 0$; this complex has the line L for its directrix. To the spears, therefore, correspond (1, 1) the lines of the special congruence

$$\mathcal{C}: \qquad q_{13} - q_{42} = 0, \qquad q_{12} = 0.$$

Let the terms *Moebius cycle*, *Laguerre cycle*, and *Lie cycle* designate respectively (1) a cycle with nonzero radius or a spear, (2) any cycle except t_∞, (3) any cycle or spear in the Moebius plane. Since the Moebius group of circle transformations in the plane, regarded as an extended group of contact transformations, transforms point turbines into point turbines and Moebius cycles into Moebius cycles, its $\mathfrak{M}$-image in S_3 is a group of ∞^6 collineations that transform the congruence $\mathcal{B}$ into itself and also transform the linear complex $\mathcal{A}$ into itself. The large (seven-parameter) Laguerre group in the plane is a group of contact transformations that convert spears into spears and Laguerre cycles into Laguerre cycles; consequently, this Laguerre group is isomorphic with a group of ∞^7 collineations in S_3 that leave invariant both the special congruence $\mathcal{C}$ and the complex $\mathcal{A}$. Lie's 10-parameter group of contact transformations transforming Lie cycles into Lie cycles is isomorphic with the group of ∞^{10} collineations in S_3

that leave $\mathcal{A}$ invariant. Kasner's 15-parameter group of turbine transformations in the plane is isomorphic with the group of projectivities (collineations and correlations) in S_3.

Various other representations of the Moebius, Laguerre, and Lie groups have been given, for instance in [3, 4].

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QUEENS COLLEGE

ON THE PATHS WITH MONGE'S EQUATIONS OF THE SECOND DEGREE AS CONDITIONS OF INTERSECTION

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1. **Introduction.** The problem on the conditions of intersection of a family of curves was suggested by S. Lie[2] and systematically studied by F. Engel.[3] Under the latter's initiative extensive researches have been made on the subject.[4] It is the aim of the present paper to point out the relations between this type of problems and the geometry of paths, thus obtaining the geometrical interpretation of some important invariants of a system of paths. To be definite, we consider in an $(n+1)$-dimensional space a $2n$-parameter family of paths. By using Pfaffian forms, we can define the tensor of the lowest order of the paths in a simple way. This tensor is essentially the one which plays a rôle in the so-called inverse problem of the calculus of variations.[5] It turns out that its vanishing is a necessary and sufficient condition for the conditions of intersection of neighboring paths to be given by a system of Monge's equations of the second degree.

2. **The tensor.** Let (x, y^i) $(i=1, \cdots, n)$ be coordinates of an $(n+1)$-dimensional space. A $2n$-parameter family of paths is defined by a system of ordinary differential equations of the form

$$(1) \quad y^{i''} = F^i(x, y^j, y^{j'}), \quad i, j = 1, \cdots, n, \quad y^{i'} = \frac{dy^i}{dx}, \quad y^{i''} = \frac{d^2y^i}{dx^2},$$

which can also be written in a Pfaffian system

$$(2) \qquad dy^i - p^i dx = 0, \qquad dp^i - F^i(x, y^j, p^j)dx = 0,$$

where p^i are auxiliary variables. To simplify our calculations we introduce the following Pfaffian forms

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[1] The author wishes to express his thanks to Professor Shiing-shen Chern for suggesting this problem.

[2] S. Lie and G. Scheffers, *Geometrie der Berührungstransformationen*, vol. I, Leipzig, 1896, p. 269.

[3] F. Engel, *Eine neue Methode zur Invariantentheorie der Differentialgleichungen*, Berichte über die Verhandlungen der Königlich Sachsischen Gesellschaft der Wissenschaften vol. 57 (1905) pp. 161–232.

[4] P. Steinbach, *Pfaffsche Gleichungen als Schnittbedingung von Kurvenscharen*, Mitteilungen des Mathematischen Seminars der Universität Giessen, vol. 2, no. 18, 1930.

[5] J. Douglas, *Solution of the inverse problem of the calculus of variations*, Proc. Nat. Acad. Sci. U.S.A. vol. 25 (1939) pp. 631–637.

$$\begin{aligned} &\omega^i = dy^i - p^i dx, \\ &\pi^i = dp^i - F^i dx - (\partial F^i/\partial p^k + \delta^i_k H)(dy^k - p^k dx)/2, \end{aligned} \tag{3}$$

the function $H = H(x, y^j, p^j)$ being any particular solution of the partial differential equation

$$2n\frac{dH}{dx} + nH^2 - 4\frac{\partial F^i}{\partial y^i} - \frac{\partial F^i}{\partial p^j}\frac{\partial F^j}{\partial p^i} + 2\frac{d}{dx}\left(\frac{\partial F^i}{\partial p^i}\right) = 0. \tag{4}$$

Here, as throughout this paper, Latin indices run from 1 to n, repeated indices denote summation, and the operator d/dx denotes total differentiation with respect to x along any of the paths, that is,

$$d/dx = \partial/\partial x + p^j\partial/\partial y^j + F^j\partial/\partial p^j.$$

By a simple calculation we find[6]

$$\begin{aligned} (\omega^i)' &= [dx\pi^i] + \frac{1}{2}\left(\frac{\partial F^i}{\partial p^k} + \delta^i_k H\right)[dx\omega^k], \\ (\pi^i)' &= Q^i_j[dx\omega^j] + \frac{1}{2}\left(\frac{\partial F^i}{\partial p^j} - \delta^i_j H\right)[dx\pi^j] \\ &\quad + \frac{1}{2}\left(\frac{\partial^2 F^i}{\partial p^k \partial p^l} + \delta^i_k\frac{\partial H}{\partial p^l}\right)[\omega^k\pi^l] + Q^i_{jk}[\omega^j\omega^k], \end{aligned} \tag{5}$$

where

$$\begin{aligned} Q^i_j = {} & \frac{\partial F^i}{\partial y^j} + \frac{1}{4}\frac{\partial F^i}{\partial p^k}\frac{\partial F^k}{\partial p^j} - \frac{1}{2}\frac{d}{dx}\left(\frac{\partial F^i}{\partial p^j}\right) \\ & - \frac{1}{n}\delta^i_j\left\{\frac{\partial F^m}{\partial y^m} + \frac{1}{4}\frac{\partial F^m}{\partial p^l}\frac{\partial F^l}{\partial p^m} - \frac{1}{2}\frac{d}{dx}\frac{\partial F^m}{\partial p^m}\right\}, \end{aligned} \tag{6}$$

while Q^i_{jk} are functions formed from F^i and H whose expressions do not interest us. The n^2 quantities Q^i_j constitute a tensor in the following sense. Let

$$\bar{x} = \bar{x}(x, y^j), \qquad \bar{y}^i = \bar{y}^i(x, y^j)$$

be a point transformation which carries our paths to a system of paths of the space $(\bar{x}, \bar{y}^i)$. Suppose the new paths be defined by the Pfaffian system

$$d\bar{y}^i - \bar{p}^i d\bar{x} = 0, \qquad d\bar{p}^i - \bar{F}^i(\bar{x}, \bar{y}^j, \bar{p}^j)d\bar{x} = 0, \tag{7}$$

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[6] E. Goursat, *Leçons sur le problème de Pfaff*, Paris, 1922, pp. 15–20.

from which the forms $\bar{\omega}^i$, $\bar{\pi}^i$ and the functions $\bar{H}$, $\bar{Q}^i_j$ can be defined in a similar way. Under this assumption there exist functions u, u_i, u^i_j, r^i_j, s^i_j such that

$$\begin{aligned} d\bar{x} &= u\,dx + u_i\omega^i, \\ \bar{\omega}^i &= u^i_j\omega^j, \qquad (8) \\ \bar{\pi}^i &= r^i_j\pi^j + s^i_j\omega^j. \end{aligned}$$

Taking the bilinear covariants of the above equations and comparing the coefficients on both sides, we obtain

$$\begin{aligned} r^i_j &= \frac{1}{u} s^i_j, \\ \frac{du^i_j}{dx} &= us^i_j + \frac{1}{2} uu^k_j\left(\frac{\partial\bar{F}^i}{\partial\bar{p}^k} + \delta^i_k\bar{H}\right) - \frac{1}{2} u^i_l\left(\frac{\partial F^l}{\partial p^j} + \delta^l_j H\right), \\ \frac{d}{dx}\left(\frac{1}{u}u^i_j\right) &= \frac{1}{2}u^l_j\left(\frac{\partial\bar{F}^i}{\partial\bar{p}^l} - \delta^i_l\bar{H}\right) - \frac{1}{2u}u^i_l\left(\frac{\partial F^l}{\partial p^j} - \delta^l_j H\right) + s^i_j, \qquad (9) \\ \frac{d}{dx}s^i_k &= uu^j_k\bar{Q}^i_j - \frac{1}{u}u^i_jQ^j_k + \frac{u}{2}s^j_k\left(\frac{\partial\bar{F}^i}{\partial\bar{p}^j} - \delta^i_j\bar{H}\right) \\ &\quad - \frac{1}{2}s^i_j\left(\frac{\partial F^j}{\partial p^k} + \delta^j_k H\right). \end{aligned}$$

Let v^i_j and s be defined by the equations

$$u^i_j v^j_k = \delta^i_k, \qquad s = s^i_j v^j_i,$$

δ^i_k being the well known Kronecker symbol. Then the first three sets of equations of (9) give

$$s^j_i = \frac{s}{n}u^j_i, \tag{10}$$

while the last set gives, after the elimination of ds^j_i/dx,

$$u^i_j\left(\frac{ds}{dx} + \frac{s^2u}{n} + su\bar{H}\right) = uu^k_j\bar{Q}^i_k - \frac{1}{u}u^i_kQ^k_j. \tag{11}$$

Multiplying (11) by v^j_i and contracting, we get

$$ds/dx + s^2u/n + su\bar{H} = 0,$$

since $\bar{Q}^i_i = Q^i_i = 0$. Finally, we have $\bar{Q}^i_j = (1/u^2)u^i_kQ^k_lv^l_j$, which shows that the quantities Q^i_j undergo a linear transformation. The vanishing of Q^i_j has therefore a geometrical meaning.

The quantities Q^i_j differ slightly from similar ones employed by Douglas.[5] The condition $Q^i_i=0$ makes them obey a simpler transformation law.

3. Conditions of intersection of paths. Let the system of finite equations defining the family of paths be

$$y^i = G^i(x, C^\alpha), \qquad \alpha = 1, \cdots, 2n. \tag{12}$$

We can consider C^α as coordinates in a $2n$-dimensional space R_{2n} whose points are our paths. Paths are said to be neighboring if the corresponding points are infinitely near. Let Γ and Γ' be two neighboring paths defined by the parameters C^α and $C^\alpha+\delta C^\alpha$ respectively. In order that they intersect, we must have

$$\delta y^i = (\partial G^i/\partial C^\alpha)\delta C^\alpha = 0, \qquad \alpha = 1, \cdots, 2n. \tag{13}$$

The system of $n-1$ equations obtained by eliminating x from (13) gives the conditions of intersection. The $n-1$ equations are independent, otherwise the paths intersecting a given path would depend on more than $n+1$ essential parameters, which case is excluded.

We proceed to show how the conditions of intersection can be treated most simply by our symbolism. Let us introduce two differential operations d and δ, the operation d denoting the variation along the path, while δ denotes the change from one path to the other. By definition they satisfy the conditions

$$dx \neq 0, \qquad dC^\alpha = 0, \qquad \delta x = 0, \qquad \delta C^\alpha \neq 0,$$

so that

$$\begin{aligned}
&\overset{i}{\omega}(d) = 0, \quad \overset{i}{\pi}(d) = 0,\\
&\overset{i}{\omega}(\delta) = \delta y^i = (\partial G^i/\partial C^\alpha)\delta C^\alpha,\\
&\overset{i}{\pi}(\delta) = \delta p^i - \frac{1}{2}\left(\frac{\partial F^i}{\partial p^j} + \delta^i_j H\right)\delta y^j \qquad (14)\\
&\qquad = \left\{\frac{\partial}{\partial C^\alpha}\left(\frac{\partial G^i}{\partial x}\right) - \frac{1}{2}\left(\frac{\partial F^i}{\partial p^j} + \delta^i_j H\right)\frac{\partial G^j}{\partial C^\alpha}\right\}\delta C^\alpha,\\
&\qquad\qquad\qquad \alpha = 1, \cdots, 2n.
\end{aligned}$$

According to the theory of completely integrable Pfaffian systems, the last $2n$ equations can be solved in terms of δC^α, thus expressing δC^α as linear combinations of $\omega^i(\delta)$ and $\pi^i(\delta)$.

Suppose that the system giving the conditions of intersection consists only of Monge's equations of the second degree:

$$A^{\epsilon}_{\alpha\beta}(C^{\gamma})\delta C^{\alpha}\delta C^{\beta} = 0, \qquad \alpha, \beta, \gamma = 1, \cdots, 2n;\ \epsilon = 1, \cdots, n-1. \tag{15}$$

We express the left-hand side in terms of $\omega^{i}(\delta)$ and $\pi^{i}(\delta)$, getting

$$A^{\epsilon}_{\alpha\beta}(C^{\gamma})\delta C^{\alpha}\delta C^{\beta} = a^{\epsilon}_{ij}\omega^{i}(\delta)\omega^{j}(\delta) + b^{\epsilon}_{ij}\omega^{i}(\delta)\pi^{j}(\delta) + e^{\epsilon}_{ij}\pi^{i}(\delta)\pi^{j}(\delta), \tag{16}$$
$$\alpha, \beta, \gamma = 1, \cdots, 2n, \quad \epsilon = 1, \cdots, n-1,$$

where the a's, b's, and e's are functions of x and C^{α} and, for unicity of the coefficients, we can assume

$$a^{\epsilon}_{ij} = a^{\epsilon}_{ji}, \quad e^{\epsilon}_{ij} = e^{\epsilon}_{ji}. \tag{17}$$

Since (15) is evidently satisfied when $\omega^{i}(\delta)=0$, e^{ϵ}_{ij} must vanish, so that (16) becomes

$$A^{\epsilon}_{\alpha\beta}(C^{\gamma})\delta C^{\alpha}\delta C^{\beta} = a^{\epsilon}_{ij}\omega^{i}(\delta)\omega^{j}(\delta) + b^{\epsilon}_{ij}\omega^{i}(\delta)\pi^{j}(\delta). \tag{16'}$$

The left-hand side of the preceding equation is independent of x, so that the same is true of the right-hand side. Thus we have

$$d\{a^{\epsilon}_{ij}\omega^{i}(\delta)\omega^{j}(\delta) + b^{\epsilon}_{ij}\omega^{i}(\delta)\pi^{j}(\delta)\} = 0. \tag{18}$$

To expand (18) we notice that from (5) we have

$$\begin{aligned} d\omega^{i}(\delta) &= dx\pi^{i}(\delta) + (\partial F^{i}/\partial p^{k} + \delta^{i}_{k}H)dx\omega^{k}(\delta)/2, \\ d\pi^{i}(\delta) &= Q^{i}_{j}dx\omega^{j}(\delta) + (\partial F^{i}/\partial p^{j} - \delta^{i}_{j}H)dx\pi^{j}(\delta)/2. \end{aligned} \tag{19}$$

Substituting these expressions into (18) we get

$$\begin{aligned} &\left\{da^{\epsilon}_{ik} + \left[a^{\epsilon}_{ij}\left(\frac{\partial F^{j}}{\partial p^{k}} + \delta^{j}_{k}H\right) + b^{\epsilon}_{ij}Q^{j}_{k}\right]dx\right\}\omega^{i}(\delta)\omega^{k}(\delta) + b^{\epsilon}_{ij}dx\pi^{i}(\delta)\pi^{j}(\delta) \\ &\quad + \left\{db^{\epsilon}_{ik} + \frac{1}{2}\left(4a^{\epsilon}_{ik} + b^{\epsilon}_{ij}\frac{\partial F^{j}}{\partial p^{k}} + b^{\epsilon}_{jk}\frac{\partial F^{j}}{\partial p^{i}}\right)dx\right\}\omega^{i}(\delta)\pi^{k}(\delta) = 0, \\ &\qquad \epsilon = 1, 2, \cdots, n-1. \end{aligned}$$

It follows that

$$b^{\epsilon}_{ij} + b^{\epsilon}_{ji} = 0,$$

$$db^{\epsilon}_{ik} + \frac{1}{2}\left(4a^{\epsilon}_{ik} + b^{\epsilon}_{ij}\frac{\partial F^{j}}{\partial p^{k}} + b^{\epsilon}_{jk}\frac{\partial F^{j}}{\partial p^{i}}\right)dx = 0,$$

$$\begin{aligned} 2da^{\epsilon}_{ik} + \Bigg\{&a^{\epsilon}_{ij}\left(\frac{\partial F^{j}}{\partial p^{k}} + \delta^{j}_{k}H\right) \\ &+ a^{\epsilon}_{kj}\left(\frac{\partial F^{j}}{\partial p^{i}} + \delta^{j}_{i}H\right) + b^{\epsilon}_{ij}Q^{j}_{k} + b^{\epsilon}_{kj}Q^{j}_{i}\Bigg\}dx = 0. \end{aligned} \tag{20}$$

On account of the symmetry of a^ϵ_{ik} and the skew-symmetry of b^ϵ_{ik} in the lower indices the second set of equations of (20) gives

$$a^\epsilon_{ik} = 0. \tag{21}$$

The last two equations of (20) then become

$$\begin{gathered} db^\epsilon_{ik} + \frac{1}{2}\left(b^\epsilon_{ij}\frac{\partial F^j}{\partial p^k} + b^\epsilon_{jk}\frac{\partial F^j}{\partial p^i}\right)dx = 0, \\ b^\epsilon_{ij}Q^j_k + b^\epsilon_{kj}Q^j_i = 0. \end{gathered} \tag{22}$$

As a conclusion of our discussions we see that the existence of b^ϵ_{ij} satisfying (22) is a necessary condition for the family of paths to have the prescribed property, that is, the conditions of intersection be given by a system of Monge's equations of the second degree.

4. **The main theorem.** Our main theorem can be stated as follows:

THEOREM. *The conditions of intersection of neighboring paths of the family defined by* (2) *consist only of Monge's equations of the second degree if and only if all the components of the tensor* Q^i_j *vanish.*

In order to prove the theorem we first establish two simple lemmas.

LEMMA 1. *Let* K^ϵ *be* $n-1$ *skew-symmetric matrices of order* n *whose first rows are linearly independent, and* R *be any matrix of the same order. Then the equations*

$$K^\epsilon R - R'K^\epsilon = 0, \quad \epsilon = 1, \cdots, n-1, \; R' = \text{transpose of } R, \tag{23}$$

imply that R *is a scalar matrix.*

PROOF. By hypothesis, there exist $n-1$ linear combinations C^ϵ of K^ϵ having their first rows identical to

$$(0, 1, 0, \cdots, 0, 0), (0, 0, 1, \cdots, 0, 0), \cdots, (0, 0, \cdots, 0, 1)$$

respectively. As the K's satisfy (23), so do all the C's, that is,

$$C^\epsilon R - R'C^\epsilon = 0.$$

By actually writing out the elements of the matrices, we can justify this lemma by a short calculation.

LEMMA 2. *Let* $b^\epsilon_{ij}x^iy^j = 0$, $\epsilon = 1, \cdots, n-1$, *be* $n-1$ *independent equations in* x^i, y^j, *with skew-symmetric coefficients*

$$b^\epsilon_{ij} + b^\epsilon_{ji} = 0.$$

Let Q *be a square matrix satisfying*

$$B^{\epsilon}Q - Q'B^{\epsilon} = 0, \tag{24}$$

where

$$B^{\epsilon} = (b^{\epsilon}_{ij}).$$

Then Q is a scalar matrix.

Proof. We can find x_0^i such that the $n-1$ linear equations in y^i,

$$b^{\epsilon}_{ij}x_0^{i}y^{j} = 0,$$

are linearly independent. Let P be any nonsingular matrix with x_0^i as its first row. The first rows of PB^{ϵ} are

$$(b^{\epsilon}_{i1}x_0^{i},\ b^{\epsilon}_{i2}x_0^{i},\ \cdots,\ b^{\epsilon}_{in}x_0^{i}),$$

respectively, and hence are linearly independent. It is easy to see that the matrices $PB^{\epsilon}P'$ satisfy all the hypotheses for K^{ϵ} in Lemma 1. By writing (24) in the form

$$(PB^{\epsilon}P')(P'^{-1}QP') - (P'^{-1}QP')'(PB^{\epsilon}P') = 0$$

it follows that $P'^{-1}QP'$, and hence Q, is a scalar matrix.

Proof of the main theorem. Suppose that the conditions of intersection of neighboring paths of the family consist only of Monge's equations of the second degree. There exist b^{ϵ}_{ij} $(\epsilon = 1, \cdots, n-1)$ satisfying (22) and having the property that the $n-1$ equations

$$b^{\epsilon}_{ij}\omega^{i}(\delta)\pi^{j}(\delta) = 0$$

are independent. From Lemma 2 it follows that

$$Q^{i}_{j} = (1/n)\delta^{i}_{j}Q^{l}_{l} = 0.$$

Conversely, suppose $Q^i_j = 0$. We regard $b^{\epsilon}_{ik} = -b^{\epsilon}_{ki}$ as functions of x, C^{α} and consider the system of $(n-1)n/2$ partial differential equations

$$\partial b_{ik}/\partial x + (1/2)(b_{ij}\partial F^{j}/\partial p^{k} + b_{jk}\partial F^{j}/\partial p^{i}) = 0,$$

where

$$\frac{\partial F^{j}}{\partial p^{l}} = \frac{\partial F^{j}(x, G^{m}, \partial G^{m}/\partial x)}{\partial p^{l}}.$$

Let b^{ϵ}_{ij} be $n-1$ sets of integrals satisfying the initial conditions that all $(b_{ij})_{x=x_0}$ vanish except

$$(b^{\epsilon}_{1\ \epsilon+1})_{x=x_0} = -(b^{\epsilon}_{\epsilon+1\ 1})_{x=x_0} = 1.$$

The quantities b^{s}_{ij} so obtained not only satisfy (22) but also possess the property that the equations

$$b^{s}_{ij}\omega^{i}(\delta)\pi^{j}(\delta) = 0$$

are independent. That these equations give the conditions of intersection of neighboring paths follows from our discussion. Therefore the theorem is proved.

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ERRATA, VOLUME 50

Page 31, line 7. Instead of “Abstract 50-1-14” read “Abstract 50-1-13.”

R. v. Mises, *Integral theorems in three-dimensional potential flow.* Page 603, display (17) should read $\int[\mathbf{r}(\mathbf{q}\cdot\mathbf{n})-(\mathbf{r}/2)\times(\mathbf{z}\times\mathbf{n})dS]$ $=\int[\mathbf{r}(\mathbf{q}\cdot\mathbf{n})-\mathbf{q}(\mathbf{r}\cdot\mathbf{n})/2+\mathbf{n}(\mathbf{r}\cdot\mathbf{q})/2]dS=0$.

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